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Computation of Best Monotone Approximations*

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Abstract. A numerical procedure to compute the best uniform approximation to a given continuous function by algebraic polynomials with nonnegative rth derivative is presented and analyzed. The method is based on discretization and linear programming. Several numerical experiments are discussed.

1. Introduction. The general problem considered in this paper is the best uniform approximation of a given continuous function f over a finite closed interval $[a, b]$ by polynomials of degree at most n whose rth derivative is constrained to be nonnegative on $[a, b]$. Denote by $\mathcal{P}_n$ the set of all algebraic polynomials $a_0 + a_1x + \cdots + a_nx^n$ of degree at most n and set $K_r = \{p \in \mathcal{P}_n: p^{(r)}(x) \geqq 0 \text{ for } a \leqq x \leqq b\}$ where r is an integer, $1 \leqq r \leqq n$. K_r is a closed convex cone in the $(n + 1)$-dimensional space $\mathcal{P}_n$. We wish to find the best uniform approximation to f from K_r; i.e., we seek $p^* \in K_r$ such that $\|f - p^*\| = \min_{p \in K_r} \|f - p\|$, where the measure of the error is the uniform norm $\|f - p\| = \max_{a \leq x \leq b} |f(x) - p(x)|$. The existence of such a best approximation was easily established; recently it was shown in [7] that the best approximation is unique. The main goal of this paper is to present and analyze a method to compute the best approximation. The original interest was in approximation by monotonic polynomials (approximation from K_1) and approximation by convex polynomials (approximation from K_2); however, the method can be extended easily to the general problem of approximation from K_r. In Section 2, a symmetry result is established. In Section 3, a numerical procedure based on discretization and linear programming is presented and analyzed. Section 4 contains several numerical examples and a discussion of some interesting features which they exhibit.

2. Symmetry. Recall that a function f is called even (odd) on $[-b, b]$ if $f(-x) = f(x)$ (respectively, $f(-x) = -f(x)$) for all x in $[-b, b]$. In the classical problem of uniform approximation on $[-b, b]$ to an even (odd) function with no constraints, i.e., $\min_{p \in \mathcal{P}_n} \|f - p\|$, the solution p^*, which we will call the best unconstrained approximation, is also even (odd). An analogous result holds for approximation from K_r.

THEOREM 1. *Let f be continuous on $[-b, b]$ and set*

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$$K_r = \{p \in \mathcal{P}_n : p^{(r)}(x) \geqq 0 \text{ for } -b \leqq x \leqq b\},$$

where $1 \leqq r \leqq n$.

(i) *If r is an even integer and if f is even on* $[-b, b]$, *then the best approximation to f from* K_r *is also even.*

(ii) *If r is an odd integer and if f is odd on* $[-b, b]$, *then the best approximation to f from* K_r *is also odd.*

Proof. (i) Assume r is an even integer and f is even. Let p^* be the best approximation to f from K_r and set $q(x) = p^*(-x)$ for all x in $[-b, b]$. Then

$$q^{(r)}(x) = (-1)^r[p^*]^{(r)}(-x) \geqq 0 \quad \text{for } -b \leqq x \leqq b,$$

i.e., q is in K_r. Now,

$$\begin{aligned} \|f - q\| &= \max_{-b \leqq x \leqq b} |f(x) - q(x)| \\ &= \max_{-b \leqq x \leqq b} |f(-x) - p^*(-x)| = \|f - p^*\|. \end{aligned}$$

Hence, by uniqueness, $q = p^*$ and so

$$p^*(-x) = p^*(x) \quad \text{for } -b \leqq x \leqq b.$$

(ii) The proof of (ii) is accomplished by setting $q(x) = -p^*(-x)$ and proceeding as in the proof of (i).

The following characterization theorem, due to Lorentz and Zeller [7], is fundamental. Recall that a point x in $[a, b]$ at which $[f - p](x) = \pm\|f - p\|$ is called an extremal point of p and a point x at which $p^{(r)}(x) = 0$ is called a constraint point. The union of the extremal points and the constraint points is the set of critical points.

THEOREM 2 (LORENTZ AND ZELLER). *Let f be continuous on* $[a, b]$ *and*

$$K_r = \{p \in \mathcal{P}_n : p^{(r)}(x) \geqq 0 \text{ for } a \leqq x \leqq b\}.$$

Then $p^* \in K_r$ *is the best approximation to f from* K_r *if and only if there exist extremal points* $x_1, \cdots, x_s$ *of* p^*, *constraint points* $x_{s+1}, \cdots, x_t$ *of* p^* (*where* $t \leqq n + 2$), *and positive constants* $\lambda_1, \cdots, \lambda_t$ *such that*

$$\sum_{i=1}^{s} \lambda_i \sigma(x_i) p(x_i) + \sum_{i=s+1}^{t} \lambda_i p^{(r)}(x_i) = 0 \tag{2.1}$$

for all $p \in \mathcal{P}_n$, *where*

$$\begin{aligned} \sigma(x_i) = \operatorname{sgn}[f(x_i) - p^*(x_i)] &= +1 \quad \text{if } [f - p^*](x_i) = +\|f - p^*\|, \\ &= -1 \quad \text{if } [f - p^*](x_i) = -\|f - p^*\|. \end{aligned}$$

Proof. [7, p. 5].

3. Computational Procedure; Discretization Error. We next present a numerical procedure to compute the best approximation from K_r; this method is described briefly in [5, p. 27]. The problems of particular interest are approximation by monotonic polynomials and approximation by convex polynomials. The discussion will be carried out for approximation by convex polynomials; the analysis for the more general problem of approximation from K_r is similar. Let $X_m = \{x_0, x_1, \cdots, x_m\}$, where $a = x_0 < x_1 < \cdots < x_m = b$, be a discrete subset of $[a, b]$. We consider the constrained problem on X_m:

$$\min_{p \in \mathcal{P}_n} \max_{x_j \in X_m} |f(x_j) - p(x_j)|,$$

subject to $p''(x_j) \geqq 0$ for all $x_j \in X_m$. Setting $p(x) = a_0 + a_1 x + \cdots + a_n x^n$, this becomes

$$\min_{a_0, \cdots, a_n} \max_{x_j \in X_m} \left| f(x_j) - \sum_{i=0}^{n} a_i (x_j)^i \right| \tag{3.1}$$

subject to

$$\sum_{i=2}^{n} a_i (i)(i-1)(x_j)^{i-2} \geqq 0, \qquad j = 0, 1, \cdots, m. \tag{3.2}$$

This is equivalent to the linear programming problem

$$\min \lambda, \tag{3.3}$$

subject to

$$\begin{aligned} -\lambda - \sum_{i=0}^{n} a_i (x_j)^i &\leqq -f(x_j), \\ -\lambda + \sum_{i=0}^{n} a_i (x_j)^i &\leqq f(x_j), \\ -\sum_{i=2}^{n} a_i (i)(i-1)(x_j)^{i-2} &\leqq 0, \qquad j = 0, 1, \cdots, m. \end{aligned} \tag{3.4}$$

λ represents the deviation (maximum error) over the discrete set X_m. The problem (3.3), (3.4) can be solved by linear programming techniques; several examples are presented and discussed in the next section. Let us now consider the relationship of the discretized problem (3.1), (3.2) to the original problem of approximation on the interval $[a, b]$. Two lemmas will be needed.

LEMMA 1. *For any* $p \in \mathcal{P}_n$,

$$\max_{a \leqq x \leqq b} |p'(x)| \leqq \frac{2n^2}{b - a} \max_{a \leqq x \leqq b} |p(x)|.$$

Proof. This is Markov's inequality stated for $[a, b]$; see, for example, [1, p. 91].

LEMMA 2. *Let* X_m, $m = 1, 2, \cdots$, *be a sequence of discrete subsets of* $[a, b]$ *such that*

$$\delta_m = \max_{y \in [a,b]} \min_{x_j \in X_m} |y - x_j| \to 0$$

as $m \to \infty$. *If* p_m *is a solution of the discretized problem* (3.1), (3.2) *then* $\{p_m : m = 1, 2, \cdots\}$ *is uniformly bounded on* $[a, b]$.

Proof.

$$\max_{x_j \in X_m} |f(x_j) - p_m(x_j)| \leqq \max_{x_j \in X_m} |f(x_j) - 0| \leqq \|f\|.$$

So

$$\max_{x_j \in X_m} |p_m(x_j)| \leqq 2\, \|f\| \quad \text{for all } m.$$

Let $I_0, \cdots, I_n$ be $n+1$ disjoint subintervals of $[a, b]$ with spacing at least ϵ between any two of them. Then, since $\delta_m \to 0$ as $m \to \infty$, there exists M such that for all $m \geqq M$ there are points $x_{i_0}, \cdots, x_{i_n}$ of X_m satisfying $x_{i_j} \in I_j$, $j = 0, \cdots, n$. Using the Lagrange interpolating polynomial form, we can write

$$p_m(x) = \sum_{k=0}^{n} p_m(x_{i_k}) \prod_{j=0;\, j\neq k}^{n} \frac{x - x_{i_j}}{x_{i_k} - x_{i_j}}.$$

Hence,

$$|p_m(x)| \leqq \sum_{k=0}^{n} |p_m(x_{i_k})| \prod_{j=0;\, j\neq k}^{n} \left|\frac{x - x_{i_j}}{x_{i_k} - x_{i_j}}\right|$$
$$\leqq (n+1)\cdot 2\cdot ||f||\cdot\left[\frac{b-a}{\epsilon}\right]^n$$

independent of m and x. Hence, $\{p_m\}$ is uniformly bounded.

In general, p_m will not satisfy the constraints on all of $[a, b]$, i.e., p_m'' may be negative. However, the violation is easily bounded.

THEOREM 3. *Let $X_m = \{x_0, x_1, \cdots, x_m\}$, $m = 1, 2, \cdots$, be a sequence of discrete subsets of $[a, b]$ with $a = x_0 < x_1 < \cdots < x_m = b$ such that*

$$\delta_m = \max_{y\in[a,b]} \min_{x_j\in X_m} |y - x_j| \to 0 \quad \text{as } m \to \infty.$$

If p_m is a solution of the discretized problem (3.1), (3.2), *then there exists a constant B independent of m such that*

$$p_m''(y) \geqq -\delta_m^2\cdot B \quad \text{for all } y \text{ in } [a, b].$$

Proof. Let y be an interior point of $[a, b]$ at which p_m'' assumes its minimum value. (If p_m'' assumes its minimum at a or b, the conclusion follows trivially.) Let x_j be a closest point in X_m to y. Using Taylor's formula,

$$p_m''(x_j) = p_m''(y) + (x_j - y)p_m'''(y) + \frac{(x_j - y)^2}{2!} p_m^{\text{iv}}(z),$$

where z is between x_j and y. Since $p_m'''(y) = 0$, we obtain

$$p_m''(y) = p_m''(x_j) - \frac{(x_j - y)^2}{2} p_m^{\text{iv}}(z)$$
$$\geqq -\frac{\delta_m^2}{2} \max_{a\leqq z\leqq b} |p_m^{\text{iv}}(z)|.$$

Four applications of Lemma 1 show that there exists a constant M such that

$$\max_{a\leqq z\leqq b} |p_m^{\text{iv}}(z)| \leqq M\cdot \max_{a\leqq z\leqq b} |p_m(z)|.$$

Since $\{p_m\}$ is uniformly bounded on $[a, b]$, the conclusion follows.

Let

$$\omega(f;\, \delta) = \sup\{|f(x) - f(y)|: x, y \in [a, b], |x - y| \leqq \delta\}$$

denote the modulus of continuity of f on $[a, b]$. The next theorem is the main result on the discretization error.

THEOREM 4. *Let* $X_m = \{x_0, x_1, \cdots, x_m\}$, $m = 1, 2, \cdots$, *be a sequence of discrete subsets of* $[a, b]$ *with* $a = x_0 < x_1 < \cdots < x_m = b$ *and such that*

$$\delta_m = \max_{y \in [a,b]} \min_{x_i \in X_m} |y - x_i| \to 0 \quad \text{as } m \to \infty .$$

Let p_m *be a solution of the discretized problem* (3.1), (3.2) *and* p^* *the best approximation to the continuous function* f *from* K_2. *Then,* $\{p_m\}$ *converges uniformly to* p^* *as* $m \to \infty$. *Furthermore, there exist constants C and D independent of m such that*

$$\|f - p^*\| - \|f - p_m\| \leqq C \cdot \delta_m^2 \tag{3.5}$$

and

$$\|f - p_m\| - \|f - p^*\| \leqq \omega(f;\, \delta_m) + D \cdot \delta_m . \tag{3.6}$$

Proof. By Lemma 2, $\{p_m\}$ is uniformly bounded and, hence, there exists a uniformly convergent subsequence $\{p_{m_k}\}$ with limit, say, p_0. We first show $p_0 \in K_2$. If not, then there is a point y in $[a, b]$ such that $p_0''(y) = -\epsilon < 0$. Two applications of Lemma 1 imply the existence of a constant R such that

$$\|p_{m_k}'' - p_0''\| \leqq R \cdot \|p_{m_k} - p_0\|,$$

and, hence, there exists K such that $\|p_{m_k}'' - p_0''\| < \epsilon/2$ for all $k \geqq K$. So $p_{m_k}''(y) < -\epsilon/2$ for all $k \geqq K$ which contradicts Theorem 3.

We next show $\|f - p_0\| = \|f - p^*\|$. Since $p_0 \in K_2$, clearly, $\|f - p^*\| \leqq \|f - p_0\|$. Assume $\|f - p^*\| = \|f - p_0\| - \epsilon$ where $\epsilon > 0$. Let $y \in [a, b]$ be a point where $|f(y) - p_0(y)| = \|f - p_0\|$. Then

$$\begin{aligned} \|f - p_0\| &= |f(y) - p_0(y)| \\ &\leqq |f(y) - f(x)| + |f(x) - p_{m_k}(x)| \\ &\quad + |p_{m_k}(x) - p_{m_k}(y)| + |p_{m_k}(y) - p_0(y)|, \end{aligned}$$

where $x \in X_{m_k}$ and $|x - y| \leqq \delta_{m_k}$,

$$\begin{aligned} &\leqq \omega(f;\, \delta_{m_k}) + \max_{x \in X_{m_k}} |f(x) - p_{m_k}(x)| \\ &\quad + |p_{m_k}'(z)| \cdot |x - y| + \|p_{m_k} - p_0\|, \end{aligned}$$

where z is between x and y. Lemmas 1, 2 imply the existence of a constant T such that $\max_{a \leqq z \leqq b} |p_{m_k}'(z)| \leqq T$ for all k. So for k large enough, we obtain

$$\|f - p_0\| \leqq \max_{x \in X_{m_k}} |f(x) - p_{m_k}(x)| + \epsilon/2.$$

Hence,

$$\|f - p^*\| = \|f - p_0\| - \epsilon \leqq \max_{x \in X_{m_k}} |f(x) - p_{m_k}(x)| - \epsilon/2$$

which contradicts the definition of p_{m_k}. So $\|f - p_0\| = \|f - p^*\|$. Since the best approximation from K_2 is unique, $p_0 = p^*$. Since every convergent subsequence has limit p^*, the sequence $\{p_m\}$ has limit p^*.

Now, let $q_m(x) = p_m(x) + B \cdot \delta_m^2 \cdot x^2/2$ where B is the constant of Theorem 3. Then $q_m \in K_2$ and, hence,

$$||f - p^*|| \leqq ||f - q_m|| \leqq ||f - p_m|| + ||p_m - q_m||$$
$$\leqq ||f - p_m|| + C \cdot \delta_m^2,$$

where $C = (B/2) \max\{a^2, b^2\}$, which establishes (3.5).

Let $y \in [a, b]$ be a point where $|f(y) - p_m(y)| = ||f - p_m||$, and let $x \in X_m$ satisfy $|y - x| \leqq \delta_m$. Then

$$\begin{aligned} ||f - p_m|| &= |f(y) - p_m(y)| \\ &\leqq |f(y) - f(x)| + |f(x) - p_m(x)| + |p_m(x) - p_m(y)| \\ &\leqq \omega(f; \delta_m) + \max_{x \in X_m} |f(x) - p_m(x)| + |p_m'(z)| \cdot |x - y| \\ &\leqq \omega(f; \delta_m) + \max_{x \in X_m} |f(x) - p^*(x)| + D \cdot \delta_m, \end{aligned}$$

where $D = \sup_m ||p_m'|| < \infty$ by Lemmas 1, 2,

$$\leqq \omega(f; \delta_m) + ||f - p^*|| + D \cdot \delta_m$$

which establishes (3.6). This completes the proof.

It follows from Theorem 4 that if f satisfies a Lipschitz condition on $[a, b]$, i.e., $|f(y) - f(x)| \leqq E \cdot |x - y|$ for all x, y in $[a, b]$, then (3.5) and (3.6) imply the existence of a constant F independent of m such that

$$|\, ||f - p^*|| - ||f - p_m||\, | \leqq F/m.$$

4. Numerical Examples. A number of numerical examples were performed to illustrate the procedure described in the previous section to compute the best convex approximation; also, several examples illustrating the computation of the best monotonic approximation (approximation from K_1) were run. A solution of the problem (3.1), (3.2) was obtained by applying the revised simplex method to the dual of the linear programming problem (3.3), (3.4); the reader unfamiliar with linear programming terminology and techniques may consult [2].

For a given convex function f, it often happens that the solution of the problem of finding the best uniform approximation (with no constraints applied) will turn out to be convex; this solution then will also be the best approximation from K_2. If it were known a priori that this were the case, then the Remes exchange algorithms for unconstrained uniform approximation could be used. However, it is not at all obvious when the best unconstrained approximation will turn out to be convex; this is an interesting problem in itself. Of course, similar remarks apply to the more general problem of approximation from K_r. To construct examples for which the unconstrained approximation would not turn out to satisfy the constraints, the problem (3.1), (3.2) was solved for X_m with uniform spacing .1. If there were no points of X_m at which $p_m''(x) = 0$, then this was taken as an indication that the best unconstrained approximation on $[a, b]$ would turn out to be convex and the example was abandoned. If there were points of X_m at which $p_m''(x) = 0$, then the example was continued for spacing .01 and .005. In all examples, points at which $p_m''(x) = 0$ continued to appear; in fact, near those for spacing .1.

In Examples 1–6, exhibited in Tables 1–6, the problem was to find the best convex

Table 1. $f(x) = x^6$

$h_m=2\delta_m$	a_0	a_1	a_2	a_3	a_4	a_5	λ_m	critical points
.1	-.07373	-.00102	0	.00260	1.0	-.00159	.07373	-1.0(+),-.8(-),0.0(+.c),.8(-),1.0(+)
.01	-.07406	.00001	0	-.00003	1.0	.00002	.07406	-1.0(+),-.82(-),0.0(+,c),.82(-),1.0(+)
.005	-.07407	0	0	0	1.0	0	.07407	-1.0(+),-.815(-),0.0(+,c),.815(-),1.0(+)
exact	-.07407	0	0	0	1.0	0	.07407	-1.0(+),-.81651(-),0.0(+,c),.81651(-),1.0(+)

Table 2. $f(x) = |x|$

$h_m=2\delta_m$	a_0	a_1	a_2	a_3	a_4	a_5	λ_m	critical points
.1	.10656	0	1.2	0	-.2	0	.10656	-1.0(-,c),-.4(+),0.0(-),.4(+),1.0(-,c)
.01	.10760	0	1.2	0	-.2	0	.10760	-1.o(-,c),-.45(+),0.0(-),.45(+),1.0(-,c)
.005	.10761	0	1.2	0	-.2	0	.10761	-1.0(-,c),-.445(+),0.0(-),.445(+),1.0(-,c)
exact	.10761	0	1.2	0	-.2	0	.10761	-1.0(-,c),-.44628(+),0.0(-),.44628(+),1.0(-,c)

Table 3. $f(x) = \begin{cases} -\exp(-x^2/2) & \text{if } -1 \le x \le 0 \\ f(-x) & \text{if } 0 < x \le 1 \end{cases}$

$h_m=2\delta_m$	a_0	a_1	a_2	a_3	a_4	a_5	λ_m	critical points
.1	-.99753	0	.47216	0	-.07869	0	.00247	-1.0(-,c),-.6(+),0.0(-),.6(+),1.o(-,c)
.01	-.99750	0	.47216	0	-.07869	0	.00250	-1.0(-,c),-.64(+),0.0(-),.64(+),1.0(-,c)
.005	-.99750	0	.47216	0	-.07869	0	.00250	-1.0(-,c),-.635(+),0.0(-),.635(+),1.0(-,c)

Table 4. $f(x) = 1 - \sqrt{1-x}$

$h_m=2\delta_m$	a_0	a_1	a_2	a_3	a_4	a_5	λ_m	critical points
.1	-.02807	.52062	0	-.06959	.32096	.21345	.04262	-1.0(-,c),-.4(+),0.0(c),.5(+),.9(-),1.0(+)
.01	-.03651	.52281	-.00781	-.08460	.32160	.21912	.04977	-1.0(-,c),-.4(+),.06(c),.52(+),.95(-),1.0(+)
.005	-.03658	.52284	-.00784	-.08477	.32164	.21920	.04984	-1.0(-,c),-.4(+),.06(c),.52(+),.945(-),1.0(+)

Table 5. f(x) = ln2.1 - ln(x+1.1)

$h_m=2\delta_m$	a_0	a_1	a_2	a_3	a_4	a_5	λ_m	critical points
.1	.66719	-.94425	0	-.02825	.85507	-.50457	.04519	-1.0(+),-.9(-),-.6(+),0.0(c),.6(+),1.0(-,c)
.01	.66684	-.94495	.00028	-.02629	.85514	-.50523	.04580	-1.0(+),-.9(-),-.58(+),.01(c),.57(+),1.0(-,c)
.005	.66677	-.94500	.00027	-.02605	.85521	-.50534	.04587	-1.0(+),-.91(-),-.58(+),.01(c),.575(+),1.0(-,c)

Table 6. f(x) = exp(-3(x+1))

$h_m=2\delta_m$	a_0	a_1	a_2	a_3	a_4	a_5	λ_m	critical points
.1	.05333	-.14333	.17442	-.24321	.27349	-.10857	.00364	-1.0(+),-.9(-),-.5(+),.1(-),.6(+),1.0(-,c)
.01	.05336	-.14324	.17406	-.24296	.27382	-.10882	.00376	-1.0(+),-.87(-),-.49(+),.07(-),.65(+),1.0(-,c)
.005	.05336	-.14324	.17406	-.24295	.27382	-.10881	.00376	-1.0(+),-865(-),-495(+),.06(-),.65(+),1.0(-,c)

Table 7. $f(x) = x^7$

$h_m=2\delta_m$	a_0	a_1	a_2	a_3	a_4	a_5	a_6	λ_m
.1	0	.10528	0	-.60928	0	1.46228	0	.04172
.01	0	.19359	0	-.82758	0	1.59099	0	.04300
.005	0	.19971	0	-.84272	0	1.59998	0	.04303

critical points

-1.0(-), -.8(+), -.4(c), .4(c), .8(-), 1.0(+)
-1.0(-), -.82(+), -.40(c), .40(c), .82(-), 1.0(+)
-1.0(-), -.815(+), -.40(c), .40(c), .815(-), 1.0(+)

Table 8. $f(x) = \sin x^3$

$h_m=2\delta_m$	a_0	a_1	a_2	a_3	a_4	a_5	a_6	λ_m
.1	-.00091	0	.00457	1.13028	-.00678	-.28083	.00312	.00798
.01	.00002	0	-.00011	1.12878	.00017	-.27917	.00008	.00814
.005	.00054	0	-.00296	1.12876	.00463	-.27915	-.00220	.00814

critical points

-1.0(+), -.9(-), -.6(+), 0.0(c), .6(-), .9(+), 1.0(-)
-1.0(+), -.89(-), -.56(+), 0.0(c), .56(-), .89(+), 1.0(-)
-1.0(+), -.89(-), -.555(+), 0.0(c), .555(-), .89(+), 1.0(-)

Table 9. $f(x) = \begin{cases} \exp(-x^4) & \text{if } -1 \le x \le 0 \\ -f(-x) & \text{if } 0 < x \le 1 \end{cases}$

$h_m=2\delta_m$	a_0	a_1	a_2	a_3	a_4	a_5	a_6	λ_m
.1	.00914	0	-.05700	.59812	.09300	.04973	-.04515	.01573
.01	.00017	0	-.00108	.60382	.00183	.04458	-.00092	.01628
.005	.00061	0	-.00394	.60387	.00668	.04455	-.00335	.01629

critical points

-1.0(+), -.9(-), -.5(+), 0.0(c), .5(-), .9(+), 1.0(-)
-1.0(+), -.86(-), -.5(+), 0.0(c), .5(-), .86(+), 1.0(-)
-1.0(+), -.865(-), -.495(+), 0.0(c), .495(-), .865(+), 1.0(-)

Table 10. $f(x) = 1 - \sqrt{1-x^3}$

$h_m=2\delta_m$	a_0	a_1	a_2	a_3	a_4	a_5	a_6	λ_m
.1	-.02236	.12468	.40478	-.08510	-1.40152	.63902	1.31016	.03037
.01	-.03786	.24555	.57496	-.41906	-1.88319	.85323	1.60623	.06016
.005	-.03787	.24553	.57500	-.41881	-1.88300	.85299	1.60601	.06017

critical points

-1.0(c), -.9(+), -.2(c), -.1(+), .4(-), .7(+), .9(-), 1.0(+)
-1.0(c), -.87(+), -.27(c), -.18(+), .39(-), .79(+), .98(-), 1.0(+)
-1.0(c), -.875(+), -.28(c), -.175(+), .395(-), .795(+), .98(-), 1.0(+)

Table 11. $f(x) = 1 - \exp(-x^5)$

$h_m=2\delta_m$	a_0	a_1	a_2	a_3	a_4	a_5	a_6	λ_m
.1	.01495	.12863	-.29483	-.52903	1.41533	1.54861	-1.67853	.02698
.01	.01540	.14030	-.28744	-.56071	1.40938	1.56579	-1.68042	.02982
.005	.01541	.14043	-.28764	-.56108	1.41009	1.56600	-1.68094	.02985

critical points

-1.0(-), -.9(+), -.7(-), -.3(+), .2(-), .3(c), .7(-), 1.0(+)
-1.0(-), -.93(+), -.62(-), -.34(+), .21(-), .26(c), .73(-), 1.0(+)
-1.0(-), -.935(+), -.62(-), -.34(+), .21(-), .26(c), .725(-), 1.0(+)

Table 12. $f(x) = \ln(1.1+x^3)$

$h_m=2\delta_m$	a_0	a_1	a_2	a_3	a_4	a_5	a_6	λ_m
.1	.13731	.36643	-.78856	-.88069	2.83139	2.00568	-2.91687	.07444
.01	.14957	.47414	-.90758	-1.18251	3.20917	2.20123	-3.15905	.10183
.005	.14971	.47581	-.90914	-1.18729	3.21448	2.20435	-3.16255	.1o221

critical points

-1.0(-), -.9(+), -.7(-), -.4(+), .2(-), .3(c), .9(-), 1.0(c)
-1.0(-), -.96(+), -.76(-), -.36(+), .19(-), .3(c), .9(-), 1.0(c)
-1.0(-), -.955(+), -.76(-), -.36(+), .19(-), .305(c), .905(-), 1.o(c)

approximation to the specified function $f(x)$ by polynomials of degree at most five over $[a, b] = [-1, 1]$. In Examples 7–12, exhibited in Tables 7–12, the problem was to find the best monotonic approximation (from K_1) to $f(x)$ by polynomials of degree at most six over $[-1, 1]$. The sets X_m were taken to be equally spaced subsets of $[-1, 1]$ with $m + 1$ points. The labels in the tables will be explained with reference to Example 1. In Example 1, the problem was to find the best uniform approximation on $[-1, 1]$ to $f(x) = x^6$ by a convex fifth-degree polynomial $\sum_{i=1}^{5} a_i x^i$. The numerical results (rounded off to 5 decimal places) of solving the discretized problem for spacing $h_m = 2\delta_m = .1, .01, .005$ are exhibited in Table 1. λ_m is the deviation of p_m over X_m. Under the heading "critical points", $-1.0(+)$ indicates that -1.0 is a plus extremal point, $-.8(-)$ indicates that $-.8$ is a minus extremal, and $0.0(+, c)$ indicates that 0.0 is a plus extremal and also a constraint point. In this example, by using symmetry and the numerical results, one can guess that the best convex approximation on $[-1, 1]$ is $x^4 - \lambda$ where λ is the deviation. This can be verified by using the characterization theorem, Theorem 2. The error curve is $e(x) = x^6 - (x^4 - \lambda)$. -1.0, 0.0, and 1.0 are plus extremals. Minus extremals occur at $-(2/3)^{1/2}$ and $+(2/3)^{1/2}$ where $e'(x) = 0$. 0.0 is a constraint point. The linear relationship (2.1) of the characterization theorem is (setting $R = (2/3)^{1/2}$)

$$R^4 p(-1) - p(-R) + 2(1 - R^4)p(0) - p(R) + R^4 p(1) + (R^2 - R^4)p''(0) = 0$$

which can be checked for $p(x) = 1, x, \cdots, x^5$. The deviation is $\lambda = 2/27$. This information, also rounded off, is included in Table 1 on the line with "exact" under h_m.

It is interesting to note that in several of the examples the number of critical points is less than $n + 2$; in the unconstrained problem the number of extremal points is always $n + 2$. Also, a variety of possible orders for the critical points is exhibited. For instance, Examples 4 and 5 have the order: plus extremal, constraint point, plus extremal, whereas Examples 8 and 9 have the order: plus extremal, constraint point, minus extremal. Example 7 has two constraint points in succession. In several examples, a constraint point coincided with an extremal point; in all such cases the adjacent extremal points were of type opposite that of the constraint-extremal point.

It can be seen from several of the examples that the analogue of the symmetry theorem, Theorem 2, is not true for constrained approximation over a discrete set X_m. The best constrained approximation over X_m is not unique in general; e.g., if we call $p(x)$ the solution of (3.1), (3.2) with $h_m \equiv 2\delta_m = .1$ in Example 1, then $p(-x)$ would also be a solution.

Notice that as h_m decreases in a particular example, the deviation λ_m is nondecreasing. This is true because $X_m \subset X_{m'}$ for $m' > m$ and a solution of (3.1), (3.2) over $X_{m'}$ would be a candidate for a solution of (3.1), (3.2) over X_m.

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1. E. W. Cheney, *Introduction to Approximation Theory*, McGraw-Hill, New York, 1966. MR **36** #5568.

2. G. F. Hadley, *Linear Programming*, Addison-Wesley Series in Industrial Management, Addison-Wesley, Reading, Mass., 1962. MR **24** #B1669.

3. P. LaFata & J. B. Rosen, "An interactive display for approximation by linear programming," *Comm. ACM*, v. 13, 1970, pp. 651–659. MR **42** #2712.

4. J. T. Lewis, *Approximation With Convex Constraints*, Doctoral Thesis, Brown University, Providence, R.I., 1969.

5. J. T. Lewis, *Approximation With Convex Constraints*, Technical Report #11, University of Rhode Island, Kingston, R.I., 1970. (Submitted for publication.)

6. G. G. Lorentz & K. L. Zeller, "Gleichmässige Approximation durch monotone Polynome," *Math. Z.*, v. 109, 1969, pp. 87–91. MR **39** #3189.

7. G. G. Lorentz & K. L. Zeller, "Monotone approximation by algebraic polynomials," *Trans. Amer. Math. Soc.*, v. 149, 1970, pp. 1–18.

8. J. R. Rice, "Approximation with convex constraints," *J. Soc. Indust. Appl. Math.*, v. 11, 1963, pp. 15–32. MR **28** #2387.