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Computation of Betti Numbers of Monomial Ideals Associated with Cyclic Polytopes

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Abstract. We give a combinatorial formula for the Betti numbers which appear in a minimal free resolution of the Stanley–Reisner ring $k[\Delta(\mathcal{P})] = A/I_{\Delta(\mathcal{P})}$ of the boundary complex $\Delta(\mathcal{P})$ of an odd-dimensional cyclic polytope $\mathcal{P}$ over a field k. A corollary to the formula is that the Betti number sequence of $k[\Delta(\mathcal{P})]$ is unimodal and does not depend on the base field k.

Introduction

Let $A = k[x_1, x_2, \ldots, x_v]$ denote the polynomial ring in v variables over a field k, which will be considered to be the graded algebra $A = \bigoplus_{n\geq 0} A_n$ over k with the standard grading, i.e., each $\deg x_i = 1$. Let $\mathbf{Z}$ (resp. $\mathbf{Q}$) denote the set of integers (resp. rational numbers). We write $A(j)$, $j \in \mathbf{Z}$, for the graded module $A(j) = \bigoplus_{n\in\mathbf{Z}} [A(j)]_n$ over A with $[A(j)]_n := A_{n+j}$. Let I be an ideal of A generated by homogeneous polynomials and let R be the quotient algebra A/I. When R is regarded as a graded module over A with the quotient grading, it has a graded *finite free resolution*

$$0 \to \bigoplus_{j\in\mathbf{Z}} A(-j)^{\beta_{h_j}} \xrightarrow{\varphi_h} \cdots \xrightarrow{\varphi_2} \bigoplus_{j\in\mathbf{Z}} A(-j)^{\beta_{1_j}} \xrightarrow{\varphi_1} A \xrightarrow{\varphi_0} R \to 0, \qquad (1)$$

where each $\bigoplus_{j\in\mathbf{Z}} A(-j)^{\beta_{i_j}}$, $1 \leq i \leq h$, is a graded free module of rank $0 \neq \sum_{j\in\mathbf{Z}} \beta_{i_j} < \infty$, and where every φ_i is degree-preserving. Moreover, there is a unique such resolution which minimizes each β_{i_j}; such a resolution is called *minimal*. If a finite free resolution (1) is minimal, then the *homological dimension* $\mathrm{hd}_A(R)$ of R over A is the nonnegative integer h and $\beta_i = \beta_i^A(R) := \sum_{j\in\mathbf{Z}} \beta_{i_j}$ is called the ith *Betti number* of R over A.

When R is a Stanley–Reisner ring, i.e., $R = A/I$ where I is generated by square-free monomials, its Betti numbers can be studied not only from an algebraic viewpoint but also from a topological and combinatorial one. It is an interesting problem to determine all the Betti numbers of the Stanley–Reisner rings for a good class of simplicial complexes.

In this paper, we give a combinatorial formula for the Betti numbers of the Stanley–Reisner ring of the boundary complex of an odd-dimensional cyclic polytope. Cyclic polytopes are important in combinatorics and have many good properties. For an even-dimensional cyclic polytope, its associated Stanley–Reisner ring has a pure minimal free resolution. Thus its Betti numbers can be easily computed from its Hilbert function. See [Sc]. On the other hand, when a cyclic polytope has an odd dimension, its associated Stanley–Reisner ring does not have a pure minimal free resolution. We need much deeper analysis to calculate the Betti numbers.

1. Simplicial Complexes and Hochster's Formula

We first recall some notation on simplicial complexes and Hochster's topological formula on Betti numbers of Stanley–Reisner rings. We refer the reader to, e.g., [BHe], [H1], [Ho], and [St] for detailed information about combinatorial and algebraic background.

(1.1) A *simplicial complex* Δ on the *vertex set* $V = \{x_1, x_2, \dots, x_v\}$ is a collection of subsets of V such that

(i) $\{x_i\} \in \Delta$ for every $1 \le i \le v$ and
(ii) $\sigma \in \Delta, \tau \subset \sigma \Rightarrow \tau \in \Delta$.

Each element σ of Δ is called a *face* of Δ. Let $\#(\sigma)$ denote the cardinality of a finite set σ. We set $d = \max\{\#(\sigma) \mid \sigma \in \Delta\}$ and define the *dimension* of Δ to be $\dim \Delta = d - 1$.

Given a subset W of V, the *restriction* of Δ to W is the subcomplex

$$\Delta_W = \{\sigma \in \Delta \mid \sigma \subset W\}$$

of Δ. In particular, $\Delta_V = \Delta$ and $\Delta_\varnothing = \{\varnothing\}$.

Let $\tilde{H}_i(\Delta; k)$ denote the ith *reduced simplicial homology group* of Δ with the coefficient field k. Note that $\tilde{H}_{-1}(\Delta; k) = 0$ if $\Delta \neq \{\varnothing\}$ and

$$\tilde{H}_i(\{\varnothing\}; k) = \begin{cases} 0 & (i \geq 0), \\ k & (i = -1). \end{cases}$$

(1.2) Let $A = k[x_1, x_2, \dots, x_v]$ be the polynomial ring in v variables over a field k. Here, we identify each $x_i \in V$ with the indeterminate x_i of A. Define I_Δ to be the ideal of A which is generated by square-free monomials $x_{i_1}x_{i_2}\cdots x_{i_r}$, $1 \le i_1 < i_2 < \cdots < i_r \le v$, with $\{x_{i_1}, x_{i_2}, \dots, x_{i_r}\} \notin \Delta$. We say that the quotient algebra $k[\Delta] := A/I_\Delta$ is the *Stanley–Reisner ring* of Δ over k. In what follows we consider A to be the graded algebra $A = \bigoplus_{n \ge 0} A_n$ with the standard grading, i.e., each $\deg x_i = 1$, and may regard $k[\Delta] = \bigoplus_{n \ge 0} (k[\Delta])_n$ as a graded module over A with the quotient grading.

(1.3) Let $h = \mathrm{hd}_A(k[\Delta])$ denote the homological dimension of $k[\Delta]$ over A and consider a graded minimal free resolution

$$0 \longrightarrow \bigoplus_{j\in\mathbf{Z}} A(-j)^{\beta_{h_j}} \xrightarrow{\varphi_h} \cdots \xrightarrow{\varphi_2} \bigoplus_{j\in\mathbf{Z}} A(-j)^{\beta_{1_j}} \xrightarrow{\varphi_1} A \xrightarrow{\varphi_0} k[\Delta] \longrightarrow 0$$

of $k[\Delta]$ over A. It is known that $v - d \leq h \leq v$.

We say that a simplicial complex Δ (or a Stanley–Reisner ring $k[\Delta]$) is *Cohen–Macaulay* (resp. *Gorenstein*) over a field k if $h = v - d$ (resp. $h = v - d$ and $\beta_{v-d} = 1$). Hochster's formula [Ho, Theorem (5.1)] guarantees that

$$\beta_{i_j} = \sum_{W\subset V, \#(W)=j} \dim_k \tilde{H}_{j-i-1}(\Delta_W; k).$$

Thus, in particular,

$$\beta_i^A(k[\Delta]) = \sum_{W\subset V} \dim_k \tilde{H}_{\#(W)-i-1}(\Delta_W; k).$$

Some combinatorial and algebraic applications of Hochster's formula have been studied. See, e.g., [Ba], [BH1], [BH2], [?], [H2]–[H4], [TH1], and [TH2].

2. Cyclic Polytopes

In this section we briefly summarize the definition and basic facts of cyclic polytopes according to [BL] and [Br]. See those references for detailed information.

(2.1) Let $\mathbf{R}$ denote the set of real numbers. For any subset M of the d-dimensional Euclidean space $\mathbf{R}^d$, there is a smallest convex set containing M. We call this convex set the *convex hull* of M and denote it by $\mathrm{conv}\, M$. For $d \geq 2$ the *moment curve* in $\mathbf{R}^d$ is the curve parametrized by

$$t \mapsto x(t) := (t, t^2, \ldots, t^d), \qquad t \in \mathbf{R}.$$

By a *cyclic polytope* $C(v, d)$, where $v \geq d + 1$ and $d \geq 2$, we mean a polytope $\mathcal{P}$ of the form $\mathcal{P} = \mathrm{conv}\{x(t_1), \ldots, x(t_v)\}$, where $t_1, \ldots, t_v$ are distinct real numbers. It is well known that $C(v, d)$ is a simplicial d-polytope with the vertex set $\{x(t_1), \ldots, x(t_v)\}$, and its face lattice is independent of the particular values of t. Therefore its boundary complex is a simplicial complex and has the same combinatorial structure for any choices of vertices. We denote it by $\Delta(C(v, d))$.

Let $V = \{x_1, \ldots, x_v\}$ be the vertex set of $C(v, d)$. Let W be a proper subset of V. A subset X of W of the form $X = \{x_i, x_{i+1}, \ldots, x_j\}$ is said to be a *contiguous subset* of W if $i > 1$, $j < v$, $x_{i-1} \notin W$, and $x_{j+1} \notin W$. The set X is a *left end-set* of W if $i = 1$ and $x_{j+1} \notin W$, and a *right end-set* of W if $j = v$ and $x_{i-1} \notin W$. We say that X is a *component* of W if X is a contiguous subset or an end-set of W. A subset X of W is said to be *even* (resp. *odd*) if the number of elements in X is even (resp. odd). The set W can be written uniquely in the form $W = Y_1 \cup X_1 \cup \cdots \cup X_n \cup Y_2$, where X_i, $1 \leq i \leq n$, is a contiguous subset of W, and Y_i, $i = 1, 2$, is an end-set of W or an empty set. We quote two facts which are necessary later. We may abuse notation and call a subset W of V itself a face of $C(v, d)$ if conv W is a face of $C(v, d)$.

Lemma 2.1 [Br, Theorem 13.7]. *Let W be an m-element subset of V, where $m \leq d$. Then W is an $(m-1)$-face of $C(v, d)$ if and only if the number of odd contiguous subsets of W is at most $d - m$.*

Lemma 2.2 [Br, Corollary 13.8]. *Let m be an integer such that $1 \leq m \leq [d/2]$. Then all m-element subsets of V are $(m-1)$-faces of $C(v, d)$.*

3. Betti Numbers of Stanley–Reisner Rings Associated with Cyclic Polytopes

In this section we compute the Betti numbers of a minimal free resolution of the Stanley–Reisner ring $k[\Delta(C(v, d))]$ of the boundary complex $\Delta(C(v, d))$ of the cyclic polytope $C(v, d)$.

We fix a field k. If the dimension d is even, a minimal free resolution of $k[\Delta]$ is pure and the Betti numbers can be easily computed from the Hilbert function of $k[\Delta]$.

Proposition 3.1 [Sch]. *Let Δ be the boundary complex $\Delta(C(v, d))$ of the cylic polytope $C(v, d)$, where $d \geq 2$ is even. Then a minimal free resolution of $k[\Delta]$ over A is of the form*

$$0 \to A(-v) \to A\left(-v + \frac{d}{2} + 1\right)^{\beta_{v-d-1}} \to \cdots \to A\left(-\frac{d}{2} - 2\right)^{\beta_2} \to A\left(-\frac{d}{2} - 1\right)^{\beta_1}$$
$$\to A \to k[\Delta] \to 0,$$

where, for $1 \leq i \leq v - d - 1$,

$$\beta_i = \binom{v - d/2 - 1}{d/2 + i}\binom{d/2 + i - 1}{d/2} + \binom{v - d/2 - 1}{i - 1}\binom{v - d/2 - i - 1}{d/2}.$$

Our formula on β_i in Proposition 3.1 is, in fact, a little bit different from the one in [Sc]. However, it is easy to show that they are coincident.

If the dimension d is odd, the minimal free resolution of $k[\Delta]$ is not pure, and the situation is much more complicated.

Now we state the main theorem in this paper.

Theorem 3.2. *Let Δ be the boundary complex $\Delta(C(v, d))$ of the cyclic polytope $C(v, d)$, where $d \geq 3$ is odd. Then a minimal free resolution of $k[\Delta]$ over A is of* *the form*

$$0 \to A(-v) \to A\left(-v+\left[\frac{d}{2}\right]+2\right)^{b_{v-d-1}} \oplus A\left(-v+\left[\frac{d}{2}\right]+1\right)^{b_1} \to \cdots$$
$$\to A\left(-\left[\frac{d}{2}\right]-2\right)^{b_2} \oplus A\left(-\left[\frac{d}{2}\right]-3\right)^{b_{v-d-2}}$$
$$\to A\left(-\left[\frac{d}{2}\right]-1\right)^{b_1} \oplus A\left(-\left[\frac{d}{2}\right]-2\right)^{b_{v-d-1}} \to A \to k[\Delta] \to 0,$$

where, for $1 \leq i \leq v-d-1$,

$$b_i = \binom{v-[d/2]-2}{[d/2]+i}\binom{[d/2]+i-1}{[d/2]}.$$

Even if the geometric realization $|\Delta|$ of a simplicial complex Δ is a sphere, a Betti number of the Stanley–Reisner ring $k[\Delta]$ may depend on the base field k in general. See Example 3.3 of [TH1]. However, as for the boundary complexes of cyclic polytopes we have the following result:

Corollary 3.3. *Let Δ be the boundary complex $\Delta(C(v,d))$ of the cyclic polytope $C(v,d)$, where $d \geq 2$. Then all the Betti numbers of the Stanley–Reisner ring $k[\Delta]$ are independent of the base field k.*

We prepare several lemmas to prove the theorem. We put $\Delta = \Delta(C(v,d))$ and $V = \{1,2,\ldots,v\}$ for simplicity, and fix an odd integer $d \geq 3$.

Lemma 3.4. *If v is odd and $W = \{1,3,5,\ldots,v\}$, then*

$$\tilde{H}_{[d/2]-1}(\Delta_W; k) = 0.$$

Proof. We have $\tilde{H}_{[d/2]-1}(\Delta_W; k) \cong \tilde{H}_{[d/2]}(\Delta_{V-W}; k)$ by the Alexander duality theorem (see, e.g., p. 76 of [St]). Since $V - W = \{2,4,\ldots,v-1\}$, if σ is a subset of $V-W$ with $\#(\sigma) > [d/2]$, then σ does not belong to Δ by Lemma 2.1. Thus we have $\tilde{H}_{[d/2]}(\Delta_{V-W}; k) = 0$. □

Lemma 3.5. *If v is even and $W = \{1,3,5,\ldots,v-1\}$, then*

$$\tilde{H}_{[d/2]-1}(\Delta_W; k) = 0.$$

Proof. All the maximal faces of Δ_W are of the form $\{1\} \cup \sigma$, where $1 \notin \sigma$, $\sigma \subset W$, $\#(\sigma) = [d/2]$. Thus Δ_W is a cone with apex $\{1\}$. Hence we have $\tilde{H}_{[d/2]-1}(\Delta_W; k) = 0$. □

Lemma 3.6. *If v is even and $W = \{2,4,6,\ldots,v\}$, then*

$$\tilde{H}_{[d/2]-1}(\Delta_W; k) = 0.$$

Proof. As in Lemma 3.5, Δ_W is a cone with apex $\{v\}$. Hence we have $\tilde{H}_{[d/2]-1}(\Delta_W; k) = 0$. $\square$

Lemma 3.7. *If v is odd and $W = \{2, 4, 6, \ldots, v-1\}$, then*

$$\dim_k \tilde{H}_{[d/2]-1}(\Delta_W; k) = \binom{[v/2]-1}{[d/2]}.$$

Proof. Let

$$0 \to C_d \to C_{d-1} \to \cdots \to C_1 \to C_0 \to C_{-1} \to 0$$

be the augmented chain complex of the simplicial complex Δ_W over k. Then we have $C_{[d/2]} = 0$ and, for $j < [d/2]$, all the $(j+1)$-subsets of W form a basis of C_j as a vector space by Lemmas 2.1 and 2.2. Thus we have $\tilde{H}_j(\Delta_W; k) = 0$ for all $j < [d/2] - 1$. Hence, the Euler–Poincaré formula (see, e.g., p. 223 of [BHe]) gives

$$\begin{aligned}
&\dim_k \tilde{H}_{[d/2]-1}(\Delta_W; k) \\
&= (-1)^{[d/2]}\left(\binom{[v/2]}{0} - \binom{[v/2]}{1} + \binom{[v/2]}{2} - \cdots + (-1)^{[d/2]}\binom{[v/2]}{[d/2]}\right) \\
&= \binom{[v/2]-1}{[d/2]}.
\end{aligned}$$

$\square$

Lemma 3.8. *Let W be a nonempty proper subset of V with a unique decomposition*

$$W = Y_1 \cup X_1 \cup X_2 \cup \cdots \cup X_n \cup Y_2$$

for some $n \geq 0$, where X_i, $1 \leq i \leq n$, is a contiguous subset and Y_i, $i = 1, 2$, is an end-set or an empty set. Then

$$\dim_k \tilde{H}_{[d/2]-1}(\Delta_W; k) = \begin{cases} \binom{n-1}{[d/2]} & \text{if } Y_1 = \varnothing \text{ and } Y_2 = \varnothing, \\ 0 & \text{otherwise}, \end{cases}$$

where we define $\binom{n-1}{[d/2]} = 0$ if $n - 1 < [d/2]$.

Proof. We prove the lemma by induction on the number v of vertices. First let $v = d+1$. Then $C(v, d)$ is a d-simplex. Thus $\tilde{H}_{[d/2]-1}(\Delta_W; k) = 0$ for every subset $W \subset V$. Since

$$n \leq \left[\frac{v-1}{2}\right] = \left[\frac{d}{2}\right],$$

the lemma holds.

Next let $v > d+1$. Let

$$V - W = X'_1 \cup X'_2 \cup \cdots \cup X'_{n+1}$$

be a unique decomposition, where X'_i, $1 \le i \le n+1$, is a component of $V - W$. Suppose $X'_i (1 \le i \le n+1)$ with $\#(X'_i) \ge 2$ exists. Let j be an element of X'_i. Put $V' = V - \{j\}$. Note that $W \subset V'$. We consider the simplicial complex $\Delta' = \Delta(C(v-1, d))$ on the vertex set V'. Then we have $\Delta'_W = \Delta_W$ by Lemma 2.1. Thus we have $\tilde{H}_j(\Delta_W; k) = \tilde{H}_j(\Delta'_W; k)$. By the induction hypothesis, we are done in this case.

We put $X_0 := Y_1$, $X_{n+1} := Y_2$. Next suppose X_i $(0 \le i \le n+1)$ with $\#(X_i) \ge 2$ exists. Let j be an element of X_i. Put $V' = V - \{j\}$. We consider the simplicial complex $\Delta' := \Delta(C(v-1, d))$ on the vertex set V'. Then we have $\Delta'_{V-W} = \Delta_{V-W}$ by Lemma 2.1. By Alexander duality, we have

$$\tilde{H}_{[d/2]-1}(\Delta_W; k) \cong \tilde{H}_{[d/2]}(\Delta_{V-W}; k) = \tilde{H}_{[d/2]}(\Delta'_{V-W}; k) \cong \tilde{H}_{[d/2]-1}(\Delta'_W; k).$$

Thus we are done in this case.

In the remaining case we may assume $\#(X_i) = 1$ for $1 \le i \le n$, $\#(X'_i) = 1$ for $1 \le i \le n+1$ and $\#(Y_i) \le 1$ for $i = 1, 2$. However, in this case we have the desired result by Lemmas 3.4–3.7. □

Proof of Theorem 3.2. Since $k[\Delta]$ is Gorenstein (see, e.g., Corollary 5.5.6 of [BHe]), we have $\mathrm{hd}_A\, k[\Delta] = v - d$. Let

$$0 \to F_{v-d} \to \cdots \to F_2 \to F_1 \to A \to k[\Delta] \to 0$$

be a minimal free resolution of $k[\Delta] = A/I_\Delta$ over A. By Lemma 2.2, we have $\min\{\alpha \in \mathbf{Z};\ (I_\Delta)_\alpha \ne 0\} = [d/2]+1$. Then F_1 has a direct summand of the form $A(-[d/2]-1)^{b_1}$ with $b_1 > 0$ and F_i, $1 \le i \le v-d-1$, may have $A(-[d/2]-i)^{b_i}$ with $b_i \ge 0$ as a direct summand. We have $F_{v-d} = A(-v)$ and F_i, $1 \le i \le v-d-1$, may have $A(-v+[d/2]+(v-d-i))^{b_{v-d-i}} = A(-[d/2]-i-1)^{b_{v-d-i}}$ as a direct summand by the self-duality of the minimal free resolution (see, e.g., p. 59 of [St]). By Proposition 1.1 of [BH2] we can easily check that other shiftings do not appear, since $k[\Delta]$ is Gorenstein. Thus we obtain the desired form of the minimal free resolution of $k[\Delta]$.

We now determine the graded Betti numbers b_i, $1 \le i \le v-d-1$. By Hochster's formula we have

$$b_i = \beta_{i_{[d/2]+i}} = \sum_{W \subset V, \#(W) = [d/2]+i} \dim_k \tilde{H}_{[d/2]-1}(\Delta_W; k).$$

Let $c_i(n)$ denote the number of $([d/2]+i)$-subsets W of V such that W has a unique decomposition $W = X_1 \cup X_2 \cdots \cup X_n$ where X_i, $1 \le i \le n$, is a contiguous subset of W. Then $c_i(n)$ is the number of positive integer solutions of the system of the equations

$$\begin{cases} x_1 + x_2 + \cdots + x_n = \left[\dfrac{d}{2}\right] + i, \\ y_1 + y_2 + \cdots + y_{n+1} = v - \left[\dfrac{d}{2}\right] - i. \end{cases}$$

Thus we have

$$c_i(n) = \binom{[d/2]+i-1}{n-1} \binom{v-[d/2]-i-1}{n}.$$

By Lemma 3.8 and combinatorial identities in Appendix 3 of [Br] we have

$$\begin{aligned} b_i &= \sum_{n\geq 1} c_i(n)\binom{n-1}{[d/2]} \\ &= \sum_{n\geq 1}\binom{v-[d/2]-i-1}{n}\binom{[d/2]+i-1}{n-1}\binom{n-1}{[d/2]} \\ &= \sum_{n\geq 1}\binom{v-[d/2]-i-1}{n}\binom{[d/2]+i-1}{[d/2]}\binom{i-1}{n-[d/2]-1} \\ &= \binom{[d/2]+i-1}{[d/2]}\binom{v-[d/2]-2}{i+[d/2]}. \end{aligned}$$

□

4. Unimodality of Betti Number Sequences

In this section we show unimodality of the Betti number sequence $(\beta_0, \beta_1, \ldots, \beta_{v-d})$ of the Stanley–Reisner ring $k[\Delta(C(v,d))]$ associated with the cyclic polytope $C(v,d)$. Since this sequence is symmetric, i.e., $\beta_i = \beta_{v-d-i}$ for every $0 \leq i \leq v-d$, the unimodality means $\beta_0 \leq \beta_1 \leq \cdots \leq \beta_{[(v-d)/2]}$.

Lemma 4.1. *Suppose d is odd. With the same situation as in Theorem 3.2 we have the following*:

(1) *If $v-d$ is even, then*

$$b_1 < b_2 < \cdots < b_{(v-d)/2} > b_{(v-d+2)/2} > \cdots > b_{v-d-1}.$$

(2) *If $v-d$ is odd, then*

$$b_1 < b_2 < \cdots < b_{(v-d-1)/2} > b_{(v-d+1)/2} > \cdots > b_{v-d-1}.$$

Proof. This lemma is clear from the following observation.

$$\begin{aligned} \frac{b_i}{b_{i+1}} = \frac{i(i+1+[d/2])}{([d/2]+i)(v-2[d/2]-2-i)} > 1 \\ &\Leftrightarrow v-2\left[\frac{d}{2}\right]-2-i < i\left(\frac{[d/2]+i+1}{[d/2]+i}\right) \\ &\Leftrightarrow v-2\left[\frac{d}{2}\right]-2-i \leq i \\ &\Leftrightarrow \frac{v-2[d/2]-2}{2} \leq i \\ &\Leftrightarrow \frac{v-d-1}{2} \leq i. \end{aligned}$$

□

Corollary 4.2. *Let Δ be the boundary complex $\Delta(C(v, d))$ of the cyclic polytope $C(v, d)$. Then, the Betti number sequence $(\beta_0(k[\Delta]), \beta_1(k[\Delta]), \ldots, \beta_{v-d}(k[\Delta]))$ of the Stanley–Reisner ring $k[\Delta]$ over A is unimodal.*

Proof. Suppose d is odd. Then $\beta_i = b_i + b_{v-d-i}$ for $1 \leq i \leq v - d - 1$. Now the corollary is clear from the above lemma.

Suppose d is even. By Proposition 3.1 and Theorem 3.2, we see $\beta_i(k[\Delta]) = \beta_i(k[\Delta(C(v + 1, d + 1))])$ for $0 \leq i \leq v - d$. Thus we can reduce this case to the odd-dimensional case. □

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Note added in proof. After submitting this article, we obtained the "stacked polytope" version of our results. See [TH3].