

COMPUTABLE OPTIMAL VALUE BOUNDS FOR  
GENERALIZED CONVEX PROGRAMS

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SUMMARY

It has been shown by Fiacco that convexity or concavity of the optimal value of a parametric nonlinear programming problem can readily be exploited to calculate global parametric upper and lower bounds on the optimal value function. The approach is attractive because it involves manipulation of information normally required to characterize solution optimality. We briefly describe a procedure for calculating and improving the bounds as well as its extensions to generalized convex and concave functions. Several areas of applications are also indicated.

INTRODUCTION

We are concerned here with parametric nonlinear programming problems of the form

$$\min_x f(x,t) \quad \text{s.t.} \quad g(x,t) \geq 0, h(x,t) = 0 \quad P(t)$$

where  $f$  is a real valued function,  $g$  and  $h$  are vector valued functions, and  $t$  is a parameter vector. The optimal value function of  $P(t)$  is defined by

$$f^*(t) = \min \{ f(x,t) : x \in R(t) \}$$

where  $R(t)$  is the feasible set of the problem  $P(t)$  given by

$$R(t) = \{ x : g(x,t) \geq 0, h(x,t) = 0 \}$$

In this paper we describe a procedure, originally proposed by Fiacco (refs.1,2), for calculating piecewise-linear continuous global upper and lower parametric bounds on the convex (or concave) optimal value  $f^*$ . We also show how these bounds can be improved in a systematic manner until a desired accuracy, as measured by the maximal deviation from the optimal value over the interval of parameter values, is achieved. Extensions of this approach to generalized convex and concave optimal value functions are discussed as well and current experience with applications is described.

## COMPUTABLE BOUNDS ON CONVEX OPTIMAL VALUE FUNCTIONS

Consider the parametric problem  $P(t)$  and assume that its optimal value function,  $f^*(t)$ , is convex. This will be the case if  $P(t)$  is a jointly convex program, i.e. if  $f$  is jointly convex in  $x$  and  $t$ , the components of  $g$  are jointly concave in  $x$  and  $t$ , and those of  $h$  are jointly linear affine in  $x$  and  $t$  (ref. 3). The assumptions on  $g$  and  $h$  can actually be generalized by requiring only that the map  $R$  is convex (ref. 4).

Suppose now that we have evaluated  $f^*(t)$  and its slope at two distinct values  $t^1$  and  $t^2$  of the parameter  $t$ , where (for simplicity)  $t$  is assumed to be a scalar. Then, global definitional properties of convex functions immediately provide global parametric continuous, piecewise-linear bounds via linear supports and linear interpolation on the graph of  $f^*$  over the line segment  $(t^1, t^2)$ . This is illustrated in figure 1.

Practical implementation of bounds calculations requires only the information provided by most standard nonlinear programming algorithms. In particular, the solution of the problem  $P(t)$  as well as the associated optimal Lagrange multipliers must be determined for two distinct parameter values. The Lagrange multipliers will coincide with derivatives of  $f^*$  in case  $f^*$  is differentiable and with subgradients of  $f^*$  in case when  $f^*$  is nondifferentiable and convex. In both cases the multipliers can be used to compute the lower bounds on  $f^*$ .

Clearly, if  $f^*$  is convex on the convex set  $S \subseteq \mathbb{R}^r$ , then any supporting hyperplane of the epigraph at any  $t \in S$  provides a global lower bound on  $f^*$  over  $S$ . Both upper and lower bounds calculations obviously apply over any interval  $(t^1, t^2)$  in  $S$ , provided that  $f^*$  is convex over  $(t^1, t^2)$ . A standard technique for studying  $f^*$  over  $(t^1, t^2)$  is to consider  $t(a) = at^1 + (1-a)t^2$  and view  $f^*$  as a function of the scalar parameter  $a \in (0,1)$ . This allows for the simultaneous perturbation of all components  $t_i$  of  $t$ , which are now linear affine functions of the scalar parameter  $a$ .

A byproduct of this practical approach is the observation that if the feasible point to set map  $R$  is convex then  $x(a) = ax^1 + (1-a)x^2 \in R(t(a))$  if  $x^1 \in R(t^1)$  and  $x^2 \in R(t^2)$ . This leads to the simple calculation of a feasible parametric vector  $x(a)$  of a problem  $P(t(a))$  whenever the condition is met. Hence we also obtain the upper bound  $\bar{f}(a) = f(x(a), t(a))$  on  $f^*(t(a))$  over  $(t^1, t^2)$ . Since the calculation of  $x(a)$  does not depend on  $f$ , this does not require  $f$  to be convex. If  $f$  is jointly convex in  $(x, t)$ , then  $f^*$  is convex and  $\bar{f}(a)$  is a convex bound on or above  $f^*$  and below the linear upper bound given in figure 1.

The parametric bounds on the optimal value function  $f^*$  described above were constrained to one-dimensional perturbations of the parameter vector  $t$ . However, it is a simple matter to extend these bounds to multi-dimensional perturbations of  $t$ .

Suppose, for example, that  $f^*$  is convex on the convex set  $S \subseteq \mathbb{R}^r$  and that we are interested in bounds on  $f^*$  for  $t$  in some polyhedron  $M$  contained in  $S$  which is determined by its extreme points  $t^1, t^2, \dots, t^\ell$ . To obtain these bounds we need only to determine the values and subgradients of  $f^*$  at  $\ell$  points  $t^1, \dots, t^\ell$ . This information will be available if we compute optimal solutions  $x^*(t^i)$  and Lagrange multipliers for  $\ell$  nonlinear programs  $P(t^i)$ ,  $i=1, \dots, \ell$ , similarly to the case of one-dimensional perturbations. If, in addition,  $R$  is a convex map, then we can

calculate a feasible parametric vector  $x(a)$ , for the problem  $P(t(a))$  with  $t(a) \in M$ , as a convex combination of  $l$  solutions  $x^*(t^i)$ ,  $i=1, \dots, l$  as well as a sharper convex upper bound  $\bar{f}(a)$  on  $f^*$ .

The described approach for calculating parametric upper and lower bounds on convex  $f^*$  can be extended to the case of concave optimal value function  $f^*$ . The well known sufficient conditions for concavity of  $f^*$  require that  $f$  be concave in  $t$  for  $t \in S$  and the feasible set  $R(t) = R_0$  for all  $t \in S$  (that is  $R(t)$  must be fixed). This result can be generalized to programs with perturbed feasible sets  $R(t)$  by assuming that the map  $R$  is concave (ref. 4).

If we now assume, similarly to the convex case, that  $f^*$  is concave over the interval  $(t^1, t^2)$  and the values and slopes of  $f^*$  are known at two distinct points  $t^1$  and  $t^2$ , then a linear interpolation on the graph of  $f^*$  will provide a lower bound while a piecewise linear upper bound will be determined by the slopes of  $f^*$ . Figure 2 illustrates these bounds.

### REFINEMENTS OF OPTIMAL VALUE BOUNDS

In the previous section we described a procedure for calculating piecewise-linear optimal value bounds on convex or concave  $f^*(t)$  over the interval  $(t^1, t^2)$  of parameter values. We also showed that a parametric feasible solution vector  $x(a)$ ,  $a \in (0,1)$ , is an immediate by-product of this approach. This remarkably regular behavior is exploitable in a number of ways as will be shown next.

Consider a convex  $f^*(t(a))$ , where  $t(a) = at^1 + (1-a)t^2$ , and view it as a function of the scalar parameter  $a \in (0,1)$  with upper and lower bounds on  $f^*$  as depicted in figure 1. Suppose that we solve the program  $P(t(a^*))$  at some intermediate value  $a^* \in (0,1)$ . Then, this additional solution of  $P(t(a^*))$  enables us to easily calculate sharper piecewise-linear continuous upper and lower bounds on  $f^*$ . These new bounds on  $f^*$  along with previous bounds are illustrated in figure 3.

Moreover, we can calculate a more accurate piecewise-linear continuous feasible estimate  $x(a)$  of the parametric solution vector, which in this case is the linear interpolation between contiguous optimal solutions of  $P(t(a))$  at three values  $a=0, a^*, 1$ . The feasible solution  $x(a)$  allows, in turn, the computation of a sharper piece-convex continuous upper bound on  $f^*$ , given by  $\bar{f}(a) = f(x(a), t(a))$ .

Similar sharper piecewise-linear continuous upper and lower bounds can be computed for a concave optimal value function  $f^*$  by solving an additional program  $P(t(a^*))$  at some intermediate value  $a^* \in (0,1)$ . The improved bounds will be analogous to those depicted in figure 3.

It is clear from figure 3 that by repeatedly solving the program  $P(t(a))$  at intermediate values of  $a$ , the bounds on  $f^*$  may be quickly and significantly improved. The value  $a^*$  of the parameter at which the problem was solved is the value where the deviation between the current upper bound  $U$  and lower bound  $L$ , i.e.,  $U(a) - L(a)$ , is the maximum over the considered interval  $(0,1)$ . This is an appealing choice, although other choices might be dictated by other criteria or user interest; eg. it might be important to know  $f^*(a)$  accurately only for certain subintervals or certain choices of  $a$ .

## EXTENSIONS OF BOUNDS TO GENERALIZED CONVEX OPTIMAL VALUE FUNCTIONS

The approaches for calculating parametric optimal value bounds described earlier can be extended in several ways to include much wider classes of parametric programs. This means that optimal value bounds are much more widely applicable than is apparent from the results of the previous sections.

The first extension is obtained by considering structured classes of generalized convex and concave optimal value functions. Suppose that map  $R$  is convex and that  $f$  is quasiconvex in  $(x,t)$  for  $t \in (t^1, t^2)$ . Then,  $f^*$  is also quasiconvex (ref. 5) and therefore a constant upper bound of  $f^*$ , given by  $\max \{f^*(t^1), f^*(t^2)\}$ , is readily available as well as a sharper quasiconvex upper bound  $\bar{f}(a) = f(x(a), t(a))$ . Additional classes of convex and nonconvex programs for which parametric upper bounds on  $f^*$  can be computed are those where the objective function and, consequently, the optimal value function  $F^*$  are strongly convex, strictly quasiconvex and strictly pseudoconvex (ref. 5).

Analogous results can be obtained in the concave case. Assume, for example, that the feasible set  $R$  is arbitrary and fixed and that  $f$  is quasiconcave in  $t$  for  $t \in (t^1, t^2)$ . Then, quasiconcavity of  $f^*$  follows (ref. 5) and  $\min \{f^*(t^1), f^*(t^2)\}$  is a constant lower bound on  $f^*$ .

The second extension is possible by considering generalized convex programs which are transformable into standard convex programs. Consider program  $P(t)$  with a convex feasible map  $R$  and an  $F$ -convex objective function  $f$ . That means that the composed function  $f_F(x,t) = F\{f(x,t)\}$  is convex in  $(x,t)$  where  $F$  is a continuous, one-to-one function (ref. 6). Thus the optimal value function  $f_F^*$  of a modified problem  $P_F(t)$

$$\min f_F(x,t) \quad \text{s.t.} \quad g(x,t) \geq 0, h(x,t) = 0 \quad P_F(t)$$

is convex and therefore piecewise-linear upper and lower bounds on  $f_F^*$ , given by  $L(t) \leq f_F^*(t) \leq U(t)$  can be calculated. Then, since  $f_F^*(t) = F\{f^*(t)\}$ , one immediately obtains the following bounds on  $f^*(t)$  (provided that  $F$  is nondecreasing):  $F^{-1}\{L(t)\} \leq f^*(t) \leq F^{-1}\{U(t)\}$ . These bounds are in general nonlinear and nonconvex but, nevertheless, can be calculated without difficulty once the program  $P_F(t)$  has been solved.

## EXPERIENCE WITH APPLICATIONS

Several preliminary studies have been conducted to investigate some of the more immediate computational and practical implications of the outlined approach for generating global parametric upper and lower optimal value bounds. The procedure for calculating optimal value bounds for both convex and concave optimal value functions was implemented by Fiacco and Ghaemi (ref. 7) as an additional module in the penalty-function based sensitivity-analysis computer program SENSUMT.

Fiacco and Ghaemi (ref. 8) studied a geometric programming model of a stream water pollution abatement system and calculated bounds on the convex optimal value (defined as the annual cost of operation) of an equivalent convex program. The indicated water pollution bounds calculation involved the perturbation of a single right-hand-side parameter, the allowable oxygen deficit level in the final reach of the stream, that proved to be the most influential parameter in the prior sensitivity study.

Subsequently, Fiacco and Kyparisis (refs. 9, 10) utilized SENSUMT to calculate bounds on the optimal value function for the same water pollution abatement model when 30 (not all right-hand-side) most influential constraint parameters were perturbed simultaneously. In this application, the optimal value function was not convex in full neighborhood of the base value of the parameter vector. However, it was possible to show that the restriction of  $f^*(t)$  to the subset of parameters involved in the desired perturbation is convex.

In another study involving the convex equivalent of a geometric programming model of a power system energy model, to find the turbine exhaust annulus and condenser system design that minimizes total annual fixed plus operating cost, Fiacco and Ghaemi (ref. 11) used SENSUMT to obtain bounds on the optimal value function for a variety of single objective function and constraint parameter changes. A novelty of this analysis is the exploitation of problem structure to calculate a nonlinear lower bound on the optimal value function. In addition, parametric bounds are computed on the optimal value which is concave for certain perturbations of objective function parameters.

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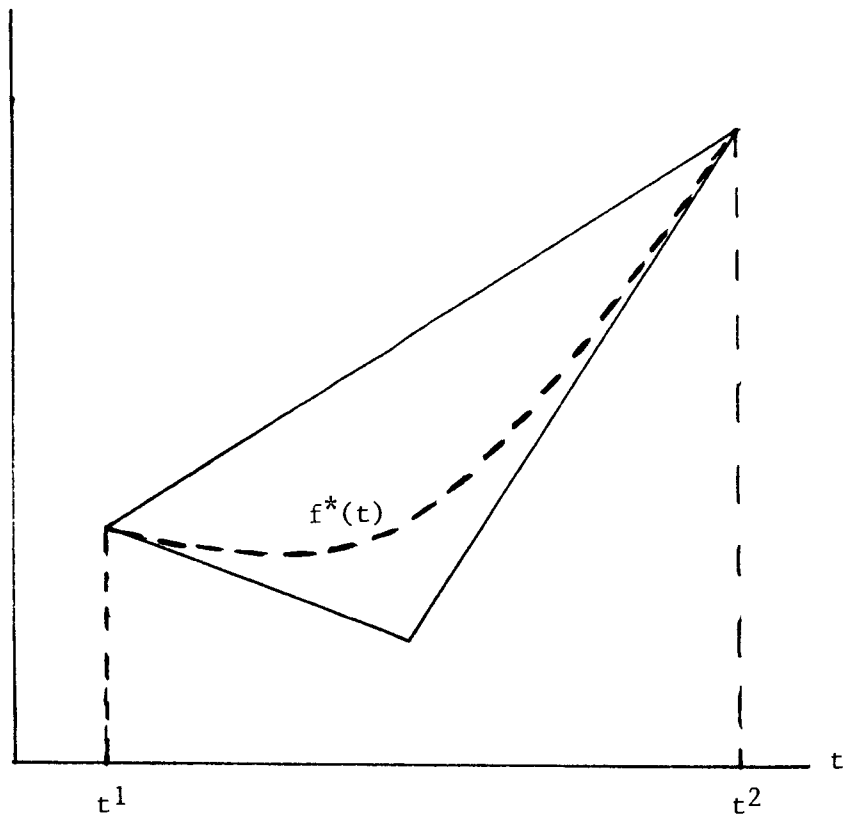


Figure 1. Optimal value bounds on convex  $f^*$ .

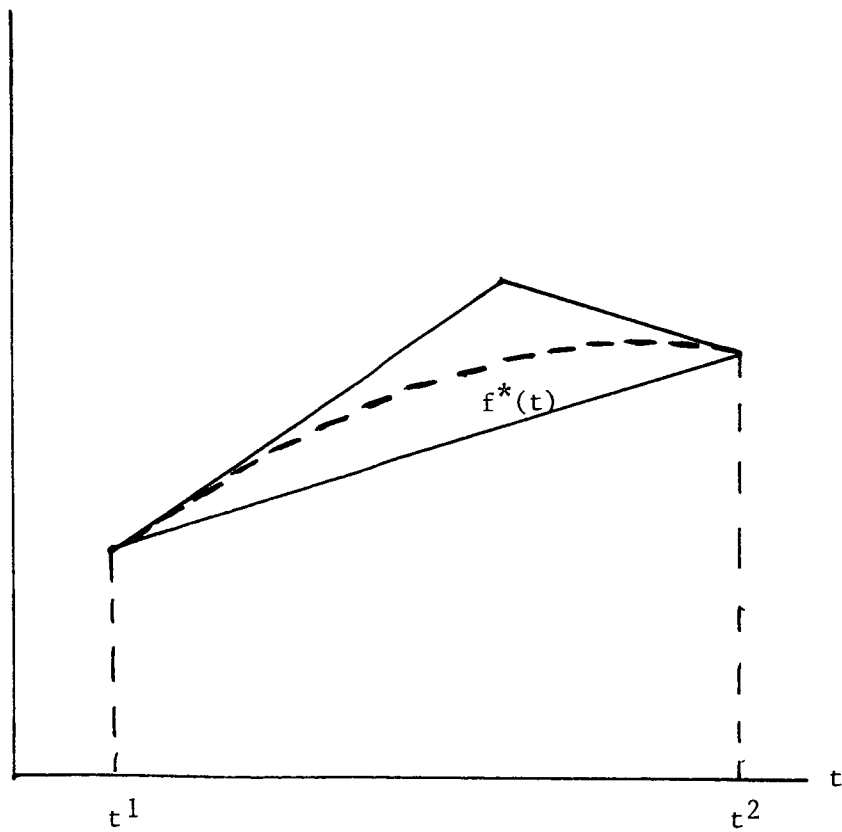


Figure 2. Optimal value bounds on concave  $f^*$ .

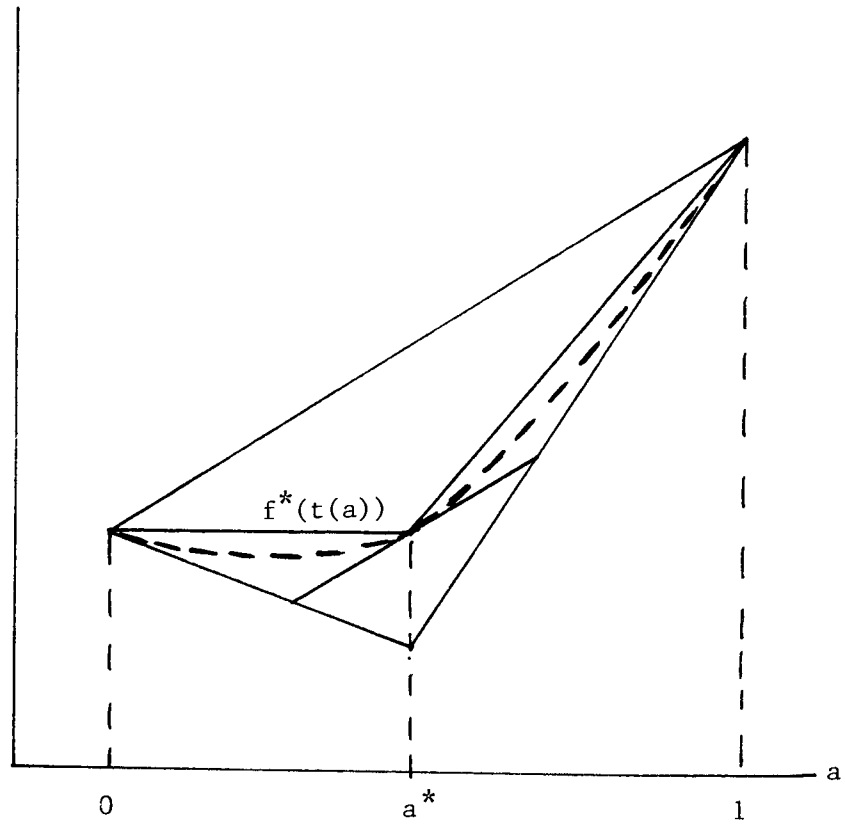


Figure 3. Improved optimal value bounds on convex  $f^*$ .