

# COMPUTATION OF BESSEL AND AIRY FUNCTIONS AND OF RELATED GAUSSIAN QUADRATURE FORMULAE \*

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## Abstract.

Procedures are described for the high-precision calculation of the modified Bessel function  $K_\nu(x)$ ,  $0 < \nu < 1$ , and the Airy function  $\text{Ai}(x)$ , for positive arguments  $x$ , as prerequisites for generating Gaussian quadrature rules having these functions as weight function.

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*Key words:* Modified Bessel function, Airy function, Gaussian quadrature.

## 1 Introduction.

Integrals involving modified Bessel functions  $K_\nu$  for  $\nu = \frac{1}{3}$  and the Airy function  $\text{Ai}$  occur naturally in some physics applications (see, e.g., Gordon [5, 6]); the weight function  $K_0$  has also found use in the asymptotic estimation of oscillatory integral transforms (see Wong [12], Gautschi [4, Example 6.1, p. 94]). Efficient evaluation of such integrals calls for Gaussian quadrature rules having these functions as weight function. Such rules (involving up to 10 points) were already given in [6] for the modified Bessel function  $K_{\frac{1}{3}}$  and in [5, 9] for the Airy function. It appears, however, that the latter are in error because of an incorrect calculation of the relevant moments. Here we develop Gaussian quadrature formulae for both weight functions with up to  $n = 40$  points. The main task is to find the first  $n$  recursion coefficients in the three-term recurrence relation for the relevant orthogonal polynomials. These can be obtained to arbitrary precision from the known moments by symbolic computation, or else, to standard machine precisions, by general procedures developed earlier in [3]. To apply these procedures, it is important to have routines that calculate modified Bessel functions and the Airy function to high accuracy. Such routines are described in Section 2 for Bessel functions, and in Section 3 for the Airy function. Section 4 discusses the computation of the respective Gaussian quadrature rules. An appendix contains the first 40 of the requisite recursion coefficients to 28 decimals.

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## 2 Modified Bessel functions.

Our interest is in generating values to high accuracy of the modified Bessel function  $K_\nu(x)$ , where  $0 < \nu < 1$  and  $x > 0$ , as a preparation for generating the Gaussian quadrature rules in Sections 4.1 and 4.2. A case of particular interest is  $\nu = \frac{1}{3}$ , but the procedure we develop is applicable also for other values of  $\nu$  except those close to 0 and 1. It is not our intention, here, to develop a general-purpose production code. There are a number of such codes available in the literature (see, e.g., [8, Sections 4.1, 5.1]), that—like ours—are suitable also for high-accuracy work. What we find worth observing is the apparently novel use of integral representations and related generalized Gauss–Laguerre quadratures to compute modified Bessel functions for moderately large, and large, real arguments. The approach, in fact, is potentially useful also for complex arguments.

When  $x$  is relatively small, say  $0 < x \leq 2$ , we use, as others have done before, the representation (cf. [1, Eq. 9.6.2])

$$(2.1) \quad K_\nu(x) = \frac{1}{2}\pi \frac{I_{-\nu}(x) - I_\nu(x)}{\sin \pi\nu},$$

and evaluate  $I_{\pm\nu}(x)$  by Taylor expansion ([1, Eq. 9.6.10])

$$(2.2) \quad I_{\pm\nu}(x) = \left(\frac{1}{2}x\right)^{\pm\nu} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}x^2\right)^k}{k!\Gamma(k+1\pm\nu)}.$$

If  $\nu$  is close to 0 or 1, considerable cancellation occurs in the numerator of (2.1). This could be dealt with, if deemed necessary, by special additional procedures (cf., e.g., [10, Section II]). Here we simply assume  $0.05 \leq \nu \leq 0.95$ , which limits the loss of accuracy to at most two (or three, if  $x$  is near 2) decimal digits.

For  $x > 2$ , we use the integral representation ([1, Eq. 9.6.23])

$$(2.3) \quad K_\nu(x) = \frac{\sqrt{\pi}}{2^\nu\Gamma(\nu+\frac{1}{2})} \frac{e^{-x}}{\sqrt{x}} \int_0^\infty \left(2 + \frac{t}{x}\right)^{\nu-\frac{1}{2}} \cdot t^{\nu-\frac{1}{2}} e^{-t} dt,$$

where the integral is conveniently evaluated by generalized Gauss–Laguerre quadrature with parameter  $\alpha = \nu - \frac{1}{2}$ ,

$$(2.4) \quad \int_0^\infty \left(2 + \frac{t}{x}\right)^{\nu-\frac{1}{2}} \cdot t^{\nu-\frac{1}{2}} e^{-t} dt \simeq \sum_{k=1}^n w_k^L \left(2 + \frac{t_k^L}{x}\right)^{\nu-\frac{1}{2}}, \quad x > 2.$$

Here,  $t_k^L$ ,  $w_k^L$  are the nodes and weights of the generalized Gauss–Laguerre quadrature rule. (The dependence on  $n$  is suppressed in the notation.) These, for  $\nu = \frac{1}{3}$ , were generated by double-precision resp. quadruple-precision analogues of the procedures **recur** and **gauss** of [3]. While it would be unreasonable to expect convergence to full machine precision as  $n \rightarrow \infty$ , it was found that in double and quadruple precision, “numerical convergence” occurs to an accuracy of  $10 \times \varepsilon_{\text{dbble}}$  resp.  $1000 \times \varepsilon_{\text{quad}}$ , where  $\varepsilon_{\text{dbble}} \simeq .111 \times 10^{-15}$ ,  $\varepsilon_{\text{quad}} \simeq .963 \times 10^{-34}$  are respectively the IEEE double- and quadruple-precision machine precisions. In other words, the approximants stabilized to these accuracies at certain values of  $n$ , which are shown in Table 2.1.

Table 2.1: Number of Gauss points in (2.4), with  $\nu = \frac{1}{3}$ , required for double- and quadruple-precision accuracy.

$\nu$	0	1/6	1/3	1/2	2/3	5/6	1
$n$ double	23	23	22	1	21	22	22
$n$ quadruple	87	85	83	1	82	82	82

(The data is for  $x = 2$ ; as  $x$  increases,  $n$  decreases.)

In many applications (including the one in Section 4.1; cf. (4.7)), it is better to compute  $e^x K_\nu(x)$ . This is also the function tabulated (for  $\nu = \frac{1}{3}$ ) in [11, Table III].

We remark that our evaluation procedure is also applicable for  $x$  in the complex plane cut along the negative real axis, but we will not pursue this here any further.

### 3 The Airy function.

The Airy function is related to the modified Bessel function  $K_{\frac{1}{3}}$  as follows (cf. [1, Eq. 10.4.14]):

$$(3.1) \quad \text{Ai}(x) = \frac{1}{\pi} \sqrt{\frac{x}{3}} K_{\frac{1}{3}}(\zeta), \quad \zeta = \frac{2}{3} x^{\frac{3}{2}}.$$

Using (2.1) with  $\nu = \frac{1}{3}$ , one gets

$$(3.2) \quad \text{Ai}(x) = \frac{1}{3} \sqrt{x} [I_{-\frac{1}{3}}(\zeta) - I_{\frac{1}{3}}(\zeta)].$$

Here, for  $0 < \zeta \leq 2$ , i.e.,  $0 < x \leq 3^{2/3} = 2.08008\dots$ , both Bessel functions can be evaluated by Taylor expansion as in (2.2) with  $\nu = \frac{1}{3}$ .

For  $\zeta > 2$ , we use the integral representation (2.3) for  $K_{\frac{1}{3}}$  in conjunction with (3.1) to obtain

$$(3.3) \quad \text{Ai}(x) = \frac{1}{\sqrt{\pi}} \frac{\zeta^{-\frac{1}{6}} e^{-\zeta}}{(48)^{\frac{1}{6}} \Gamma(\frac{5}{6})} \int_0^\infty \left(2 + \frac{t}{\zeta}\right)^{-\frac{1}{6}} \cdot t^{-\frac{1}{6}} e^{-t} dt,$$

with  $\zeta$  as defined in (3.1). Now generalized Gauss–Laguerre quadrature is appropriate with Laguerre parameter  $\alpha = -\frac{1}{6}$ . According to Table 2.1 ( $\nu = \frac{1}{3}$ ), a 22-point formula yields double-precision accuracy and a 83-point formula quadruple-precision accuracy.

This procedure can be used also for complex  $x$ , at least in the sector  $|\arg x| < \frac{2}{3}\pi$ .

### 4 Gauss quadratures.

#### 4.1 Gauss quadrature with Bessel weight function.

We define the weight function

$$(4.1) \quad w(x) = \frac{2}{\pi} \cos(\frac{1}{2}\nu\pi) K_\nu(x), \quad 0 < x < \infty.$$

Its moments can be calculated explicitly by (cf. [7, 6.561.16])

$$(4.2) \quad \mu_k = \int_0^\infty x^k w(x) dx = \frac{1}{\pi} \cos(\frac{1}{2}\nu\pi) \cdot 2^k \Gamma(\frac{1}{2}(k+1+\nu)) \Gamma(\frac{1}{2}(k+1-\nu)).$$

In particular, for  $k = 0$ ,

$$(4.3) \quad \mu_0 = \frac{1}{\pi} \cos(\frac{1}{2}\nu\pi) \cdot \Gamma(\frac{1}{2}(1+\nu)) \Gamma(\frac{1}{2}(1-\nu)) = 1$$

by virtue of the reflection formula ([1, 6.1.17]) for  $\Gamma(z)\Gamma(1-z)$ ,  $z = \frac{1}{2}(1+\nu)$ . Thus, (4.1) is a normalized weight function.

In principle, the first  $2n$  moments  $\mu_k$  of  $w$  can be used to generate the  $n$ -point Gauss formula for  $w$ . It is well known, however, that this becomes quickly unstable as  $n$  increases. A way around this problem is to employ a symbolic computation package combined with extended-precision arithmetic. A Maple script, named `cheb.mws`<sup>1</sup>, has been developed for this purpose and has been used to produce the required recursion coefficients  $\alpha_k, \beta_k$  (cf. (4.4) below) for  $0 \leq k \leq 39$  to as many as 100 decimal digits<sup>2</sup>. It can be accessed at

<http://www.cs.purdue.edu/archives/2001/wxg/codes/cheb.mws>.

(A text file `cheb.txt` can be found at the same URL.)

Here, we describe a stable numerical procedure—a four-pronged discretization procedure (cf. [4, Section 6])—that discretizes the inner product for the weight function  $w$  and generates the corresponding discrete orthogonal polynomials. If the discretization is chosen judiciously, the discrete orthogonal polynomials converge to the desired ones as the discretization is made increasingly finer. The first  $n$  recursion coefficients  $\alpha_k, \beta_k$  in the three-term recurrence relation

$$(4.4) \quad \begin{aligned} \pi_{k+1}(x) &= (x - \alpha_k)\pi_k(x) - \beta_k\pi_{k-1}(x), \quad k = 0, 1, \dots, n-1, \\ \pi_0(x) &= 1, \quad \pi_{-1}(x) = 0, \end{aligned}$$

for the monic orthogonal polynomials (where  $\beta_0 = \int_0^\infty w(x) dx$ ) are computed by the Stieltjes procedure ([4, Section 6.3]).

The discretization we choose makes use of a composition of the positive real axis into three subintervals,  $\mathbb{R}_+ = (0, x_0] \cup [x_0, x_1] \cup [x_1, \infty)$ , with  $x_0, x_1$  still to be selected such that  $0 < x_0 \leq 1, 1 < x_1 < \infty$ . In the first subinterval, the behavior of  $K_\nu(x)$  for small  $x$  must be properly accounted for. One has [1, Eqs. 9.6.2 and 9.6.10]

$$(4.5) \quad K_\nu(x) = \frac{\pi}{2 \sin \nu\pi} \left\{ \frac{(\frac{1}{2}x)^{-\nu}}{\Gamma(1-\nu)} S_{-\nu}(x) - \frac{(\frac{1}{2}x)^\nu}{\Gamma(1+\nu)} S_\nu(x) \right\},$$

where

$$S_{\pm\nu}(x) = \sum_{k=0}^\infty \frac{(\frac{1}{4}x^2)^k \Gamma(1 \pm \nu)}{k! \Gamma(k+1 \pm \nu)}.$$

<sup>1</sup>This is written for Maple Release 5.

<sup>2</sup>The author is indebted to Oscar Chinellato at the Institute for Scientific Computing of the ETH Zurich, Switzerland, for translating a slightly edited version of our ORTHPOL routine `cheb` (cf. [3]) into a Maple script.

The two distinct behaviors  $x^{-\nu}$  and  $x^\nu$  as  $x \rightarrow 0$  need to be treated separately for purposes of integration. Indeed, the following composition for integrals over  $(0, x_0]$  is suggested:

$$(4.6) \quad \int_0^{x_0} p(x)K_\nu(x)dx = \frac{2^{\nu-1}\pi}{\sin \nu\pi \cdot \Gamma(1-\nu)} \int_0^{x_0} p(x)S_{-\nu}(x) \cdot x^{-\nu} dx \\ - \frac{\pi}{2^{\nu+1} \sin \nu\pi \cdot \Gamma(1+\nu)} \int_0^{x_0} p(x)S_\nu(x) \cdot x^\nu dx.$$

The first integral on the right is approximated by an  $N$ -point Gauss–Jacobi quadrature rule relative to the interval  $(0, x_0]$  with Jacobi parameters  $\alpha = 0$ ,  $\beta = -\nu$ , the second by a similar  $N$ -point Gauss–Jacobi rule with parameters  $\alpha = 0$ ,  $\beta = \nu$ . In the second interval, we apply the ordinary  $N$ -point Gauss–Legendre rule transformed to the interval  $[x_0, x_1]$ . For the last interval, we write

$$(4.7) \quad \int_{x_1}^{\infty} p(x)K_\nu(x)dx = e^{-x_1} \int_0^{\infty} p(x_1+t)[e^{x_1+t}K_\nu(x_1+t)] \cdot e^{-t} dt$$

and approximate the integral on the right by an  $N$ -point Gauss–Laguerre quadrature rule, the function  $e^x K_\nu(x)$ ,  $x \geq x_1$ , being computed from the integral representation (2.3) as discussed in the text following (2.3).

We call this a “four-pronged” discretization procedure since four different quadrature rules are employed to discretize the integral  $\int_0^{\infty} p(x)K_\nu(x)dx$  (and with it the inner product relative to the weight function (4.1)): two Gauss–Jacobi rules for approximating the two integrals on the right of (4.6), the Gauss rule over  $[x_0, x_1]$ , and the Gauss–Laguerre rule to deal with the integral on the right of (4.7).

The parameters  $x_0$ ,  $x_1$  are chosen in an attempt to reduce the number  $N$  of quadrature terms required to achieve a given accuracy. Some limited experimentation suggested the choice  $x_0 = 1$  and  $x_1 = 10$ . When  $\nu = \frac{1}{3}$  and  $n = 10$ , then  $N = 41$  will yield a relative accuracy of 1000 times the double machine precision (about 12 decimal-digit accuracy), and  $N = 71$  a relative accuracy of  $10^5$  times the quadruple machine precision (about 29 decimal digits). For  $n = 40$  the respective numbers are both  $N = 81$ .

The values of the recursion coefficients  $\alpha_k$ ,  $\beta_k$  (for  $\nu = \frac{1}{3}$ ) to 28 decimal digits are given in the appendix, Table A1, for  $k = 0, 1, \dots, 39$ . These allow us to generate the respective Gauss and Gauss–Radau quadrature rules for up to 40 points by well-known eigenvalue/eigenvector techniques [4, Section 4]. Double- and quadruple-precision fortran programs producing these coefficients are accessible at <http://www.cs.purdue.edu/archives/2001/wxg/codes/> in the files `d0Pbess.f` and `q0Pbess.f`. The coefficients themselves to 28 decimals can be found in the file `coeffbess` at the same URL. The file `ORTHPOLq` contains the quadruple-precision routines of the package `ORTHPOL` in [3].

#### 4.2 Gauss quadrature with Airy weight function.

We define a weight function proportional to the one in Eq. (1.4) of [9],

$$(4.8) \quad w(x) = \frac{2^{\frac{2}{3}}\pi}{3^{\frac{5}{6}}\Gamma(\frac{2}{3})} x^{-\frac{2}{3}} e^{-x} \text{Ai}(\left(\frac{3}{2}x\right)^{\frac{2}{3}}), \quad 0 < x < \infty.$$

By (3.1), one has

$$(4.9) \quad w(x) = \frac{2^{\frac{1}{3}}}{3\Gamma(\frac{2}{3})} x^{-\frac{1}{3}} e^{-x} K_{\frac{1}{3}}(x).$$

As in [9], we use [7, 6.621.3] to obtain for the moments (correctly)

$$(4.10) \quad \mu_k = \int_0^\infty x^k w(x) dx = \frac{2^{\frac{2}{3}} \sqrt{\pi}}{\Gamma(\frac{2}{3})} \frac{k! \Gamma(k + \frac{1}{3})}{(6k + 1) 2^k \Gamma(k + \frac{1}{6})}.$$

In particular,

$$(4.11) \quad \mu_0 = \frac{2^{\frac{2}{3}} \sqrt{\pi}}{\Gamma(\frac{2}{3})} \frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{1}{6})} = 1$$

by virtue of the duplication formula [1, 6.1.19] for  $\Gamma(2z)$ ,  $z = \frac{1}{6}$ . Thus, the weight function (4.8) is normalized.

The three-term recurrence relation (4.4) for this weight function can again be obtained by symbolic computation using the Maple script `cheb.mws` (cf. §4.1). Numerically, on the other hand, we may use, similarly as in Section 4.1, a three-pronged discretization method. First, integration against the weight function is “regularized” by means of the change of variable  $x \mapsto x^3/3$ ; thus,

$$(4.12) \quad \int_0^\infty p(x) w(x) dx = \frac{2^{\frac{2}{3}} \pi}{3^{\frac{1}{6}} \Gamma(\frac{2}{3})} \int_0^\infty p(\frac{1}{3} x^3) \text{Ai}(2^{-\frac{2}{3}} x^2) \cdot e^{-x^3/3} dx.$$

The discretization is effected by using the decomposition  $\mathbb{R}_+ = (0, 2] \cup [2, 6] \cup [6, \infty)$  and  $N$ -point Gauss–Legendre quadrature on the first two intervals, and  $N$ -point Gauss quadrature relative to the weight function  $e^{-x^3/3}$ ,  $0 < x < \infty$ , on the last interval (after transforming the integral over  $[6, \infty)$  to one over  $[0, \infty)$ ). The latter quadrature rules have been generated by a “general-purpose” discretization method [4, p. 95]; see also [2]. In this way, when  $n = 10$ , then  $N = 51$  (in double precision) yields about 12-digit accuracy,  $N = 91$  (in quadruple precision) an accuracy of about 29 digits. For  $n = 40$ , the respective numbers are  $N = 81$  and  $N = 161$ .

Values of the recursion coefficients  $\alpha_k, \beta_k$  to 28 decimal digits are given in the appendix, Table A2, for  $k = 0, 1, \dots, 39$ . These again permit the generation of Gauss and Gauss–Radau quadrature rules with up to 40 points. Double- and quadruple-precision fortran programs producing these coefficients are accessible at

<http://www.cs.purdue.edu/archives/2001/wxg/codes/> in the files `d0Pairy.f` and `q0Pairy.f`. The coefficients themselves to 28 decimals can be found in the file `coeffairy` at the same URL.

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## A Appendix.

Recursion coefficients  $\alpha_k$  and  $\beta_k$  for the orthogonal polynomials relative to the weight function  $\frac{\sqrt{x}}{\pi}K_{\frac{1}{3}}(x)$ ,  $0 < x < \infty$ , are given in Table A.1.

Table A.1: Recursion coefficients for orthogonal polynomials with Bessel weight function.

k	dalpha(k)	dbeta(k)
0	0.5773502691896257645091487805D+00	0.100000000000000000000000000000D+01
1	0.2540341184434353363840254634D+01	0.555555555555555555555555555556D+00
2	0.4530325099277556972940619392D+01	0.308000000000000000000000000000D+01
3	0.6525263205367024754584421146D+01	0.7597308896010194711493412792D+01
4	0.8522091681572316495410571818D+01	0.1411128895064359048476764685D+02
5	0.1051987069032579681709327397D+02	0.2262328375651997214118489043D+02
6	0.1251820534200411842698661310D+02	0.3313393449511827909015205175D+02
7	0.1451689745070847589982540347D+02	0.4564360198457644797420242203D+02
8	0.1651583533890971294501705203D+02	0.6015251169387165444574183086D+02
9	0.1851495070370021650146786956D+02	0.7666081512279027237029582199D+02
10	0.2051419913459637659996186934D+02	0.9516861964153151276420251970D+02
11	0.2251355035096639733155330055D+02	0.1156760045129536402241437743D+03
12	0.2451298290692544758973945221D+02	0.1381830301661429736830519992D+03
13	0.2651248113403138074584401742D+02	0.1626897438888321271102896829D+03
14	0.2851203328421788988619681952D+02	0.1891961834909268127004429781D+03
15	0.3051163035337998449498607362D+02	0.2177023797559084152522105542D+03
16	0.3251126530928971516451325688D+02	0.2482083581355438536773868170D+03
17	0.3451093256931540393887725062D+02	0.2807141399544099117867506745D+03
18	0.3651062763776024473491489160D+02	0.3152197432866833050342266185D+03
19	0.3851034684821817685385767070D+02	0.3517251836077464523819711369D+03
20	0.4051008717681441792603844069D+02	0.3902304742873398067597234709D+03
21	0.4250984610438625709490911393D+02	0.4307356269688528086704175367D+03
22	0.4450962151314044127608953558D+02	0.4732406518652590307625711171D+03
23	0.4650941160804019999493643228D+02	0.5177455579930047052041031595D+03
24	0.4850921485622134798647588526D+02	0.5642503533590167393634950613D+03
25	0.5050902993974754185328926380D+02	0.6127550451118075040591250302D+03
26	0.5250885571836803795271883128D+02	0.6632596396647438874918855282D+03
27	0.5450869119986848180007865562D+02	0.7157641427974924632092938047D+03
28	0.5650853551625092002779691310D+02	0.7702685597401779256832448017D+03
29	0.5850838790443559309423523307D+02	0.8267728952437190489066061745D+03
30	0.6050824769050409076746501085D+02	0.8852771536390157066778742762D+03
31	0.6250811427674077725581328007D+02	0.9457813388870707655384939225D+03
32	0.6450798713090365394190828550D+02	0.1008285454621685966632728510D+04
33	0.6650786577728518444594287897D+02	0.1072789504186032141840176389D+04
34	0.6850774978922060972058434023D+02	0.1139293490664133569567757833D+04
35	0.7050763878277471470954744945D+02	0.1207797416908104118798814682D+04
36	0.7250753241139409466091921010D+02	0.1278301285561814669842441005D+04
37	0.7450743036135516422801459683D+02	0.1350805099081546598221771377D+04
38	0.7650733234787168021992493344D+02	0.1425308859754087068026168299D+04
39	0.7850723811175176515929034805D+02	0.1501812569712642681574641394D+04

Recursion coefficients  $\alpha_k$  and  $\beta_k$  for the orthogonal polynomials relative to the weight function  $\frac{2^{2/3}\pi}{3^{5/6}\Gamma(2/3)} x^{-2/3} e^{-x} \text{Ai}((\frac{3}{2}x)^{2/3})$ ,  $0 < x < \infty$ , are given in Table A.2.

Table A.2: Recursion coefficients for orthogonal polynomials with Airy weight function.

k	dalpha(k)	dbeta(k)
0	0.1428571428571428571428571429D+00	0.1000000000000000000000000000D+01
1	0.1110508830215072565133764644D+01	0.6750392464678178963893249608D-01
2	0.2103923262303366469719046650D+01	0.6176435520631454946477366322D+00
3	0.3100612923619468060204041749D+01	0.1665222369579728763993598104D+01
4	0.4098522990053350300259677356D+01	0.3211665437933813301442673108D+01
5	0.5097048017912237952211191366D+01	0.5257439791428266998569894003D+01
6	0.6095934772142735289405067651D+01	0.7802762871816141245078916230D+01
7	0.7095055742615174969695761115D+01	0.1084775582766331463731785213D+02
8	0.8094338704979790304572522853D+01	0.1439249405127950300891982957D+02
9	0.9093739247462993136699966123D+01	0.1843702812922128928773098512D+02
10	0.1009322834609956529095674925D+02	0.2298139391050911313320103004D+02
11	0.1109278611899427796389513021D+02	0.2802561787656439530189513139D+02
12	0.1209239842573524183149667965D+02	0.3356972023701598312985079015D+02
13	0.1309205489613779605430552664D+02	0.3961371682566247092476763052D+02
14	0.1409174772819666777679886468D+02	0.4615762031917471854658917880D+02
15	0.1509147092378366705795857263D+02	0.5320144105213568478003200469D+02
16	0.1609121978528171983953788068D+02	0.6074518758045262957376738563D+02
17	0.1709099057396944013793558492D+02	0.6878886708190032036430935116D+02
18	0.1809078027208983877445351834D+02	0.7733248564781360694249859341D+02
19	0.1909058641334036072040544031D+02	0.8637604849999474726610235545D+02
20	0.2009040695967678119625359998D+02	0.9591956015498898485337591888D+02
21	0.2109024021017825188614733837D+02	0.1059630245505277814479273789D+03
22	0.2209008473255468030080111379D+02	0.1165064451442633467187277061D+03
23	0.2308993931093330244858796206D+02	0.1275498249918669408772461453D+03
24	0.2408980290553974280008571454D+02	0.1390931668095257606646414852D+03
25	0.2508967462119770022815575463D+02	0.1511364730244838704145147093D+03
26	0.2608955368245429586520635128D+02	0.1636797458163074470540297192D+03
27	0.2708943941374431823158236292D+02	0.1767229871508726145735111346D+03
28	0.2808933122342959521382761789D+02	0.1902661988085847580652764775D+03
29	0.2908922859084928131852207332D+02	0.2043093824079820406036281648D+03
30	0.3008913105573190129755639056D+02	0.2188525394256132521262541885D+03
31	0.3108903820947633067457385995D+02	0.2338956712128841886680652998D+03
32	0.3208894968792388105605120154D+02	0.2494387790104189467091905899D+03
33	0.3308886516532915095237870707D+02	0.2654818639603698571811680251D+03
34	0.3408878434930151025223927020D+02	0.2820249271170230956579194238D+03
35	0.3508870697653776378539261556D+02	0.2990679694559797076870358157D+03
36	0.3608863280920376927294370407D+02	0.3166109918821391090615739319D+03
37	0.3708856163185149561174724958D+02	0.3346539952366705633992391624D+03
38	0.3808849324878032081407976838D+02	0.3531969803031251160025604749D+03
39	0.3908842748176883766030660897D+02	0.3722399478128140407767669942D+03



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