

Computation of a Multivariate F Distribution*

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Abstract. Methods for evaluating the joint cumulative probability integral associated with random variables $F_k = (X_k/r_k)/(Y/s)$, $k = 1, 2, \dots, n$, are considered where the X_k and Y are independently $\chi^2(r_k)$ and $\chi^2(s)$, respectively. For $n = 2$, series representations in terms of incomplete beta distributions are given, while a quadrature with efficient procedures for the integrand is presented for $n \geq 2$. The results for $n = 2$ are applied to the evaluation of the correlated bivariate F distribution.

Introduction. If X_k , $k = 1, \dots, n$, and Y are independent χ^2 random variables with r_k , $k = 1, \dots, n$, and s degrees of freedom, respectively, then the random variables

$$F_k = \frac{X_k/r_k}{Y/s}, \quad k = 1, \dots, n,$$

have a joint cumulative distribution $P(0 \leq F_1 \leq f_1, \dots, 0 \leq F_n \leq f_n \mid s, r_k, k = 1, \dots, n)$ given by

$$(1) \quad I_n(\hat{\alpha}, \hat{c}, \beta) = \int_0^\infty \frac{e^{-z} z^{\beta-1}}{\Gamma(\beta)} \prod_{k=1}^n \frac{\gamma(\alpha_k, c_k z)}{\Gamma(\alpha_k)} dz, \quad \beta > 0, \alpha_k > 0,$$

where

$$c_k = f_k \alpha_k / \beta, \quad k = 1, \dots, n, \quad \alpha_k = r_k / 2 \quad \text{and} \quad \beta = s / 2$$

and the incomplete gamma function is defined by

$$\gamma(\alpha, x) = \int_0^x e^{-t} t^{\alpha-1} dt, \quad \alpha > 0, x \geq 0.$$

This distribution arises in many applications, most of which are special cases of (1). Among these are the inverted Dirichlet distribution, the maximum F distribution or multivariate t^2 distribution [25, p. 159] and an important class where $\alpha_k = \alpha$ for all k . References [5], [6], [7], [10], [15], [16], [18], [20] contain recent results which utilize this multivariate F .

The analysis below forms the basis for a procedure by which I_n can be evaluated numerically over a wide range of parameters.

Analysis. We first take the case where $\mu \equiv \beta + \sum_{k=1}^n \alpha_k > 1$ and show that the integrand

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$$(2) \quad g(z) = \frac{e^{-z} z^{\beta-1}}{\Gamma(\beta)} \prod_{k=1}^n \frac{\gamma(\alpha_k, c_k z)}{\Gamma(\alpha_k)}$$

is bell-shaped with a single maximum at $z_0 > 0$. The integration then proceeds from z_0 to the left and right in the form

$$(3) \quad I_n = R_1 + \sum_{i=0}^{N_1} \int_{z_0 - (i+1)\sigma}^{z_0 - i\sigma} g(z) dz + \sum_{i=0}^{N_2} \int_{z_0 + i\sigma}^{z_0 + (i+1)\sigma} g(z) dz + R_2,$$

where $z_i = z_0 - (i+1)\sigma$ and R_1 are replaced by zero if z_i is negative. This formula sums quadratures over lengths σ which approximate the "spread" of $g(z)$. This step is necessary in order to include a large, but not excessive, portion of the area when parameters vary widely. R_1 and R_2 are truncation errors given by

$$R_1 = \int_0^{z_L} g(z) dz, \quad R_2 = \int_{z_U}^{\infty} g(z) dz$$

which can be bounded in the form

$$(4) \quad \begin{aligned} R_1 < B_1 &= 0 && \text{if } z_L = 0, \\ &= \frac{\gamma(\beta, z_L)}{\Gamma(\beta)} \prod_{k=1}^n \frac{\gamma(\alpha_k, c_k z_L)}{\Gamma(\alpha_k)} && \text{if } z_L > 0, \end{aligned} \quad R_2 < B_2 = \frac{\Gamma(\beta, z_U)}{\Gamma(\beta)},$$

where $z_L = \max\{0, z_0 - (N_1 + 1)\sigma\}$ and $z_U = z_0 + (N_2 + 1)\sigma$. With these bounds, a relative error test (which can be made after each quadrature addition) is appropriate for the truncation error since

$$\frac{R_1 \text{ or } R_2}{\text{exact sum}} \leq \frac{R_1 \text{ or } R_2}{\text{accumulated sum}} < \frac{B_1 \text{ or } B_2}{\text{accumulated sum}}.$$

The length σ over which each quadrature is taken is estimated by the Laplace method for an asymptotic form. That is, we write $y(z) = \ln g(z)$, $y'(z_0) = 0$,

$$(5) \quad I_n = \int_0^{\infty} e^{y(z)} dz \sim e^{y(z_0)} \int_{-\infty}^{+\infty} \exp\left\{-\frac{1}{2} |y''(z_0)| (z - z_0)^2\right\} dz = e^{y(z_0)} \left(\frac{2\pi}{|y''(z_0)|}\right)^{1/2}$$

and

$$(6) \quad \sigma = (|y''(z_0)|)^{-1/2}.$$

Now, we show that there is only one maxima of $g(z)$, give an equation for z_0 and compute $y''(z_0)$ for (6). Logarithmic differentiation of (2) together with the confluent series form of $\gamma(\alpha, x)$,

$$(7) \quad \gamma(\alpha, x) = \frac{e^{-x} x^\alpha}{\alpha} \Phi(1, 1 + \alpha; x), \quad \Phi(a, c; x) = \sum_{k=0}^{\infty} \frac{(a)_k x^k}{(c)_k k!}, \quad c \neq 0, -1, \dots,$$

yield

$$y'(z) = \frac{g'(z)}{g(z)} = \left[-1 + \frac{\beta - 1}{z} + \sum_{k=1}^n \frac{\alpha_k}{z \Phi(1, 1 + \alpha_k; c_k z)} \right].$$

Since $\mu > 1$ guarantees that $g(0) = 0$, the nonzero extrema occur at the roots of

$$(8) \quad z = \beta - 1 + \sum_{k=1}^n \frac{\alpha_k}{\Phi(1, 1 + \alpha_k; c_k z)} \equiv f(z).$$

The relations

$$(9) \quad \begin{aligned} \Phi'(1, 1 + \alpha; x) &= \frac{1}{1 + \alpha} \Phi(2, 2 + \alpha; x) > 0, \quad x \geq 0, \alpha > 0, \\ \Phi(1, 1 + \alpha; x) &\sim \Gamma(1 + \alpha)e^{-x}x^{-\alpha}, \quad \text{for } x \rightarrow \infty, \end{aligned}$$

establish the monotone decreasing behavior of the right side of (8) and hence the uniqueness of the root $z_0 > \max\{0, \beta - 1\}$ for $\mu > 1$.

Since

$$g(0) = 0, \quad g(z) > 0 \quad \text{for } z > 0, \quad \text{and} \quad \lim_{z \rightarrow \infty} g(z) = 0,$$

$g(z_0)$ is a maximum of $g(z)$ and

$$(10) \quad g''(z_0) = -\frac{1}{z_0} - \sum_{k=1}^n \frac{\alpha_k c_k}{(1 + \alpha_k)z_0} \frac{\Phi(2, 2 + \alpha_k; c_k z_0)}{\Phi^2(1, 1 + \alpha_k; c_k z_0)}.$$

If $1 > \mu > 0$, the integral I_n exists but (8) has no solution and $g' < 0$. The integrand is therefore monotone decreasing from ∞ to 0 and the integration scheme must account for the infinite singularity at $z = 0$ when $z = 0$ is in the interval of integration. If $2 > \mu > 1$, $g(0) = 0$ but the infinite derivative of $g(z)$ at $z = 0$ causes polynomial type integration schemes to converge very slowly. For $\mu \geq 2$ this problem is less severe. In many cases, the truncation on the left completely eliminates the problem of singular behavior. If $\mu = 1$, no singularity at $z = 0$ is encountered, and the integrand decreases monotonically to zero.

Computational Considerations. Due to the wide range of numbers which can be generated, the integrand $g(z)$ is conveniently evaluated in the form

$$(11) \quad g(z) = e^{h(z)} \prod_{k=1}^n \frac{\gamma(\alpha_k, c_k z)}{\Gamma(\alpha_k)}$$

to prevent underflow or overflow during a direct evaluation. Here

$$h(z) = -z + (\beta - 1) \ln z - \ln \Gamma(\beta)$$

and the ratios γ/Γ are generated by the following scheme. If $x < 1 + \alpha$, we use (7) for Φ in the form

$$\Phi(1, c; x) = \sum_{k=1}^N A_k + R_N, \quad R_N \leq \frac{A_{N+1}}{1 - x/(c + N)}, \quad x < c,$$

$$A_1 = 1, \quad A_{k+1} = A_k x / (c + k - 1), \quad k = 1, 2, \dots, N,$$

with $c = 1 + \alpha$ and compute $\gamma(\alpha, x)/\Gamma(\alpha)$ from (7),

$$\frac{\gamma(\alpha, x)}{\Gamma(\alpha)} = \Phi(1, 1 + \alpha; x) \exp\{-x + \alpha \ln x - \ln \Gamma(\alpha + 1)\}.$$

On the other hand, if $x \geq 1 + \alpha$, we compute K so that $c = 1 + \alpha + K > x$, generate Φ as above and recur backward with

$$y_K = S\Phi(1, 1 + \alpha + K; x),$$

$$y_{k-1} = \frac{x}{\alpha + k} y_k + S, \quad k = K, K - 1, \dots, 1,$$

which is a modification of the two term recurrence relation for γ . Then, $y_0 = S\Phi(1, 1 + \alpha; x)$ where S is a scale factor on the order of the underflow limit of the machine. Finally,

$$\frac{\gamma(\alpha, x)}{\Gamma(\alpha)} = (\exp\{-x + \alpha \ln x - \ln \Gamma(\alpha + 1)\}/S) \cdot y_0$$

takes care of the scaling. Refinements to take care of the tails of γ/Γ make the routine faster and eliminate unnecessary computation when the spread of $g(z)$ is large and some ratios in (11) are 1 to the word length of the machine. Thus, we seek $X(\alpha)$ such that, for $x > X$, $\gamma(\alpha, x)/\Gamma(\alpha) = 1$ to the word length of the machine. This value can be estimated from the asymptotic expansion

$$\frac{\gamma(\alpha, x)}{\Gamma(\alpha)} \sim 1 - \frac{\alpha e^{-x} x^{\alpha-1}}{\Gamma(\alpha + 1)} = 1 - 10^{-E}$$

where we take $x = c\alpha$ and solve for c in the relation

$$F(c) = -\alpha(c - 1) + (\alpha - 1) \ln c + E \ln 10 + \frac{1}{2} \ln \alpha = 0$$

after using the Stirling approximation for $\Gamma(\alpha + 1)$. If we take $c = 1$ to start Newton's method, the convergence is from above and the termination at c_0 always provides a conservative estimate of $X(\alpha) = c_0\alpha$.

Values of $X(\alpha)$ were generated and an empirical form

$$X(\alpha) = \frac{c_1\alpha^3 + c_2\alpha^2 + c_3\alpha + c_4}{\alpha^2 + c_5\alpha + c_6}$$

was fitted to maximum errors of approximately 1% by a linear least squares analysis for the ranges $1 \leq \alpha \leq 200$, $200 \leq \alpha \leq 10,000$, $10,000 \leq \alpha \leq 100,000$ with $E = 14$ for a CDC 6600 computer. If $\alpha \leq 1$, we increase α by one and proceed for $\alpha > 1$ entering the backward recursive loop at least one time, even when $x < 1 + \alpha$. For the lower tail, the brackets $\{ \}$ in the exponentials can be tested for underflow. As described, γ/Γ is a millisecond significant digit routine on the CDC 6600.

$z = \max\{0, \beta - 1\}$ provides a starting value for Newton's iteration in the solution of (8) for z_0 when $\mu > 1$. Then,

$$f'(z) = - \sum_{k=1}^n \frac{\alpha_k c_k}{(1 + \alpha_k)} \frac{\Phi(2, 2 + \alpha_k; c_k z)}{\Phi^2(1, 1 + \alpha_k; c_k z)}$$

is needed in addition to (8) and

$$y''(z_0) = [f'(z_0) - 1]/z_0.$$

The convergence criteria for Newton's method need not be severe since z_0 is not needed very accurately.

For the bound B_1 , the procedure described above is used in computing the ratios γ/Γ needed in (4). The bound B_2 can be computed by

$$B_2 = \frac{\Gamma(\beta, x)}{\Gamma(\beta)} = 1 - \frac{\gamma(\beta, x)}{\Gamma(\beta)}$$

when I_n is relatively large (say ≥ 0.1 as measured a priori by (5)) even though several

significant figures will be lost when the relative error test is severe. However, B_2 must be generated another way when I_n is much smaller because all significance will be lost in this relation and the relative error test cannot be met. In these cases, the continued fraction

$$S_{\nu+1} = 0,$$

$$S_k = x + \frac{(k - \beta_0)S_{k+1}}{k + S_{k+1}}, \quad k = \nu, \nu - 1, \dots, 1,$$

$$A_1 = e^x x^{-\beta_0} \frac{\Gamma(\beta_0, x)}{\Gamma(\beta_0)} = \frac{1/\Gamma(\beta_0)}{S_1} \cdot S, \quad 0 < \beta_0 \leq 1,$$

with the forward recurrence scheme for $\beta = \beta_0 + N > 1$,

$$T_1 = \frac{1/\Gamma(\beta_0)}{\beta_0} \cdot S$$

$$A_{k+1} = A_k + T_k, \quad k = 1, 2, \dots, N = \left\{ \begin{array}{l} [\beta] \\ [\beta] - 1 \end{array} \right\}$$

$$T_{k+1} = T_k \cdot x / (\beta_0 + k),$$

$$\frac{\Gamma(\beta, x)}{\Gamma(\beta)} = (\exp\{-x + \beta_0 \ln x\} / S) \cdot A_{N+1}$$

has proven satisfactory. S is a positive scale factor on the order of the underflow limit of the machine.

The asymptotic form (5) produced the right order of magnitude for I_n in all cases tested with one to three significant digits quite common.

In the case that $\alpha_k = \alpha$ and $c_k = c$ for all k , the computer time required is much less since the product in (1) reduces to a simple power and only one gamma ratio need be evaluated for each z . In this case, the sums (8), (10), and (11) as well as the product in (4) collapse to one term. In [13], Gupta obtained (1) for this case and gave percent point tables for c with $\alpha = \beta = \text{integer}$ for $1 \leq \alpha \leq 25$ and $1 \leq n \leq 10$. Samples from these tables were checked using this procedure on a CDC 6600 computer with a Romberg integration routine. The functional relationships were extended in tabular form in [2] to values of n up to 1000 for various values of c in $1.1 \leq c \leq 3.0$. Values of $\beta = \alpha$ up to 4700 appear in the body of the tables.

Special Cases. If one writes the gamma functions as integrals and exchanges the order of integration, the result is an n -fold integral

$$I_n = c_1^{\alpha_1} \dots c_n^{\alpha_n} \frac{\Gamma(\mu)}{\Gamma(\beta)\Gamma(\alpha_1) \dots \Gamma(\alpha_n)} \int_0^1 \dots \int_0^1 \frac{v_1^{\alpha_1-1} \dots v_n^{\alpha_n-1} dv_1 \dots dv_n}{(1 + c_1 v_1 + \dots + c_n v_n)^\mu}$$

which can be identified as an n -fold Lauricella sum F_A of n variables [4, pp. 114, 115]. Although special manipulative formulae exist for this function, the results do not appear numerically advantageous for $n > 1$. However, this integral is also an inverted Dirichlet distribution and some recursive results are given in [27] for α_k even.

For $n = 1$, we have very simple results in terms of the incomplete beta distribution

$$(12) \quad \int_0^\infty \frac{e^{-z} z^{\beta-1}}{\Gamma(\beta)} \frac{\gamma(\alpha, cz)}{\Gamma(\alpha)} dz = I_x(\alpha, \beta), \quad \alpha > 0, \beta > 0,$$

$$\int_0^\infty \frac{e^{-z} z^{\beta-1}}{\Gamma(\beta)} \frac{\Gamma(\alpha, cz)}{\Gamma(\alpha)} dz = I_{1-x}(\beta, \alpha),$$

$$x = c/(1+c), \quad 1-x = 1/(1+c).$$

For $n = 2$, the Lauricella function reduces to the Appell F_2 function of two variables [4, p. 28], [8, p. 230]. This function also has special manipulative properties, but some additional formulae in terms of readily computable functions can be derived. We start with

$$I_2(\hat{\alpha}, \hat{c}, \beta) = \int_0^\infty \frac{e^{-z} z^{\beta-1}}{\Gamma(\beta)} \frac{\gamma(\alpha_1, c_1 z)}{\Gamma(\alpha_1)} \frac{\gamma(\alpha_2, c_2 z)}{\Gamma(\alpha_2)} dz$$

and use various representations of $\gamma(\alpha_1, c_1 z)$ and $\gamma(\alpha_2, c_2 z)$ to derive sums in terms of incomplete beta distributions. Note the symmetry in (α_1, c_1) and (α_2, c_2) . The representations presented below are obtained by applying (7) in addition to

$$\frac{\gamma(\alpha, x)}{\Gamma(\alpha)} = 1 - \frac{\Gamma(\alpha, x)}{\Gamma(\alpha)}, \quad \frac{\Gamma(n, x)}{\Gamma(n)} = e^{-x} \sum_{k=0}^{n-1} \frac{x^k}{k!},$$

to one or both gamma functions followed by reduction with (12). The results are

$$(13) \quad I_2 = \frac{(1-u)^\beta}{\alpha_1} \frac{\Gamma(\beta + \alpha_1)}{\Gamma(\beta)\Gamma(\alpha_1)} \sum_{k=0}^\infty \frac{(\beta + \alpha_1)_k}{(1 + \alpha_1)_k} u^{k+\alpha_1} I_v(\alpha_2, k + \beta + \alpha_1),$$

$$u = c_1/(1+c_1), \quad 1-u = 1/(1+c_1),$$

$$y = c_2/(1+c_1+c_2), \quad 1-y = (1+c_1)/(1+c_1+c_2),$$

$$(14) \quad I_2 = I_u(\alpha_1, \beta) - \frac{(1-u)^\beta}{\alpha_1} \frac{\Gamma(\beta + \alpha_1)}{\Gamma(\beta)\Gamma(\alpha_1)} \sum_{k=0}^\infty \frac{(\beta + \alpha_1)_k}{(1 + \alpha_1)_k} u^{k+\alpha_1} I_{1-y}(k + \beta + \alpha_1, \alpha_2),$$

$$(15) \quad I_2 = I_v(\alpha_2, \beta) - (1-u)^\beta \sum_{k=0}^{\alpha_1-1} \frac{(\beta)_k}{k!} u^k I_v(\alpha_2, k + \beta),$$

$$v = c_2/(1+c_2), \quad 1-v = 1/(1+c_2), \quad \alpha_1 \text{ a positive integer,}$$

$$(16) \quad I_2 = 1 - I_{1-u}(\beta, \alpha_1) - I_{1-y}(\beta, \alpha_2)$$

$$+ (1-u)^\beta \sum_{k=0}^{\alpha_1-1} \frac{(\beta)_k}{k!} u^k I_{1-y}(k + \beta, \alpha_2), \quad \alpha_1 \text{ a positive integer.}$$

(15) was obtained by Tiao and Guttman in [27]. Sequences of incomplete beta functions $I_x(p+k, q)$ or $I_x(p, q+k)$, $k = 0, 1, 2, \dots$, can easily be generated by the methods described in [1], [3], [11], and [12]. The series in (13) and (14) will be slowly convergent if β is large and c_1 is large. On the other hand, (15) and (16) are computationally appealing if either α_1 or α_2 is an integer due to the symmetry in (α_1, c_1) and (α_2, c_2) . The series (13) has all positive terms and one expects significant digit results. The incomplete beta function in (14) helps convergence since $I_x(p+k, q)$ decreases monotonically to zero for $k \rightarrow \infty$ while $I_x(p, q+k)$ increases monotonically to 1 for $k \rightarrow \infty$.

If $P_{ab}(a_1 \leq F_1 \leq b_1, \dots, a_n \leq F_n \leq b_n | \beta, \alpha_k, k = 1, \dots, n)$ is desired, expansion of the product in

$$P_{ab} = \int_0^\infty \frac{e^{-z} z^{\beta-1}}{\Gamma(\beta)} \prod_{k=1}^n \frac{\gamma(\alpha_k, c_{kb}z) - \gamma(\alpha_k, c_{ka}z)}{\Gamma(\alpha_k)} dz,$$

with

$$c_{kb} = \alpha_k b_k / \beta, \quad c_{ka} = \alpha_k a_k / \beta,$$

gives results in terms of I_n . The special case, where $\alpha_k = \alpha, a_k = a, b_k = b$, is

$$(17) \quad P_{ab} = \sum_{k=0}^n (-1)^k \binom{n}{k} I_n(\alpha, \hat{c}_k, \beta),$$

where

$$(18) \quad I_n(\alpha, \hat{c}_k, \beta) = \int_0^\infty \frac{e^{-z} z^{\beta-1}}{\Gamma(\beta)} \left[\frac{\gamma(\alpha, c_b z)}{\Gamma(\alpha)} \right]^{n-k} \left[\frac{\gamma(\alpha, c_a z)}{\Gamma(\alpha)} \right]^k dz$$

and $c_b = \alpha b / \beta, c_a = \alpha a / \beta$. In this form, only two gamma ratios need to be evaluated for each value of z in (11). On the CDC 6600, with the relative error requirements on the order of 5×10^{-6} and $n = 20$, one can expect each integral of (17) to take on the order of 0.5 to 5 seconds with Romberg integration, the longer times occurring for larger k values. It is possible that an adaptive Romberg routine could reduce the computer time.

The Correlated Bivariate F Distribution. The bivariate χ^2 distribution with correlation ρ^2 for random variables X_1, X_2 , each with m degrees of freedom, can be represented as [22]

$$g(x_1, x_2) = (1 - \rho^2)^{m/2} \sum_{i=0}^\infty \frac{(m/2)_i}{i!} \rho^{2i} \frac{(x_1 x_2)^{m/2+i-1} \exp\{- (x_1 + x_2) / [2(1 - \rho^2)]\}}{[2^{m/2+i} (1 - \rho^2)^{m/2+i} \Gamma(m/2 + i)]^2}.$$

If Y is χ^2 with n degrees of freedom

$$f(y) = \frac{1}{2^{n/2} \Gamma(n/2)} y^{n/2-1} e^{-y/2}$$

and Y is independent of X_1 and X_2 , then the variables

$$F_i = \frac{X_i/m}{Y/n}, \quad i = 1, 2,$$

have a joint cumulative distribution given by

$$P(0 \leq F_1 \leq f_1, 0 \leq F_2 \leq f_2 | \rho, m, n) = \int_0^\infty f(y) \int_0^{f_2 m y / n} \int_0^{f_1 m y / n} g(x_1, x_2) dx_1 dx_2 dy.$$

In series form,

$$(19) \quad P = (1 - \rho^2)^{m/2} \sum_{i=0}^\infty \frac{(m/2)_i}{i!} \rho^{2i} I_2(\alpha_i, \hat{c}, \beta),$$

where $I_2(\hat{\alpha}_i, \hat{c}, \beta)$ is given by (3), (13), (14), (15) or (16), and

$$\beta = n/2, \quad \hat{c} = \left(\frac{f_1 m}{n(1 - \rho^2)}, \frac{f_2 m}{n(1 - \rho^2)} \right), \quad \hat{\alpha}_i = \left(\frac{m}{2} + i, \frac{m}{2} + i \right).$$

There is symmetry in the pairs (α_1, c_1) , (α_2, c_2) . For use in (1), $\alpha_1 = \alpha_2 = m/2 + i$ and there is complete symmetry in c_1 and c_2 . Equations (13) and (14), together with the quadrature in (3), do not require either α_1 or α_2 to be integers. One always expects significant digits from the positive series (13) and the quadrature of (3). In these cases, (19) also yields significant digits.

When m is even the formula

$$(1 - \rho^2)^{-m/2} = \sum_{i=0}^{\infty} \frac{(m/2)_i}{i!} \rho^{2i}, \quad 0 \leq \rho^2 < 1,$$

with (19) and (16) gives

$$(20) \quad P = 1 - (1 - \rho^2)^{m/2} \sum_{i=0}^{\infty} \frac{(m/2)_i}{i!} \rho^{2i} J_2(\hat{\alpha}_i, \hat{c}, \beta),$$

where $\beta = n/2$, $\alpha_1 = \alpha_2 = m/2 + i$ in

$$\begin{aligned} J_2(\hat{\alpha}, \hat{c}, \beta) &= I_{1-u}(\beta, \alpha_1) + I_{1-v}(\beta, \alpha_2) \\ &- (1 - u)^\beta \sum_{k=0}^{\alpha_1-1} \frac{(\beta)_k}{k!} u^k I_{1-v}(\beta + k, \alpha_2), \quad \alpha_1 \text{ a positive integer.} \end{aligned}$$

At first glance, one might expect faster convergence for ρ close to 1. However the experiments detailed below indicate otherwise. Other series representations can be obtained by exchanging the order of summation in (19) and (20).

Computations with (19) and (20) were done for the first quadrant of the f_1, f_2 plane for $m, n = 2, 10, 50$ and $\rho = 0.1, 0.5, 0.9$. No extensive testing of (19) with (13) or (14) was done because of the slow convergence when β and c_1 are large. Although (19) with the quadrature in (3) worked for small ρ , the speed of (19) (with (15)) and (20) ruled out the quadrature as a satisfactory procedure for m even. The series (19) and (20) were truncated when a term was less than 5×10^{-6} of the accumulated sum (always less than 225 terms). The agreement was generally four significant figures with slight discrepancies in the fifth digit for some $\rho = 0.9$ cases. When $P < 0.1$ in (20), only absolute accuracy was obtained.

The comparisons were made on a CDC 6600 computer with incomplete beta sequences generated according to the method described in [1] and [2]. 675 comparisons inside the radii listed in Table 1 showed (19) to be approximately three times faster than (20) with running times of 2 and 6 minutes respectively. The corresponding time for the quadrature in (3) for the 450 cases associated with $\rho = 0.1$ and $\rho = 0.5$ was on the order of 30 minutes with comparable relative error requirements. The accuracy with which z_0 was isolated in (8) seemed to have a strong influence on the overall speed of computation in (3).

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TABLE 1

Radii in the f_1, f_2 Plane where $P = 0.95$ on the 45° Ray for $\rho = 0.1, 0.5, 0.9$

$m \setminus n$	2	10	50
2	40.70	7.52	5.58
	38.90	7.34	5.49
	32.91	6.64	5.07
10	34.22	4.97	3.25
	33.33	4.88	3.21
	30.39	4.58	3.07
50	30.63	4.09	2.44
	30.23	4.05	2.42
	28.89	3.90	2.35

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