Computation of Degree Constrained Rational Interpolants With Non-Strictly Positive Parametrizing Functions by Extension of a Continuation Method

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***Abstract*— A numerically stable homotopy continuation method was first proposed by Enqvist for computing degree constrained rational covariance extensions. The approach was later adapted in the works of Nagamune, and Blomqvist and Nagamune, to the Nevanlinna-Pick interpolation problem and more general complexity constrained problems. Yet the method has not been developed to the fullest extent as all the previous works limit the associated parametrizing function (in the form of a generalized pseudopolynomial) to be strictly positive definite on the unit circle, or equivalently, that all spectral zeros should lie inside the unit circle. The purpose of this paper is to show that the aforementioned restriction is unnecessary and that the method is equally applicable when some spectral zeros are on the unit circle. We show that even in this case, the modified functional of Enqvist has a stationary minimizer. Several numerical examples are provided herein to demonstrate the applicability of the method for computing degree constrained interpolants with spectral zeros on the unit circle, including solutions which may have poles on the unit circle.**

***Index Terms*— Rational interpolation with degree constraint, homotopy continuation, unbounded interpolants**

I. NOTATION AND BASIC DEFINITIONS

This section introduces the main notation used in this paper, and also recall some definitions and relevant results from the literature.

- $\overline{A}$ and ∂A denote the completion and boundary of a set A, respectively.
- $\mathbb{R}$, $\mathbb{C}$, $\mathbb{D}$ and $\mathbb{T}$ denote the set of real numbers, complex numbers, the open unit disc= $\{z \in \mathbb{C} \mid |z| < 1\}$ and the unit circle, respectively.
- $\Re\{c\}$ denotes the real part of $c \in \mathbb{C}$.
- $C^\top$ and C^* denote the transpose and conjugate transpose of a complex matrix C, respectively.
- $\mathrm{col}(a_1, \ldots, a_n) = [a_1 \ldots a_n]^{\mathrm{T}}$.
- f_* denotes the *parahermitian conjugate* of a complex function f, defined by $f_*(z) = f(z^{*-1})^*$.
- $\mathcal{C}$ denotes the Carathèodory class $\{f \in \mathcal{H} \mid \Re\{f(z)\} \geq 0 \forall z \in \mathbb{D}\}$ and $\mathcal{C}_+$ denotes the subset $\{f \in \mathcal{H} \mid \operatorname{ess\,inf}_{z\in\mathbb{D}} \Re\{f(z)\} > 0\}$ of $\mathcal{C}$ where $\mathcal{H}$ denotes the set of functions holomorphic in $\mathbb{D}$.
- $\mathcal{H}^\infty$ denotes the Hardy class of functions in $\mathcal{H}$ which are essentially bounded on $\mathbb{T}$.
- $\langle f, g\rangle$ denotes the integral $\frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(e^{i\theta}\right) g\left(e^{i\theta}\right)^* d\theta$ for complex functions f and g which are square-integrable on $\mathbb{T}$.
- $\|f\|_\infty = \operatorname{ess\,sup}_{z\in\mathbb{T}} |f(z)|$.

For $z_0, z_1, \ldots, z_n \in \mathbb{D}$, define

$$\alpha_k(z) = \frac{z + z_k}{z - z_k}$$

whenever z_k has multiplicity 1, and

$$\alpha_k(z) = \frac{z + z_k}{z - z_k} \quad \text{and} \quad \alpha_{k+j}(z) = \frac{2z}{(z - z_k)^{j+1}}$$

for $j = 1, \ldots, m-1$ when z_k has multiplicity m and $z_k = z_{k+1} = \ldots = z_{k+m-1}$. By a *generalized pseudopolynomial* we mean a complex function of the form $f(z) = a_0 + \sum_{k=1}^{n} (a_k^* \alpha_k + a_k \alpha_{k*})$, where $0 \leq n < \infty$, $a_n \neq 0$ and $(a_0, a_1, \ldots, a_n) \in \mathbb{R} \times \mathbb{C}^n$. We say that n is the *order* of the generalized pseudopolynomial f (the order is zero if f is a constant function). $\mathfrak{Q}(n, A)$ denotes the set of all generalized pseudopolynomials of order *at most* n with $(a_0, a_1, \ldots, a_n) \in \mathbb{R} \times A^n$ where $A \subseteq \mathbb{C}$. We induce a topology on this set by the $\|\cdot\|_\infty$ norm. We also define $\mathfrak{Q}_+(n, A)$ to be the set of all elements of $\mathfrak{Q}(n, A)$ which are strictly positive (> 0) on $\mathbb{T}$. The restriction of any element of $\overline{\mathfrak{Q}_+(n, A)}\backslash\{0\}$ to $\mathbb{T}$ is a rational spectral density of McMillan degree at most $2n$, thus we shall often view any such element as a spectral density instead of a generalized pseudopolynomial. Hence, to each $d \in \overline{\mathfrak{Q}_+(n, A)}\backslash\{0\}$ we may associate a *unique* outer rational function (i.e., having no roots and poles in $\mathbb{D}$) of McMillan degree at most n, denoted by $\phi(d)$, which is the unique canonical spectral factor of d satisfying: $\phi(d)(0) > 0$ and $|\phi(d)(z)|^2 = d(z)\ \forall z \in \mathbb{T}$. For details on outer functions, spectral densities and CSFs see [1], [2].

Let $\tau(z) = \Pi_{k=0}^{n}(1 - z_k^* z)$ and $H_n = \mathrm{span}\{\alpha_0, \alpha_1, \ldots, \alpha_n\}$. It will prove useful later to note that H_n has an equivalent description as $H_n = \{f \mid f = \frac{\sigma}{\tau}, \sigma \text{ is polynomial of degree at most } n\}$ [3]. Then $\mathfrak{Q}_+(n, \mathbb{C})$ can be alternatively described as $\mathfrak{Q}_+(n, \mathbb{C}) = \{g + g_* \mid g \in H_n, \Re\{g(z)\} \geq 0 \forall z \in \mathbb{D}, g(0) > 0\}$.

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II. Background and Motivation

Let there be given $\{z_0, z_1, \ldots, z_n\} \subset \mathbb{D}$ and $\{w_0, w_1, \ldots, w_n\} \subset \mathbb{C}_{0+}$, where $\mathbb{C}_{0+} = \{z \in \mathbb{C} \mid \Re\{z\} \geq 0\}$. We make the convention that non-unique z_k's are ordered sequentially. Moreover, for simplicity we shall assume $z_0 = 0$ and w_0 is real. There is no loss in generality in taking this assumption since the map $z \mapsto \frac{z - z_0}{1 - z_0^* z}$ sends any $z_0 \in \mathbb{D}$ to 0 and is a bianalytic map from $\mathbb{D}$ onto itself. Secondly, we are allowed subtract the imaginary part of w_0 from $w_1, \ldots, w_n$ without changing Problem 1 to be stated below. For further details, the reader may consult [4, Appendix A].

Consider the following degree constrained rational interpolation problem:

Problem 1: Find all $f \in \mathcal{C}$ of McMillan degree at most n such that $f(z_k) = w_k$ if z_k is of multiplicity 1, and $f^{(k+j)}(z_k) = w_{k+j}$ for $j = 1, \ldots, m-1$ if z_k is of multiplicity m and $z_k = z_{k+1} = \ldots = z_{k+m-1}$.

It is well known that the above problem has a solution if and only if a certain *(generalized) Pick matrix*, constructed from the data $\{w_0, w_1, \ldots, w_n\}$, is non-negative definite [3], [5]. Moreover, the solution is unique if the matrix is singular, otherwise there are infinitely many solutions. It has been shown in [3] (and rederived by constructive means in [6] and [7], see Remark 4 below) that when the generalized Pick matrix is (strictly) positive definite, all solutions to Problem 1 are completely parametrized by the set of generalized polynomials of degree at most n which are non-negative definite on $\mathbb{T}$. To be precise, we have the following:

Theorem 2 ([3], [6], [7]): For a given interpolation data with a positive definite Pick matrix, and any polynomial $\eta \neq 0$ of degree $\leq n$ with roots in $\mathbb{D}^c$ normalized by $\eta(0) = 1$, there exists a unique pair of polynomials (a, b) of degree $\leq n$ such that $b(0) > 0$, $a + b$ has all its roots in $\mathbb{D}^c$, the pair satisfies the relation

$$ab_* + ba_* = \kappa^2 \eta \eta_* \tag{1}$$

for a fixed $\kappa > 0$, and $f = \frac{a}{b}$ is a solution of Problem 1.

The roots of η in the above theorem are referred to in the literature as "spectral zeros."

The connection between α_k and Problem 1 lies in the Herglotz representation [8]. In this representation, any solution of Problem 1 is expressed as:

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta),$$

where μ is a non-decreasing function of bounded variation on $[-\pi, \pi]$ with $\mu(0) = 0$, called the *spectral distribution* of f. The spectral distribution has the decomposition $\mu = \mu_a + \mu_s$ where μ_a is absolutely continuous while μ_s is a piecewise constant function with at most $n - 1$ jumps. This allows us to write each interpolation condition in integral form:

$$\begin{aligned} f(z_k) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \alpha_k(e^{i\theta}) d\mu(\theta) = w_k \end{aligned}$$

and

$$\begin{aligned} \frac{1}{j!} f^{(j)}(z_k) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{2e^{i\theta}}{(e^{i\theta} - z)^{j+1}} d\mu(\theta) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \alpha_{k+j}(e^{i\theta}) d\mu(\theta) \\ &= w_{k+j}, \quad \text{for } j = 0, 1, \ldots, m-1 \end{aligned}$$

whenever $z_k = z_{k+1} = \ldots, z_{k+m-1}$.

A convex optimization approach for computing solutions of the problem for $\eta_* \eta \in \mathfrak{Q}_+(n, \mathbb{R})$ are given in the papers [9], [10]. However, the method, without modification, has features which makes it numerically unsuitable for computation of solutions with poles close to the unit circle. A modification of the method, by reparametrization and application of a homotopy continuation method, was first introduced by Enqvist [11] for the rational covariance extension problem, and subsequently adapted by Nagamune [12], and Blomqvist and Nagamune [4], [13] to Nevanlinna-Pick interpolation and moment problems. However, the approach was never studied and extended to the case where $\eta_* \eta \in \partial \mathfrak{Q}_+(n, \mathbb{C}) \backslash \{0\}$. Although theoretical results are available in [6] in the setting of generalized interpolation on $\mathcal{H}^\infty$, and an alternative treatment given in [14], [7], no "complete" algorithm has been presented for this case apart from [15]. The latter algorithm departs from the ideas of [11], [4], [12], [13] and proposes computation of all real solutions by numerically solving some non-linear equations. However, the latter approach was specific for rational interpolation problems, while the method of [9], [11] extends to more general moment problems as demonstrated in [13]. Moreover, further theoretical results have been obtained by studying the convex optimization technique for the case of spectral zeros on the unit circle [6], [14], [7] and has been applied to spectral factorization [16]. Therefore, it is of interest to investigate applicability of the homotopy continuation method of Enqvist if one allows η to have zeros on the unit circle. It has already been argued in [14] and indicated in Example 14 therein that such an extension is feasible when the solution f is bounded (has no poles on $\mathbb{T}$). The present paper provides further justification and goes on to cover the case of unbounded solutions as well. As a starting point, we focus on the rational interpolation problem as summarized in Problem 1 and show theoretically that such an extension is indeed valid. Later in Section IV, the extended homotopy continuation method is then applied on several examples for practical illustration.

III. Analysis and Main Results

Define the mapping $Q : \mathbb{R} \times \mathbb{C}^n \to \mathfrak{Q}(n, \mathbb{C})$ by:

$$Q(q_0, q_1, q_2, ..., q_n)(z) = q_0 + \sum_{k=1}^{n} \frac{1}{2}(q_k^* \alpha_k + q_k \alpha_{k*}). \tag{2}$$

Clearly Q is a bijective map. Then we define $\mathcal{Q}_{n+} \subset \mathbb{R} \times \mathbb{C}^n$ as $\mathcal{Q}_{n+} = Q^{-1}(\mathfrak{Q}_+(n, \mathbb{C}))$ and let $\mathcal{Q}_n = \overline{\mathcal{Q}_{n+}}$.

Let $\Psi = \frac{\eta_* \eta}{\tau_* \tau}$ with η is a polynomial as defined in Theorem 2. Then $\Psi \in \overline{\mathfrak{Q}_+(n, \mathbb{C})} \backslash \{0\}$ (for details, see Section III of

[3], [10]). We first consider the functional $\mathbb{J}_\Psi : \mathcal{Q}_n \to \mathbb{R} \cup \{\infty\}$ defined by:

$$\mathbb{J}_\Psi(q) = \Re\big\{w^* q - \langle \Psi, \log Q(q) \rangle\big\} \tag{3}$$

The functional was first introduced and studied for the case $\Psi \in \mathfrak{Q}_+(n,\mathbb{C})$ in [9], [10]. It was subsequently shown in [17], [14] that they continue to hold for $\Psi \in \partial\mathfrak{Q}_+(n,\mathbb{C})\backslash\{0\}$. Its properties are summarized the following:

Theorem 3 ([14], [17], [10], [9]): $\mathbb{J}_\Psi$ has the following properties for any $\Psi \in \overline{\mathfrak{Q}_+(n,\mathbb{C})}\backslash\{0\}$:

- $\mathbb{J}_\Psi$ is finite and continuous at any $q \in \mathcal{Q}_n$, except at zero. Moreover, $\mathbb{J}_\Psi((1-t)q_0 + tq_1)$ is a C^∞ function w.r.t. t for any $q_0, q_1 \in \mathcal{Q}_n$.
- $\mathbb{J}_\Psi$ is strictly convex on the closed, convex domain $\mathcal{Q}_n$.
- The functional $\mathbb{J}_\Psi$ has a unique minimum on $\mathcal{Q}_n$.

Remark 4: Strictly speaking, [17], [14], [7] consider the case $z_0 = z_1 = \ldots = z_n = 0$ and $\alpha_k(z) = \frac{2}{z^k}$ for $k = 0, 1, \ldots, n$, i.e., the rational covariance extension problem. However, as the stated in [14], [7], and as can be seen by observing that $\overline{\mathfrak{Q}_+(n,\mathbb{C})}$ lies in a finite dimensional space (i.e., $\mathrm{span}\{\alpha_{1*}, \ldots, \alpha_{n*}\} \bigoplus H_n$) and contains rational functions continuous on $\mathbb{T}$, the analysis carries over *mutatis mutandis* (one substitutes z^k with $\alpha_{k*}(z)$ and c_k with w_k, etc) without technical difficulty to the current setting.

For any $q' \in \mathcal{Q}_n$, let $D_{q'}\mathbb{J}_\Psi(q)$ denote the direction derivative of $\mathbb{J}_\Psi$ at the point q in the direction of q', i.e.,

$$D_{q'}\mathbb{J}_\Psi(q) = \lim_{h \downarrow 0} \frac{\mathbb{J}_\Psi(q + h(q'-q)) - \mathbb{J}_\Psi(q)}{h}.$$

It was first shown in [9] that whenever Ψ is positive definite on $\mathbb{T}$, $\mathbb{J}_\Psi$ has a minimizer q_{min} which is stationary (i.e., $D_{q'}\mathbb{J}_\Psi(q_{min}) = 0 \; \forall q' \in \mathcal{Q}_n$) and lies in $\mathcal{Q}_{n+}$. It then follows that b in Theorem 2 is given by $b = \tau\phi(Q(q_{min})$ and a can be found by solving the equation $a_* b + b a_* = \Psi$ [10].

As for the case where Ψ has zeros on $\mathbb{T}$, it turns out that $\mathbb{J}_\Psi$ exhibits some interesting properties as stated in the following adaptation of [7, Theorem 8]:

Theorem 5: Let η be as in Theorem 2, $\Psi = \frac{\eta_*\eta}{\tau_*\tau} \in \partial\mathfrak{Q}_+(n,\mathbb{C})\backslash\{0\}$ and $q_{min} = \underset{q \in \mathcal{Q}_n}{\arg\min}\, \mathbb{J}_\Psi$. Then:

1) $D_{q'}\mathbb{J}_\Psi(q_{min}) = 0$ for all $q' \in \mathcal{Q}_n$ if and only if the pair (a,b) as defined in Theorem 2 is such that $f = \frac{a}{b} \in \mathcal{H}^\infty$.
2) $D_{q'}\mathbb{J}_\Psi(q_{min}) > 0$ (resp., $= 0$) for all $q' \in \mathcal{Q}_{n+}$ (resp., $q' \in \partial\mathcal{Q}_{n+}$) if and only if the pair (a,b) as defined in Theorem 2 is such that $f = \frac{a}{b}$ has a pole on $\mathbb{T}$. $D_{q'}\mathbb{J}_\Psi(q_{min})$ is then given by:

$$\begin{aligned} D_{q'}\mathbb{J}_\Psi(q_{min}) &= \sum_{k=0}^{n} \Re\bigg\{\Big(w_k - \langle \alpha_k, \frac{\Psi}{Q(q_{min})}\rangle\Big)^* \\ &\qquad (q'_k - q_{min,k})\bigg\} \\ &= \sum_{l=0}^{m} K_l \Re\bigg\{\sum_{k=0}^{n} \alpha_{k*}(e^{i\theta_l}) \\ &\qquad (q'_k - q_{min,k})\bigg\}, \end{aligned} \tag{4}$$

where $m < n$, $K_0, K_1, \ldots, K_m$ are some positive constants and $\theta_0, \theta_1, \ldots, \theta_m \in (-\pi, \pi]$, with $\theta_i \neq \theta_j$ whenever $i \neq j$, are the discontinuity points of the spectral distribution of f, i.e., $e^{i\theta_0}, \ldots, e^{i\theta_m}$ are poles of f on $\mathbb{T}$.

Moreover, in both cases $\frac{b}{\tau} = \phi(Q(q_{min}))$ and all roots of $Q(q_{min})$ on $\mathbb{T}$, including multiplicities, are also roots of Ψ.

Proof: Although the proof is analogous to the proof of [7, Theorem 8] (see Remark 4), for the sake of clarity we shall here just detail a possibly not so obvious part in the adaptation of the latter proof needed to establish Point 2 of the theorem.

As in [7], we write $f = f_a + f_s$, where $f_a \in \mathcal{C} \cap \mathcal{H}^\infty$ while $f_s \in \mathcal{C}$ has one or more poles on $\mathbb{T}$. We also have the representation $f_a(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} d\mu_a$ and $f_s(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} d\mu_s$, where μ_a and μ_s are, respectively, the absolutely continuous and singular part of the spectral distribution μ of f. Since $d\mu_a(\theta) = \Re\{f_a(e^{i\theta})\}d\theta$ and $d\mu_s(\theta) = \sum_{l=0}^{m} K_l \delta(\theta - \theta_l)d\theta$ for some positive constants $K_0, K_1, \ldots, K_n$ ($\delta(x)$ denotes the Dirac delta function), we have that $f_a(z_k) = \langle f_a + f_{a*}, \alpha_{k*}\rangle$ and $f_s(z_k) = \sum_{l=0}^{m} K_l \alpha_k(e^{\theta_l})$. Thus, we obtain the relation $f(z_k) - f_a(z_k) = w_k - \langle f_a + f_{a*}, \alpha_{k*}\rangle = \sum_{l=0}^{m} K_l \alpha_k(e^{i\theta_l})$, in analogy with that obtained for the case $z_0 = z_1 = \ldots = z_n = 0$ in [7]. The relation is a key one for establishing (4). As for remaining arguments, they are straightforward to adapt from the proof of [7, Theorem 8]. ■

An important conclusion to be drawn from Theorem 5 is that, regardless of whether Ψ is positive definite on $\mathbb{T}$ or has zeros there, we always have that the polynomial b of Theorem 2 associated with Ψ is given by $b = \tau\phi(Q(q_{min}))$. Once b is computed, a is easily obtained from b by multiplying the coefficients of b by a certain matrix W which only depends on the interpolation data $w_1, w_2, \ldots, w_n$ (see, e.g., [15], [7] for further details), i.e., if $a(z) = \mathrm{col}(a_0, a_1, \ldots, a_n)^{\mathrm{T}}\mathrm{col}(1, z, \ldots, z^n)$ and $b = \mathrm{col}(b_0, b_1, \ldots, b_n)^{\mathrm{T}}\mathrm{col}(1, z, \ldots, z^n)$ then:

$$\mathrm{col}(a_0\; a_1\; \ldots\; a_n) = W\mathrm{col}(b_0, b_1, \ldots, b_n). \tag{5}$$

The only disrepancy is that when Ψ has zeros on $\mathbb{T}$, $\mathbb{J}_\Psi$ may have a minimizer which is *not* a stationary point.

Although properties of $\mathbb{J}_\Psi$ make it convenient for analysis, it is not suitable for numerical optimization, especially when q_{min} is close to or on the boundary. This is due to the fact that the condition number of the Hessian of $\mathbb{J}_\Psi$ tends to ∞ as one goes to the boundary. Define

$$D(d)(z) = \sum_{k=0}^{n} d_k z^k$$

and $\mathcal{D}_n = \{d = \mathrm{col}(d_0, d_1, \ldots, d_n) \in \mathbb{R} \times \mathbb{C}^n \mid d_0 > 0, D(d) \,\text{is outer}\}$. Then for $\Psi \in \mathfrak{Q}_+(n,\mathbb{C})$, one way to circumvent the difficulty with $\mathbb{J}_\Psi$, developed in [11], [18], [4], [12], [13], is to reformulate the optimization problem as minimizing the functional $\mathcal{J}_\Psi : \mathcal{D}_n \to \mathbb{R}$:

$$\mathcal{J}_\Psi(d) = d^* K d - \Big\langle \log \Big| \frac{D(d)}{\tau} \Big|^2, \Psi \Big\rangle.$$

where K is a positive definite matrix which is only dependent on the interpolation data $\{(z_k, w_k)\}_{k=0,1,\ldots,n}$. Since $|D(a)(e^{i\theta})|^2 = D(e^{i\theta})D(e^{i\theta})^*$ (and similarly for τ), it is easy to see that $\mathcal{J}_\Psi$ can be written as

$$\mathcal{J}_\Psi(d) = d^* K d - 2\Re\{\langle \log D(d), \Psi\rangle\} + 2\Re\{\langle \log \tau, \Psi\rangle\},$$

where the last term does not depend on d is not essential in the ensuing analysis.

It has been argued in [11], that the new functional is much better suited for numerical treatment as the hessian and its condition number does not blow up as one goes to the boundary of D_n. However, the modified optimization problem is no longer convex as the domain $\mathcal{D}_n$ is not a convex set, besides the fact that $\mathcal{J}_\Psi$ is also not convex on $\mathcal{D}_n$. Fortunately, due to the bijective correspondence between $\mathfrak{Q}_+(n,\mathbb{C})\backslash\{0\}$ and $\mathcal{D}_n$, $\mathcal{J}_\Psi$ has a unique global minimum, and it has been shown that locally convex around the global minimum. This makes it possible to find its global minimum by constructing a convex homotopy and solving a sequence of locally convex optimization problems as detailed in [11], [18], [4], [12]. We have the following new result which has only been shown previously for $\Psi \in \mathfrak{Q}_+(n,\mathbb{C})$:

Lemma 6: For $\Psi \in \partial\mathfrak{Q}_+(n,\mathbb{C})\backslash\{0\}$, $\mathcal{J}_\Psi$ again has a unique minimizer on $\mathcal{D}_n$. Moreover, this minimizer is also stationary.

Proof: Let s denote the bijective map that sends $a \in \mathcal{D}_n$ to $Q^{-1}(D(a)_* D(a)) \in \mathcal{Q}_n\backslash\{0\}$ and note the relation:

$$\mathcal{J}_\Psi(a) = \mathbb{J}_\Psi(s(a)) + 2\langle \log|\tau|, \Psi\rangle,$$

Let q_{min} be as in Theorem 5 and define $\hat{d} = s^{-1}(q_{min})$. Using the fact that $\mathbb{J}_\Psi(q_{min}) \le \mathbb{J}_\Psi(q)\ \forall q \in \mathcal{Q}_n$ (by Theorem 3), we then have that

$$\begin{aligned}\mathcal{J}_\Psi(\hat{d}) &= \mathbb{J}_\Psi(q_{min}) + 2\langle \log|\tau|, \Psi\rangle \\ &< \mathbb{J}_\Psi(q) + 2\langle \log|\tau|, \Psi\rangle \quad \forall q \in \mathcal{Q}_n\backslash\{q_{min}\} \\ &= \mathcal{J}_\Psi(s^{-1}(q)) \quad \forall q \in \mathcal{Q}_n\backslash\{q_{min}\}.\end{aligned}$$

Therefore, $\mathcal{J}_\Psi(\hat{d}) < \mathcal{J}_\Psi(d)$ for all $d \in \mathcal{D}_n\backslash\{\hat{d}\}$, implying that $\hat{d}$ is the unique minimizer of $\mathcal{J}_\Psi$. This proves the first part of the lemma.

Define the directional derivative of $\mathcal{J}_\Psi$ in the direction d' analogously to (4) and denote it by $D_{d'}\mathcal{J}_\Psi$, where $d' \in \mathcal{D}_n$. Let $\mathcal{N}_\Psi = \{d \in \mathcal{D}_n \mid \operatorname{ess\,sup}_{z\in\mathbb{T}} \Big|\frac{\Psi(z)}{D(d)(z)}\Big| < \infty\}$. Then by similar arguments to [14, Proof of Theorem 13 (Appendix)], we may show that $D_{d'}\mathcal{J}_\Psi$ is given by:

$$\begin{aligned}D_{d'}\mathcal{J}_\Psi(d) &= 2\Re\Big\{ d^* R(d' - d) \\ &\qquad - \sum_{i=0}^{n} \Big\langle \frac{g_i}{D(d)}, \Psi \Big\rangle (d'_i - d_i) \Big\},\end{aligned} \tag{6}$$

where $g_i(z) = z^i$. Let d_{min} be the unique minimizer of $\mathcal{J}_\Psi$. Since $d_{min} = s^{-1}(q_{min})$, it is obvious that $D(d_{min}) = \phi(Q(q_{min}))$. Now, let Ψ_k, $k = 1, 2, \ldots$, be a sequence such that $\Psi_k \in \mathfrak{Q}_+(n,\mathbb{C})$ for all k and Ψ_k converges to Ψ uniformly on $\mathbb{T}$, i.e., $\lim_{k\to\infty} \|\Psi - \Psi_k\|_\infty = 0$ and let $d^k_{min} = \arg\min_{d\in\mathcal{D}_n} \mathcal{J}_{\Psi_k}(d)$. Then as shown in [7, Proof of Theorem 8]:

$$\lim_{k\to\infty} \|d_{min} - d^k_{min}\|_2 = \lim_{k\to\infty} \|D(d_{min}) - D(d^k_{min})\|_\infty = 0.$$

Furthermore, as shown in [13] we have that $D_{d'}\mathcal{J}_{\Psi_k}(d^k_{min}) = 0$ for all $d' \in \mathcal{D}_n$ and for all k. Now, by the uniform convergence of Ψ_k to Ψ and $D(d^k_{min})$ to $D(d_{min})$ as noted above, we have:

$$\lim_{k\to\infty} \frac{\Psi_k(z)}{D(d^k_{min})(z)} = \frac{\Psi(z)}{D(d_{min})(z)} \quad \text{for a.a. } z \in \mathbb{T}, \tag{7}$$

with the exceptional points being the roots of $D(d_{min})$ on $\mathbb{T}$ (which are also roots of Ψ by Theorem 5). Since $\Psi_k = D(d^k_{min})D(Wd^k_{min})_* + D(d^k_{min})_* D(Wd^k_{min})$ (see the discussion on the previous page and Eq. (5)), we have that

$$\begin{aligned}\Big\|\frac{\Psi_k}{D(d^k_{min})}\Big\|_\infty &= \Big\| D(Wd^k_{min})_* \\ &\qquad + \frac{D(d^k_{min})_*}{D(d^k_{min})} D(Wd^k_{min}) \Big\|_\infty \\ &\le \|D(Wd^k_{min})_*\|_\infty \\ &\qquad + \Big\|\frac{D(d^k_{min})_*}{D(d^k_{min})}\Big\|_\infty \|D(Wd^k_{min})\|_\infty \\ &= 2\|D(Wd^k_{min})\|_\infty.\end{aligned}$$

Now, since $D(Wd^k_{min}) \stackrel{\|\cdot\|_\infty}{\to} D(Wd_{min})$ as $k \to \infty$, it follows that $\sup_{k\ge1} \|D(Wd^k_{min})\|_\infty < \infty$. Consequently,

$$\sup_{k\ge1} \Big\|\frac{\Psi_k}{D(d^k_{min})}\Big\|_\infty < \infty, \tag{8}$$

i.e., the sequence $\{\|\frac{\Psi_k}{D(d^k_{min})}\|_\infty; k = 1, 2, \ldots\}$ is uniformly bounded. Now, by plugging (7) into (6), and once again invoking the Lebesque Dominated Convergence Theorem by using (8), we get:

$$\begin{aligned}D_{d'}\mathcal{J}_\Psi(d_{min}) &= \lim_{k\to\infty} 2\Re\Big\{(d^k_{min})^* R(d' - d^k_{min}) \\ &\qquad - \sum_{i=0}^{n} \Big\langle \frac{g_i}{D(d^k_{min,i})}, \Psi\Big\rangle (d'_i - d^k_{min,i})\Big\} \\ &= \lim_{k\to\infty} D_{d'}\mathcal{J}_{\Psi_k}(d^k_{min}) \\ &= \lim_{k\to\infty} 0 = 0 \quad \text{for all } d' \in \mathcal{D}_n.\end{aligned}$$

This shows that d_{min} is a stationary point and completes the proof of the lemma. ■

Lemma 6 shows a *striking difference* between $\mathcal{J}_\Psi$ and $\mathbb{J}_\Psi$: for $\Psi \in \partial\mathfrak{Q}_+(n,\mathbb{C})\backslash\{0\}$, the minimizer of $\mathcal{J}_\Psi$ is *always stationary* while the minimizer of $\mathbb{J}_\Psi$ *may not be*. From the lemma the following is easily obtained:

Corollary 7: The functional $\mathcal{J}_\Psi$ is locally convex in a neighborhood of its unique minimizer.

Proof: Again, let d_{min} denote the unique minimizer of $\mathcal{J}_\Psi$ and let $\mathcal{N}_\Psi$ be as defined in the proof of the previous lemma. Note that $d_{\min} \in \mathcal{N}_\Psi$ since all roots of $D(d_{min})$ on $\mathbb{T}$, counting multiplicities, are also roots of Ψ. Then we may, as before, proceed by invoking the Lebesque Dominated Convergence Theorem [19] to show that the second directional derivative of $\mathcal{J}_\Psi$ at a point $d \in \mathcal{N}_\Psi$ in the direction of $d' \in \mathcal{D}_n$, defined as:

$$D^2_{d'}\mathcal{J}_\Psi(d) = \lim_{h \downarrow 0} \frac{D_{d'}\mathcal{J}_\Psi(d + h(d'-d)) - D_{d'}\mathcal{J}_\Psi(d)}{h},$$

is given by (recall that $g_i(z) = z^i$):

$$\begin{aligned} D_{d'}\mathcal{J}_\Psi(d) &= 2\Re\Big\{(d'-d)^* R(d'-d) - d^* R(d'-d) \\ &\quad + \sum_{i=0}^{n}\sum_{j=0}^{n}\langle \frac{g_i g_j}{D(d)^2}, \Psi\rangle \qquad (9) \\ &\quad \times (d'_i - d_i)(d'_j - d_j)\Big\}. \qquad (10) \end{aligned}$$

In particular, $D^2_{d'}\mathcal{J}_\Psi(d_{min})$ exists and is bounded in all directions $d' \in \mathcal{D}_n$. Since d_{min} is stationary and is the unique minimizer of $\mathcal{J}_\Psi$, as shown in the previous lemma, we may conclude that $D^2\mathcal{J}_\Psi(d_{min}) \geq 0$ for all $d' \in \mathcal{D}_n$. Hence, $\mathcal{J}_\Psi$ is convex on some sufficiently small convex neighborhood of $d_{\min}$. ■

Lemma 6 and Corollary 7 justify the use of the homotopy continuation method for finding solutions of Problem 1 corresponding to η with spectral zeros on the unit circle. Although the functional is not globally convex, we do have a stationary minimizer and local convexity around the minimizer. This is enough to allow us to use a homotopy continuation to circumvent the lack of global convexity, and solve a sequence of locally convex problems, as is done for the case where all spectral zeros are strictly inside the unit circle. In the next section, we put our assertions to the test by applying the continuation method to compute the different kinds of possible solutions as summarized in Theorem 5.

IV. Numerical examples

In this section we present numerical results of applying the continuation method for computing solutions of Problem 1 corresponding to spectral zeros on $\mathbb{T}$. However, although our results are developed for a general case, in the examples we restrict our attention to the rational covariance extension problem, i.e. $z_0 = z_1, \ldots, z_n = 0$. The reason for this is that this special problem has been the focus of our recent research efforts in approximation of second order processes and spectral factorization [16]. Moreover, to avoid complex arithmetics, we shall only consider the real case, where $w_0, w_1, \ldots, w_n \in \mathbb{R}$. We implement the homotopy continuation algorithm as described in [11] and use the stopping criteria: $e_n = \|\mathcal{J}_\Psi(\hat{d}_n) - \mathcal{J}_\Psi(\hat{d}_{n-1})\|_2 < \epsilon$ for a specified tolerance $\epsilon > 0$, where $\hat{d}_n$ denote the iterate (approximation of d_{min}) at the k-th iteration of the algorithm. In all examples, we take the step size $\rho = 0.1$ (see [11, p. 1196]) and set $\epsilon = 10^{-6}$.

Example 8: Let $w_0 = 0.21052$, $w_1 = -0.10263$ and $w_2 = -0.00671$. We choose $\eta(z) = (z-0.5)(z-1) \Leftrightarrow \Psi = \eta_*\eta = z^{-2} - 4.5z^{-1} + 7 - 4.5z + z^2$. The algorithm returns $d_{min} = \mathrm{col}(3.162283, -1.423018, 0.158116)$. Thus, $b(z) = 3.162283 - 1.423018z + 0.158116z^2$ and $a(z)$ can be computed to be $a(z) = 0.21052 - 0.10263z - 0.00671z^2$. Since all roots of b are inside the unit circle, this example illustrates the case where there are spectral zeros inside the unit circle, but where b is in the interior of $\mathcal{D}_n$ (cf. Point 1 of Theorem 5).

Example 9: Let $w_0 = 0.10088$, $w_1 = -0.00439$, $w_2 = -0.00702$ and $w_3 = -0.00294$. We choose $\eta(z) = (z-0.5)(z+1) \Leftrightarrow \Psi = \eta_*\eta = z^{-2} - 0.5z^{-1} + 3 - 0.5z - z^2$. The algorithm returns $d_{min} = \mathrm{col}(3.162283, 1.73787, -1.26443, 0.158116)$. Thus, $b(z) = 3.16323 - 1.423018z + 0.15807z^2$ and $a(z)$ can be computed to be $a(z) = 0.10087 - 0.00439z - 0.00702z^2 - 0.00293z^3$. It may be inspected b has one root very close to -1 (indeed the true b, with which this example is constructed, has a root on $\mathbb{T}$). This example serves to illustrate the case where b has roots on $\mathbb{T}$ which cancels the same corresponding roots of η. An example of this type has also been given in [14], [17].

Example 10: Let $w_0 = 1.10088$, $w_1 = 0.86164$, $w_2 = 0.492982$, $w_3 = -0.00294$ and $w_4 = -0.50097$. We choose $\eta(z) = (z-0.5)(z+e^{i\frac{\pi}{6}})(z-e^{i\frac{\pi}{6}}) \Leftrightarrow \Psi = \eta_*\eta = z^{-3} + 5.96410z^{-2} - 14.66025z^{-1} + 19.42820 + -14.66025z + 5.96410z^2 + z^3$. The algorithm returns $d_{min} = \mathrm{col}(3.16228, -6.90025, 5.78514, -1.69689, 0.15811)$. Thus, $b(z) = 3.16228 - 6.90025z + 5.78514z^2 - 1.69689z^3 + 0.15811z^4$ and $a(z) = 1.10088 - 0.86164z + 0.49298z^2 - 0.00294z^3 - 0.50097z^4$. It may be inspected b has roots very close to $e^{i\frac{\pi}{6}}$ and $e^{-i\frac{\pi}{6}}$, but a does not. The example serves to illustrate the case where $f = \frac{a}{b}$ is an unbounded solution with poles on $\mathbb{T}$ (cf. Point 2 of Theorem 5).

The examples show that indeed all solutions to Problem 1 for η corresponding to η with some spectral zeros on the unit circle can be computed with the same homotopy continuation method that was previously developed for spectral zeros exclusively inside the unit circle.

V. Conclusions

The contribution of this paper is development of theoretical results which show that a certain homotopy continuation method, originally due to Enqvist, for computing solutions of degree constrained rational interpolation problems with strictly positive parametrizing functions remains valid even when the parametrizing function Ψ is non-strictly positive definite on the unit circle. In particular, we show that, unlike the original dual functional $\mathbb{J}_\Psi$ introduced by Byrnes et. al., the modified functional $\mathcal{J}_\Psi$ of Enqvist, has the remarkable property that it continues to have a stationary minimizer when the parametrizing function is non-strictly positive definite. Several numerical examples have been provided to illustrate the validity of the theoretical results.

For the special case of the rational covariance extension problem, this method is particularly attractive since the Hessian of $\mathcal{J}_\Psi$ has a Toeplitz-plus-Hankel structure which can be inverted with fast algorithms, some of which can be implemented in parallel. Moreover, since the method has been adapted for finding strictly positive solutions of some moment problems, the development of this paper may allow for finding non-strictly positive solutions of the problems. This will be treated in forthcoming work.

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