

# Computation of Invariants of Lie Algebras by Means of Moving Frames

**Vyacheslav Boyko**

Institute of Mathematics of NAS of Ukraine, Kyiv, Ukraine

Talk is based on the joint works with *Jiri Patera* and *Roman Popovych*

A purely algebraic algorithm for computation of invariants (generalized Casimir operators) of Lie algebras is presented. It uses the Cartan's method of moving frames and the knowledge of the group of inner automorphisms of each Lie algebra. The algorithm is applied, in particular, to computation of invariants of low-dimensional Lie algebras and invariants of solvable Lie algebras of general dimension  $n < \infty$  restricted only by a required structure of the nilradical.

- BPP06 Boyko V., Patera J. and Popovych R., Computation of invariants of Lie algebras by means of moving frames, *J. Phys. A: Math. Gen.* **39** (2006), 5749–5762, math-ph/0602046.
- BPP07a Boyko V., Patera J. and Popovych R., Invariants of Lie algebras with fixed structure of nilradicals, *J. Phys. A: Math. Teor.* **40** (2007), 113–130, math-ph/0606045.
- BPP07b Boyko V., Patera J. and Popovych R., Invariants of triangular Lie algebras, *J. Phys. A: Math. Teor.* **40** (2007), 7557–7572, arXiv:0704.0937.
- BPP07c Boyko V., Patera J. and Popovych R., Invariants of triangular Lie algebras with one nilindependent diagonal element, *J. Phys. A: Math. Teor.* **40** (2007), 9783–9792, arXiv:0705.2394.
- BPP08 Boyko V., Patera J. and Popovych R., Invariants of solvable Lie algebras with triangular nilradicals and diagonal nilindependent elements, *Linear Algebra Appl.* **428** (2008), 834–854, arXiv:0706.2465.

- Popovych R., Boyko V., Nesterenko M. and Lutfullin M. Realizations of real low-dimensional Lie algebras, *J. Phys. A: Math. Gen.*, 2003, V.36, 7337–7360; see math-ph/0301029v5 for the revised and extended version.
- Nesterenko M. and Popovych R., Contractions of low-dimensional Lie algebras, *J. Math. Phys.* **47** (2006), 123515, 45 pages.

The invariants of Lie algebras are one of their defining characteristics. They have numerous applications in different fields of mathematics and physics, in which Lie algebras arise (representation theory, integrability of Hamiltonian differential equations, quantum numbers etc). In particular, the polynomial invariants of a Lie algebra exhaust its set of Casimir operators, i.e., the center of its universal enveloping algebra. That is why non-polynomial invariants are also called generalized Casimir operators, and the usual Casimir operators are seen as ‘trivial’ generalized Casimir operators. Since the structure of invariants strongly depends on the structure of the algebra and the classification of all (finite-dimensional) Lie algebras is an inherently difficult problem (actually unsolvable), it seems to be impossible to elaborate a complete theory for generalized Casimir operators in the general case. Moreover, if the classification of a class of Lie algebras is known, then the invariants of such algebras can be described exhaustively. These problems have already been solved for the semi-simple and low-dimensional Lie algebras, and also for the physically relevant Lie algebras of fixed dimensions.

The standard method of construction of generalized Casimir operators consists of integration of overdetermined systems of first-order linear partial differential equations. It turns out to be rather cumbersome calculations, once the dimension of Lie algebra is not one of the lowest few. Alternative methods use matrix representations of Lie algebras. They are not much easier and are valid for a limited class of representations.

The presented algebraic method of computation of invariants of Lie algebras is simpler and generally valid. It extends to our problem the exploitation of the Cartan’s method of moving frames in Fels–Olver version [*Acta Appl. Math.*, 1998, 1999].

We recalculated invariant bases and, in a number of cases, enhance their representation for the following Lie algebras:

- the complex and real Lie algebras up to dimension 6 [BPP06];
- the complex and real Lie algebras with Abelian nilradicals of codimension 1 [BPP07a];
- the complex indecomposable solvable Lie algebras with the nilradicals isomorphic to  $\mathfrak{J}_0^n$ ,  $n = 3, 4, \dots$  (the nonzero commutation relations between the basis elements  $e_1, \dots, e_n$  of  $\mathfrak{J}_0^n$  are exhausted by  $[e_k, e_n] = e_{k-1}$ ,  $k = 1, \dots, n - 1$ ) [BPP07a];
- the nilpotent Lie algebras  $\mathfrak{t}_0(n)$  of  $n \times n$  strictly upper triangular matrices [BPP07a, BPP07b];
- the solvable Lie algebras  $\mathfrak{t}(n)$  of  $n \times n$  upper triangular matrices and the solvable Lie algebras  $\mathfrak{st}(n)$  of  $n \times n$  special upper triangular matrices [BPP07b, BPP07c, BPP08];
- the solvable Lie algebras with nilradicals isomorphic to  $\mathfrak{t}_0(n)$  and diagonal nilindependent elements, [BPP07b, BPP07c, BPP08].

Note that earlier there exist only conjectures on invariants of two latter families of Lie algebras. Moreover, for the last family the conjecture was formulated only for partial case of a single nilindependent element.

## Preliminaries

Consider a Lie algebra  $\mathfrak{g}$  of dimension  $\dim \mathfrak{g} = n < \infty$  over the complex or real field and the corresponding connected Lie group  $G$ . Let  $\mathfrak{g}^*$  be the dual space of the vector space  $\mathfrak{g}$ . The map  $\text{Ad}^*: G \rightarrow GL(\mathfrak{g}^*)$ , defined for any  $g \in G$  by the relation

$$\langle \text{Ad}_g^* x, u \rangle = \langle x, \text{Ad}_{g^{-1}} u \rangle \quad \text{for all } x \in \mathfrak{g}^* \text{ and } u \in \mathfrak{g}$$

is called the *coadjoint representation* of the Lie group  $G$ . Here  $\text{Ad}: G \rightarrow GL(\mathfrak{g})$  is the usual adjoint representation of  $G$  in  $\mathfrak{g}$ , and the image  $\text{Ad}_G$  of  $G$  under  $\text{Ad}$  is the inner automorphism group  $\text{Int}(\mathfrak{g})$  of the Lie algebra  $\mathfrak{g}$ . The image of  $G$  under  $\text{Ad}^*$  is a subgroup of  $GL(\mathfrak{g}^*)$  and is denoted by  $\text{Ad}_G^*$ .

A function  $F \in C^\infty(\mathfrak{g}^*)$  is called an *invariant* of  $\text{Ad}_G^*$  if

$$F(\text{Ad}_g^* x) = F(x) \quad \text{for all } g \in G \text{ and } x \in \mathfrak{g}^*.$$

The set of invariants of  $\text{Ad}_G^*$  is denoted by  $\text{Inv}(\text{Ad}_G^*)$ . The maximal number  $N_{\mathfrak{g}}$  of functionally independent invariants in  $\text{Inv}(\text{Ad}_G^*)$  coincides with the codimension of the regular orbits of  $\text{Ad}_G^*$ , i.e., it is given by the difference

$$N_{\mathfrak{g}} = \dim \mathfrak{g} - \text{rank } \text{Ad}_G^*.$$

Here  $\text{rank } \text{Ad}_G^*$  denotes the dimension of the regular orbits of  $\text{Ad}_G^*$  and will be called the *rank of the coadjoint representation* of  $G$  (and of  $\mathfrak{g}$ ). It is a basis independent characteristic of the algebra  $\mathfrak{g}$ , the same as  $\dim \mathfrak{g}$  and  $N_{\mathfrak{g}}$ .

To calculate the invariants explicitly, one should fix a basis  $\mathcal{E} = \{e_1, \dots, e_n\}$  of the algebra  $\mathfrak{g}$ . It leads to fixing the dual basis  $\mathcal{E}^* = \{e_1^*, \dots, e_n^*\}$  in the dual space  $\mathfrak{g}^*$  and to the identification of  $\text{Int}(\mathfrak{g})$  and  $\text{Ad}_G^*$  with the associated matrix groups. The basis elements  $e_1, \dots, e_n$  satisfy the commutation relations  $[e_i, e_j] = \sum_{k=1}^n c_{ij}^k e_k$ ,  $i, j = 1, \dots, n$ , where  $c_{ij}^k$  are components of the tensor of structure constants of  $\mathfrak{g}$  in the basis  $\mathcal{E}$ .

Let  $x \rightarrow \check{x} = (x_1, \dots, x_n)$  be the coordinates in  $\mathfrak{g}^*$  associated with  $\mathcal{E}^*$ . Given any invariant  $F(x_1, \dots, x_n)$  of  $\text{Ad}_G^*$ , one finds the corresponding invariant of the Lie algebra  $\mathfrak{g}$  by symmetrization,  $\text{Sym} F(e_1, \dots, e_n)$ , of  $F$ . It is often called a *generalized Casimir operator* of  $\mathfrak{g}$ . If  $F$  is a polynomial,  $\text{Sym} F(e_1, \dots, e_n)$  is a usual *Casimir operator*, i.e., an element of the center of the universal enveloping algebra of  $\mathfrak{g}$ . More precisely, the symmetrization operator  $\text{Sym}$  acts only on the monomials of the forms  $e_{i_1} \cdots e_{i_r}$ , where there are non-commuting elements among  $e_{i_1}, \dots, e_{i_r}$ , and is defined by the formula

$$\text{Sym}(e_{i_1} \cdots e_{i_r}) = \frac{1}{r!} \sum_{\sigma \in S_r} e_{i_{\sigma_1}} \cdots e_{i_{\sigma_r}},$$

where  $i_1, \dots, i_r$  take values from 1 to  $n$ ,  $r \geq 2$ . The symbol  $S_r$  denotes the permutation group consisting of  $r$  elements. The set of invariants of  $\mathfrak{g}$  is denoted by  $\text{Inv}(\mathfrak{g})$ .

A set of functionally independent invariants  $F^l(x_1, \dots, x_n)$ ,  $l = 1, \dots, N_{\mathfrak{g}}$ , forms a *functional basis (fundamental invariant)* of  $\text{Inv}(\text{Ad}_G^*)$ , i.e., any invariant  $F(x_1, \dots, x_n)$  can be uniquely represented as a function of  $F^l(x_1, \dots, x_n)$ ,  $l = 1, \dots, N_{\mathfrak{g}}$ . Accordingly the set of  $\text{Sym } F^l(e_1, \dots, e_n)$ ,  $l = 1, \dots, N_{\mathfrak{g}}$ , is called a basis of  $\text{Inv}(\mathfrak{g})$ .

Our task here is to determine the basis of the functionally independent invariants for  $\text{Ad}_G^*$ , and then to transform these invariants into the invariants of the algebra  $\mathfrak{g}$ . Any other invariant of  $\mathfrak{g}$  is a function of the independent ones.



## Infinitesimal approach

Any invariant  $F(x_1, \dots, x_n)$  of  $\text{Ad}_G^*$  is a solution of the linear system of first-order partial differential equations, see e.g. [Beltrametti-Blasi1966, Abellanas-MartinezAlonso1975, Patera-Sharp-Winternitz-Zassenhaus1976]

$$X_i F = 0, \quad \text{i.e.} \quad c_{ij}^k x_k F_{x_j} = 0, \quad (1)$$

where  $X_i = c_{ij}^k x_k \partial_{x_j}$  is the infinitesimal generator of the one-parameter group  $\{\text{Ad}_G^*(\exp \varepsilon e_i)\}$  corresponding to  $e_i$ . The mapping  $e_i \rightarrow X_i$  gives a representation of the Lie algebra  $A$ .

## The algorithm

Let  $\mathcal{G} = \text{Ad}_G^* \times \mathfrak{g}^*$  denote the trivial left principal  $\text{Ad}_G^*$ -bundle over  $\mathfrak{g}^*$ . The right regularization  $\widehat{R}$  of the coadjoint action of  $G$  on  $\mathfrak{g}^*$  is the diagonal action of  $\text{Ad}_G^*$  on  $\mathcal{G} = \text{Ad}_G^* \times \mathfrak{g}^*$ . It is provided by the map

$$\widehat{R}_g(\text{Ad}_h^*, x) = (\text{Ad}_h^* \cdot \text{Ad}_{g^{-1}}^*, \text{Ad}_g^* x), \quad g, h \in G, \quad x \in \mathfrak{g}^*,$$

where the action on the bundle  $\mathcal{G} = \text{Ad}_G^* \times \mathfrak{g}^*$  is regular and free. We call  $\widehat{R}_g$  the *lifted coadjoint action* of  $G$ . It projects back to the coadjoint action on  $\mathfrak{g}^*$  via the  $\text{Ad}_G^*$ -equivariant projection  $\pi_{\mathfrak{g}^*}: \mathcal{G} \rightarrow \mathfrak{g}^*$ . Any *lifted invariant* of  $\text{Ad}_G^*$  is a (locally defined) smooth function from  $\mathcal{G}$  to a manifold, which is invariant with respect to the lifted coadjoint action of  $G$ . The function  $\mathcal{I}: \mathcal{G} \rightarrow \mathfrak{g}^*$  given by  $\mathcal{I} = \mathcal{I}(\text{Ad}_g^*, x) = \text{Ad}_g^* x$  is the *fundamental lifted invariant* of  $\text{Ad}_G^*$ , i.e.,  $\mathcal{I}$  is a lifted invariant, and any lifted invariant can be locally written as a function of  $\mathcal{I}$ . Using an arbitrary function  $F(x)$  on  $\mathfrak{g}^*$ , we can produce the lifted invariant  $F \circ \mathcal{I}$  of  $\text{Ad}_G^*$  by replacing  $x$  with  $\mathcal{I} = \text{Ad}_g^* x$  in the expression for  $F$ . Ordinary invariants are particular cases of lifted invariants, where one identifies any invariant formed as its composition with the standard projection  $\pi_{\mathfrak{g}^*}$ . Therefore, ordinary invariants are particular functional combinations of lifted ones that happen to be independent of the group parameters of  $\text{Ad}_G^*$ .

The essence of the normalization procedure by Fels and Olver can be presented in the form of the following statement.

**Proposition 1.** Let  $\mathcal{I} = (\mathcal{I}_1, \dots, \mathcal{I}_n)$  be a fundamental lifted invariant, for the lifted invariants  $\mathcal{I}_{j_1}, \dots, \mathcal{I}_{j_\rho}$  and some constants  $c_1, \dots, c_\rho$  the system  $\mathcal{I}_{j_1} = c_1, \dots, \mathcal{I}_{j_\rho} = c_\rho$  be solvable with respect to the parameters  $\theta_{k_1}, \dots, \theta_{k_\rho}$  and substitution of the found values of  $\theta_{k_1}, \dots, \theta_{k_\rho}$  into the other lifted invariants result in  $m = n - \rho$  expressions  $\hat{\mathcal{I}}_l, l = 1, \dots, m$ , depending only on  $x$ 's. Then  $\rho = \text{rank Ad}_G^*$ ,  $m = N_{\mathfrak{g}}$  and  $\hat{\mathcal{I}}_1, \dots, \hat{\mathcal{I}}_m$  form a basis of  $\text{Inv}(\text{Ad}_G^*)$ .

The *algebraic algorithm* for finding invariants of the Lie algebra  $\mathfrak{g}$  is briefly formulated in the following four steps.

1. *Construction of the generic matrix  $B(\theta)$  of  $\text{Ad}_G^*$ .*  $B(\theta)$  is the matrix of an inner automorphism of the Lie algebra  $\mathfrak{g}$  in the given basis  $e_1, \dots, e_n$ ,  $\theta = (\theta_1, \dots, \theta_r)$  is a complete tuple of group parameters (coordinates) of  $\text{Int}(\mathfrak{g})$ , and  $r = \dim \text{Ad}_G^* = \dim \text{Int}(\mathfrak{g}) = n - \dim Z(\mathfrak{g})$ , where  $Z(\mathfrak{g})$  is the center of  $\mathfrak{g}$ .

2. *Representation of the fundamental lifted invariant.* The explicit form of the fundamental lifted invariant  $\mathcal{I} = (\mathcal{I}_1, \dots, \mathcal{I}_n)$  of  $\text{Ad}_G^*$  in the chosen coordinates  $(\theta, \check{x})$  in  $\text{Ad}_G^* \times \mathfrak{g}^*$  is  $\mathcal{I} = \check{x} \cdot B(\theta)$ , i.e.,  $(\mathcal{I}_1, \dots, \mathcal{I}_n) = (x_1, \dots, x_n) \cdot B(\theta_1, \dots, \theta_r)$ .

3. *Elimination of parameters by normalization.* We choose the maximum possible number  $\rho$  of lifted invariants  $\mathcal{I}_{j_1}, \dots, \mathcal{I}_{j_\rho}$ , constants  $c_1, \dots, c_\rho$  and group parameters  $\theta_{k_1}, \dots, \theta_{k_\rho}$  such that the equations  $\mathcal{I}_{j_1} = c_1, \dots, \mathcal{I}_{j_\rho} = c_\rho$  are solvable with respect to  $\theta_{k_1}, \dots, \theta_{k_\rho}$ . After substituting the found values of  $\theta_{k_1}, \dots, \theta_{k_\rho}$  into the other lifted invariants, we obtain  $N_{\mathfrak{g}} = n - \rho$  expressions  $F^l(x_1, \dots, x_n)$  without  $\theta$ 's.

4. *Symmetrization.* The functions  $F^l(x_1, \dots, x_n)$  necessarily form a basis of  $\text{Inv}(\text{Ad}_G^*)$ . They are symmetrized to  $\text{Sym } F^l(e_1, \dots, e_n)$ . It is the desired basis of  $\text{Inv}(\mathfrak{g})$ .

Our experience on the calculation of invariants of a wide range of Lie algebras shows that the version of the algebraic method, which is based on Proposition 1, is most effective. In particular, it provides finding the cardinality of the invariant basis in the process of construction of the invariants. Indeed, the algorithm can involve different kinds of coordinate in the inner automorphism groups (the first canonical, the second canonical or special one) and different techniques of elimination of parameters (empiric techniques, with additional combining of lifted invariants, using a floating system of normalization equations etc)

Let us underline that the search of invariants of Lie algebra  $\mathfrak{g}$ , which has been done by solving a linear system of first-order partial differential equations, is replaced here by the construction of the matrix  $B(\theta)$  of inner automorphisms and by excluding the parameters  $\theta$  from the fundamental lifted invariant  $\mathcal{I} = \check{x} \cdot B(\theta)$  in some way.

## Illustrative example(s)

The six-dimensional solvable Lie algebra  $\mathfrak{g}_{6.38}^a$  with five-dimensional nilradical  $\mathfrak{g}_{3.1} \oplus 2\mathfrak{g}_1$  has the following non-zero commutation relations

$$\begin{aligned} [e_4, e_5] &= e_1, & [e_1, e_6] &= 2ae_1, & [e_2, e_6] &= ae_2 - e_3, & [e_3, e_6] &= e_2 + ae_3, \\ [a_4, e_6] &= e_2 + ae_4 - e_5, & [e_5, e_6] &= e_3 + e_4 + ae_5, & a &\in \mathbb{R}. \end{aligned}$$

Here we follow the numeration of low-dimensional Lie algebras by Mubarakzhanov. We only have modified the basis to  $K$ -canonical form, i.e. now  $\langle e_1, \dots, e_i \rangle$  is an ideal of  $\langle e_1, \dots, e_i, e_{i+1} \rangle$  for any  $i = 1, 2, 3, 4, 5$ .

The matrices of the adjoint representation  $\hat{\text{ad}}_{e_i}$  of the basis elements  $e_1, e_2, e_3, e_4, e_5$  and  $-e_6$  correspondingly have the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 2a \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 2a & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 1 & 1 & 0 & 0 \\ 0 & -1 & a & 0 & 1 & 0 \\ 0 & 0 & 0 & a & 1 & 0 \\ 0 & 0 & 0 & -1 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The inner automorphisms of  $\mathfrak{g}_{6.38}^a$  are then described by the block triangular matrix

$$B(\theta) = \prod_{i=1}^5 \exp(\theta_i \hat{\text{ad}}_{e_i}) \cdot \exp(-\theta_6 \hat{\text{ad}}_{e_6})$$

$$= \begin{pmatrix} \varepsilon^2 & 0 & 0 & -\theta_5 \varepsilon \varkappa - \theta_4 \varepsilon \sigma & -\varepsilon \theta_5 \sigma + \varepsilon \theta_4 \varkappa & -\frac{1}{2} \theta_5^2 + a \theta_4 \theta_5 - \frac{1}{2} \theta_4^2 + 2a \theta_1 \\ 0 & \varepsilon \varkappa & \varepsilon \sigma & \theta_6 \varepsilon \varkappa & \theta_6 \varepsilon \sigma & \theta_4 + \theta_3 + a \theta_2 \\ 0 & -\varepsilon \sigma & \varepsilon \varkappa & -\theta_6 \varepsilon \sigma & \theta_6 \varepsilon \varkappa & \theta_5 + a \theta_3 - \theta_2 \\ 0 & 0 & 0 & \varepsilon \varkappa & \varepsilon \sigma & \theta_5 + a \theta_4 \\ 0 & 0 & 0 & -\varepsilon \sigma & \varepsilon \varkappa & a \theta_5 - \theta_4 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where  $\varepsilon = e^{a\theta_6}$ ,  $\varkappa = \cos \theta_6$ ,  $\sigma = \sin \theta_6$ .

Therefore, a functional basis of lifted invariants is formed by

$$\mathcal{I}_1 = \varepsilon^2 x_1,$$

$$\mathcal{I}_2 = \varepsilon(\varkappa x_2 - \sigma x_3),$$

$$\mathcal{I}_3 = \varepsilon(\sigma x_2 + \varkappa x_3),$$

$$\mathcal{I}_4 = \varepsilon((-\theta_5 \varkappa - \theta_4 \sigma)x_1 + \theta_6 \varkappa x_2 - \theta_6 \sigma x_3 + \varkappa x_4 - \sigma x_5),$$

$$\mathcal{I}_5 = \varepsilon((-\theta_5 \sigma + \theta_4 \varkappa)x_1 + \theta_6 \sigma x_2 + \theta_6 \varkappa x_3 + \sigma x_4 + \varkappa x_5),$$

$$\begin{aligned} \mathcal{I}_6 = & \left( -\frac{1}{2}\theta_5^2 + a\theta_4\theta_5 - \frac{1}{2}\theta_4^2 + 2a\theta_1 \right) x_1 + (\theta_4 + \theta_3 + a\theta_2)x_2 + (\theta_5 + a\theta_3 - \theta_2)x_3 \\ & + (\theta_5 + a\theta_4)x_4 + (a\theta_5 - \theta_4)x_5 + x_6. \end{aligned}$$

The algebra  $\mathfrak{g}_{6.38}^a$  has two independent invariants. They can be easily found from first three lifted invariants by the normalization procedure. Further the cases  $a = 0$  and  $a \neq 0$  should be considered separately since there exists difference between them in the normalization procedure.

It is obvious in case  $a = 0$  that  $e_1$  generating the center  $Z(\mathfrak{g}_{6.38}^0)$  is one of the invariants. The second invariant is found via combining the lifted invariants  $\mathcal{I}_2$  and  $\mathcal{I}_3$ :  $\mathcal{I}_2^2 + \mathcal{I}_3^2 = x_2^2 + x_3^2$ . Since the symmetrization procedure is trivial for this algebra we obtain the following set of polynomial invariants

$$e_1, \quad e_2^2 + e_3^2.$$



In case  $a \neq 0$  we solve the equation  $\mathcal{I}_1 = 1$  with respect to  $e^{2a\theta_6}$  and substitute the obtained expression  $e^{2a\theta_6} = 1/x_1$  into the combinations  $\mathcal{I}_2^2 + \mathcal{I}_3^2$  and  $\exp(-2a \arctan \mathcal{I}_3/\mathcal{I}_2)$ . In view of trivial symmetrization we obtain the final basis of generalized Casimir invariants

$$\frac{e_2^2 + e_3^2}{e_1}, \quad e_1 \exp\left(-2a \arctan \frac{e_3}{e_2}\right).$$

It is equivalent to the one constructed by Campoamor-Stursberg (2005), but it contains no complex numbers and is written in a more compact form.

Below effectiveness of the algorithm is demonstrated by its application to computation of invariants of solvable Lie algebras of general dimension  $n < \infty$  restricted only by a required structure of the nilradical.

Further we use the following notations:

$\text{diag}(\alpha_1, \dots, \alpha_k)$  is the  $k \times k$  diagonal matrix with the elements  $\alpha_1, \dots, \alpha_k$  on the diagonal;

$E^k = \text{diag}(1, \dots, 1)$  is the  $k \times k$  unity matrix;

$E_{ij}^k$  (for the fixed values  $i$  and  $j$ ) is the  $k \times k$  matrix with the unit on the cross of the  $i$ -th row and the  $j$ -th column and the zero otherwise;

$J_\lambda^k$  is the Jordan block of dimension  $k$  and the eigenvalue  $\lambda$ :

$$[J_\lambda^k]_{ij} = \begin{cases} \lambda, & \text{if } j = i, \\ 1, & \text{if } j - i = 1, \\ 0, & \text{otherwise.} \end{cases} \quad i, j = 1, \dots, k,$$

i.e.

$$J_\lambda^k = \begin{pmatrix} \lambda & 1 & 0 & 0 & \dots & 0 \\ 0 & \lambda & 1 & 0 & \dots & 0 \\ 0 & 0 & \lambda & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & \lambda \end{pmatrix}, \quad \exp(\theta J_0^k) = \begin{pmatrix} 1 & \theta & \frac{1}{2!}\theta^2 & \frac{1}{3!}\theta^3 & \dots & \frac{1}{(k-1)!}\theta^{k-1} \\ 0 & 1 & \theta & \frac{1}{2!}\theta^2 & \dots & \frac{1}{(k-2)!}\theta^{k-2} \\ 0 & 0 & 1 & \theta & \dots & \frac{1}{(k-3)!}\theta^{k-3} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \theta \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

(let us note that  $J_\lambda^k = \lambda E^k + J_0^k$  and therefore  $\exp(\theta J_\lambda^k) = e^{\lambda\theta} \exp(\theta J_0^k)$ );

$R_{\mu\nu}^r$  is the real Jordan block of dimension  $r = 2k$ ,  $k \in \mathbb{N}$ , which corresponds to the pair of two complex Jordan blocks  $J_\lambda^k$  and  $J_{\lambda^*}^k$  with the complex conjugate eigenvalues  $\lambda$  and  $\lambda^*$ , where  $\mu = \operatorname{Re} \lambda$ ,  $\nu = \operatorname{Im} \lambda \neq 0$ :

$$R_{\mu\nu}^2 = \begin{pmatrix} \mu & \nu \\ -\nu & \mu \end{pmatrix}, \quad R_{\mu\nu}^{2k} = \left( \begin{array}{cccccc} R_{\mu\nu}^2 & E^2 & 0 & 0 & \cdots & 0 \\ 0 & R_{\mu\nu}^2 & E^2 & 0 & \cdots & 0 \\ 0 & 0 & R_{\mu\nu}^2 & E^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & E^2 \\ 0 & 0 & 0 & 0 & \cdots & R_{\mu\nu}^2 \end{array} \right) \left. \vphantom{R_{\mu\nu}^{2k}} \right\} k \text{ blocks};$$

$A_1 \oplus A_2$  is the direct sum  $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$  of the square matrices  $A_1$  and  $A_2$ ;

$A_1 \overset{C}{+} A_2$  is the block triangular matrix  $\begin{pmatrix} A_1 & C \\ 0 & A_2 \end{pmatrix}$ , where  $A_1 \in M_{k,k}$ ,  $A_2 \in M_{l,l}$ ,  $C \in M_{k,l}$ .

Above 0 denotes the zero matrices of different dimensions.

## Solvable algebras with Abelian ideals of codimension 1

Consider a Lie algebra  $\mathfrak{g}$  of dimension  $n$  with the Abelian ideal  $I$  of dimension  $n - 1$ . Let us suppose that the ideal  $I$  is spanned on the basis elements  $e_1, e_2, \dots, e_{n-1}$ . Then the algebra  $\mathfrak{g}$  is completely determined by the  $(n - 1) \times (n - 1)$  matrix  $M = (m_{kl})$  of restriction of the adjoint action  $\text{ad}_{e_n}$  on the ideal  $I$ . The (possibly) non-zero commutation relations of  $\mathfrak{g}$  have the form

$$[e_k, e_n] = \sum_{l=1}^{n-1} m_{lk} e_l, \quad k = 1, \dots, n - 1.$$

The matrix  $M$  is reduced to the Jordan canonical form by change of the basis in  $I$ :

$$M = J_{\lambda_1}^{r_1} \oplus \dots \oplus J_{\lambda_s}^{r_s},$$

where  $r_1 + \dots + r_s = n - 1$ ,  $r_i \in \mathbb{N}$ ,  $\lambda_i \in \mathbb{C}$ ,  $i = 1, \dots, s$ . In the real case the direct sum of two complex blocks  $J_{\lambda_i}^{r_i}$  and  $J_{\lambda_j}^{r_j}$ , where  $r_i = r_j$  and  $\lambda_i$  is conjugate of  $\lambda_j$ , is assumed as replaced by the corresponding real Jordan block  $R_{\mu\nu}^{2r_i}$  with  $\mu = \text{Re } \lambda_i$  and  $\nu = \text{Im } \lambda_i \neq 0$ . The Jordan canonical form is unique up to permutation of the Jordan blocks.

The above algebra will be denoted as  $\mathfrak{J}_{\lambda_1 \dots \lambda_s}^{r_1 \dots r_s}$ .

The Lie algebra  $\mathfrak{J}_{\lambda_1 \dots \lambda_s}^{r_1 \dots r_s}$  is decomposable iff there exists a value of  $i$  such that  $(\lambda_i, r_i) = (0, 1)$ . (Then  $e_i$  is an invariant of  $\mathfrak{J}_{\lambda_1 \dots \lambda_s}^{r_1 \dots r_s}$ .) Hence the contrary condition is supposed to be satisfied below. It should be also noted this algebra is nilpotent iff  $\lambda_1 = \dots = \lambda_s = 0$ .

## Simplest cases

Consider the simplest case for  $M$  to be a single Jordan block with the eigenvalue  $\lambda$ , i.e.  $\mathfrak{g} = \mathfrak{J}_\lambda^{n-1}$ ,  $n = 2, 4, \dots$ . The value of  $\lambda$  can be normalized to 1 in case  $\lambda \neq 0$  but it is convenient for the further consideration to avoid normalization of  $\lambda$  some time.

The non-zero commutation relations of  $\mathfrak{J}_\lambda^{n-1}$  at most are

$$[e_1, e_n] = \lambda e_1, \quad [e_k, e_n] = \lambda e_k + e_{k-1}, \quad k = 2, \dots, n-1, \quad \lambda \in \mathbb{C}.$$

(The first one is zero if  $\lambda = 0$ .) Therefore, its inner automorphisms are described by the triangular matrix

$$B(\theta) = \exp(\theta_n J_\lambda^{n-1}) \stackrel{C}{+} E^1, \quad C = (\theta_2 + \lambda\theta_1, \theta_3 + \lambda\theta_2, \dots, \theta_{n-1} + \lambda\theta_{n-2}, \lambda\theta_{n-1})^T,$$

i.e. a functional basis of lifted invariants are formed by

$$\widehat{\mathcal{I}}_k = e^{\lambda\theta_n} \mathcal{I}_k, \quad k = 1, \dots, n-1, \quad \widehat{\mathcal{I}}_n = \mathcal{I}_n + \lambda \sum_{j=1}^{n-1} \theta_j x_j,$$

where

$$\mathcal{I}_k = \sum_{j=1}^k \frac{\theta_n^{k-j}}{(k-j)!} x_j, \quad k = 1, \dots, n-1, \quad \mathcal{I}_n = \sum_{j=1}^{n-2} \theta_{j+1} x_j + x_n. \quad (2)$$

The nilpotent ( $\lambda = 0$ ) and solvable ( $\lambda \neq 0$ ) cases of  $\mathfrak{J}_\lambda^{n-1}$  should be considered further separately since there exists difference in the normalization procedure. The dimension  $n = 2$  is singular in the both cases.  $\mathfrak{J}_0^1$  is two-dimensional Abelian Lie algebra and therefore has two independent invariants, namely  $e_1$  and  $e_2$ .  $\mathfrak{J}_1^1$  is two-dimensional non-Abelian Lie algebra and therefore has no invariants. We assume below that  $n \geq 3$ .

Let us note that the adjoint representation of  $\mathfrak{J}_0^{n-1}$  is unfaithful since the center  $Z(\mathfrak{J}_0^{n-1}) = \langle e_1 \rangle \neq \{0\}$ . Therefore, there are  $n - 1$  parameters in the expression of  $B(\theta)$  excluding  $\theta_1$ , and  $\widehat{\mathcal{I}}$  coincides with  $\mathcal{I}$ . It is obvious that the element  $e_1$  generating  $Z(\mathfrak{J}_0^{n-1})$  is one of the invariants, which corresponds to  $\mathcal{I}_1 = x_1$ . Another  $(n - 3)$  invariants are found by the normalization procedure applied to the lifted invariants  $\mathcal{I}_2, \dots, \mathcal{I}_{n-1}$ . Namely, we solve the equation  $\mathcal{I}_2 = 0$  with respect to  $\theta_n$  and then substitute the obtained expression  $\theta_n = -x_2/x_1$  to the other  $\mathcal{I}$ 's. To construct polynomial invariants finally, we multiply the derived invariants by powers of the invariant  $x_1$ . Since the symmetrization procedure is trivial for this algebra, we result to the following complete set of independent generalized Casimir operators which are classical (i.e. polynomial) Casimir operators:

$$\xi_1 = e_1, \quad \xi_k = \sum_{j=1}^k \frac{(-1)^{k-j}}{(k-j)!} e_1^{j-2} e_2^{k-j} e_j, \quad k = 3, \dots, n-1. \quad (3)$$

This set completely coincides with the one determined in Lemma 1 of [Ndogmo, Wintnitz, 1994] and Theorem 4 of [Snobl, Wintnitz, 2005].

In case  $\lambda \neq 0$  the  $n - 2$  invariants of  $\mathfrak{J}_\lambda^{n-1}$  are found by the normalization procedure applied to the lifted invariants  $\widehat{\mathcal{I}}_1, \dots, \widehat{\mathcal{I}}_{n-1}$ . We solve  $\widehat{\mathcal{I}}_2 = 0$  with respect to the parameter  $\theta_n$ . Substitution of the obtained expression  $\theta_n = -x_2/x_1$  to  $\widehat{\mathcal{I}}_1$  and  $\widehat{\mathcal{I}}_k/\widehat{\mathcal{I}}_1$ ,  $k = 3, \dots, n - 1$ , results to a basis of  $\text{Inv}(\mathfrak{J}_\lambda^{n-1})$ :

$$\zeta_1 = e_1 \exp\left(-\lambda \frac{e_2}{e_1}\right), \quad \zeta_k = \frac{\xi_k}{\xi_1^{k-1}}, \quad k = 3, \dots, n - 1,$$

where  $\xi_k$ ,  $k = 1, 3, \dots, n - 1$ , are defined by (3).

This set of invariants completely coincides with the one determined in Lemma 2 of [Ndogmo, Wintenitz, 1994]. We only use exponential function instead logarithmic one in expression of the first invariant.

Let us emphasize that any basis of  $\text{Inv}(\mathfrak{J}_\lambda^{n-1})$  contains at least one transcendental invariant. The other basis invariants can be chosen rational.

The real version  $\mathfrak{J}_{(\mu,\nu)}^{n-1}$  of the complex algebra  $\mathfrak{J}_{\lambda\lambda^*}^r$ , where  $n = 2r + 1$ ,  $r \in \mathbb{N}$ ,  $\mu = \operatorname{Re} \lambda$ ,  $\nu = \operatorname{Im} \lambda \neq 0$ , has the non-zero commutation relations

$$\begin{aligned} [e_1, e_n] &= \mu e_1 - \nu e_2, & [e_2, e_n] &= \nu e_1 + \mu e_2, \\ [e_{2k-1}, e_n] &= \mu e_{2k-1} - \nu e_{2k} + e_{2k-3}, & [e_{2k}, e_n] &= \nu e_{2k-1} + \mu e_{2k} + e_{2k-2}, \quad k = 2, \dots, r. \end{aligned}$$

A complete tuple  $\widehat{\mathcal{I}}$  of lifted invariants has the form

$$\begin{aligned} \widehat{\mathcal{I}}_{2k-1} &= e^{\mu\theta_n} (\mathcal{I}_{2k-1} \cos \nu\theta_n - \mathcal{I}_{2k} \sin \nu\theta_n), & \widehat{\mathcal{I}}_{2k} &= e^{\mu\theta_n} (\mathcal{I}_{2k-1} \sin \nu\theta_n + \mathcal{I}_{2k} \cos \nu\theta_n), \\ \widehat{\mathcal{I}}_n &= \sum_{j=1}^r \left( \theta_{2j-1} (\mu x_{2j-1} - \nu x_{2j}) + \theta_{2j} (\nu x_{2j-1} + \mu x_{2j}) \right) + \sum_{j=1}^{r-1} \left( \theta_{2j+1} x_{2j-1} + \theta_{2j+2} x_{2j} \right) + x_n, \end{aligned}$$

where  $k = 1, \dots, r$ ,

$$\mathcal{I}_{2k-1} = \sum_{j=1}^k \frac{\theta_n^{k-j}}{(k-j)!} x_{2j-1}, \quad \mathcal{I}_{2k} = \sum_{j=1}^k \frac{\theta_n^{k-j}}{(k-j)!} x_{2j}.$$



The normalization procedure is conveniently applied to the following combinations of the lifted invariants  $\widehat{\mathcal{I}}_{2k-1}, \widehat{\mathcal{I}}_{2k}, k = 1, \dots, r$ :

$$\widehat{\mathcal{I}}_1^2 + \widehat{\mathcal{I}}_2^2 = (x_1^2 + x_2^2)e^{2\mu\theta_n}, \quad \arctan \frac{\widehat{\mathcal{I}}_2}{\widehat{\mathcal{I}}_1} = \arctan \frac{x_2}{x_1} + \nu\theta_n,$$

$$\frac{\widehat{\mathcal{I}}_1\widehat{\mathcal{I}}_3 + \widehat{\mathcal{I}}_2\widehat{\mathcal{I}}_4}{\widehat{\mathcal{I}}_1^2 + \widehat{\mathcal{I}}_2^2} = \frac{x_1x_3 + x_2x_4}{x_1^2 + x_2^2} + \theta_n, \quad \frac{\widehat{\mathcal{I}}_2\widehat{\mathcal{I}}_3 - \widehat{\mathcal{I}}_1\widehat{\mathcal{I}}_4}{\widehat{\mathcal{I}}_1^2 + \widehat{\mathcal{I}}_2^2} = \frac{x_2x_3 - x_1x_4}{x_1^2 + x_2^2},$$

$$\frac{\widehat{\mathcal{I}}_1\widehat{\mathcal{I}}_{2k-1} + \widehat{\mathcal{I}}_2\widehat{\mathcal{I}}_{2k}}{\widehat{\mathcal{I}}_1^2 + \widehat{\mathcal{I}}_2^2} = \frac{x_1\mathcal{I}_{2k-1} + x_2\mathcal{I}_{2k}}{x_1^2 + x_2^2}, \quad \frac{\widehat{\mathcal{I}}_2\widehat{\mathcal{I}}_{2k-1} - \widehat{\mathcal{I}}_1\widehat{\mathcal{I}}_{2k}}{\widehat{\mathcal{I}}_1^2 + \widehat{\mathcal{I}}_2^2} = \frac{x_2\mathcal{I}_{2k-1} - x_1\mathcal{I}_{2k}}{x_1^2 + x_2^2}, \quad k = 3, \dots, r.$$

We use the condition that the third combination (or second one if  $n = 3$ ) equals to 0 as a normalization equation on the parameter  $\theta_n$  and then exclude  $\theta_n$  from the other combinations. It gives the basis of  $\text{Inv}(\mathfrak{J}_{(\mu,\nu)}^{n-1})$

$$\zeta_1 = (e_1^2 + e_2^2) \exp\left(-2\frac{\mu}{\nu} \arctan \frac{e_2}{e_1}\right),$$

$$\zeta_3 = \nu \frac{e_1e_3 + e_2e_4}{e_1^2 + e_2^2} - \arctan \frac{e_2}{e_1}, \quad \zeta_4 = \frac{e_1e_4 - e_2e_3}{e_1^2 + e_2^2},$$

$$\zeta_{2k-1} = \frac{e_1\widehat{\zeta}_{2k-1} + e_2\widehat{\zeta}_{2k}}{e_1^2 + e_2^2}, \quad \zeta_{2k} = \frac{e_2\widehat{\zeta}_{2k-1} - e_1\widehat{\zeta}_{2k}}{e_1^2 + e_2^2}, \quad k = 3, \dots, r,$$

where

$$\hat{\zeta}_{2k-1} = \sum_{j=1}^k \left( -\frac{e_1 e_3 + e_2 e_4}{e_1^2 + e_2^2} \right)^{k-j} \frac{e_{2j-1}}{(k-j)!}, \quad \hat{\zeta}_{2k} = \sum_{j=1}^k \left( -\frac{e_1 e_3 + e_2 e_4}{e_1^2 + e_2^2} \right)^{k-j} \frac{e_{2j}}{(k-j)!}.$$

Therefore,  $\mathfrak{J}_{(\mu,\nu)}^2$  has unique independent invariant  $\zeta_1$  which is necessarily transcendental. In case  $n = 2r + 1 \geq 5$  any basis of  $\text{Inv}(\mathfrak{J}_{(\mu,\nu)}^{n-1})$  contains at least two transcendental invariants; the other  $n - 4$  basis invariants can be chosen rational. A quite optimal basis with minimal number of transcendental invariants is formed by  $\zeta_k$ ,  $k = 1, 3, \dots, n - 1$ .

- Casimir H.B.G. Über die Konstruktion einer zu den irreduzibelen Darstellungen halbeinfacher kontinuierlicher Gruppen gehörigen Differentialgleichung, *Proc. R. Acad. Amsterdam*, 1931, V.34, 844–846.
- Drinfel'd I.M. Centre of infinitesimal group ring, *Mat. Sb.*, 1950, V.26, 103–112.
- Beltrametti E.G. and Blasi A. On the number of Casimir operators associated with any Lie group, *Phys. Lett.*, 1966, V.20, 62–64.
- Pauri M. and Prosperi G.M. On the construction of the invariants operators for any finite-parameter Lie group, *Nuovo Cimento A*, 1966, V.43, 533–537.
- Perelomov A.M. and Popov V.S. Casimir operators for semisimple Lie group, *Izv. Akad. Nauk USSR Mat.*, 1968, V.32, 1368–1390.
- Abellanas L. and Martinez Alonso L. A general setting for Casimir invariants, *J. Math. Phys.*, 1975, V.16, 1580–1584.
- Patera J., Sharp R.T., Winternitz P. and Zassenhaus H. Invariants of real low dimension Lie algebras, *J. Math. Phys.*, 1976, V.17, 986–994.
- Zassenhaus H. On the invariants of a Lie group. I, in *Computers in Nonassociative Rings and Algebras (Special Session, 82nd Annual Meeting Amer. Math. Soc., San Ontario, 1976)*, Editors R.E. Beck and B. Kolman, New York, Academic Press, 1977, 139–155.

Functional bases of invariants were calculated for

- three-, four-, five-dimensional and nilpotent six-dimensional real Lie algebras  
Patera J., Sharp R.T., Winternitz P. and Zassenhaus H. *J. Math. Phys.*, 1976, V.17, 986–994.
- the six-dimensional real Lie algebras with four-dimensional nilradicals  
Ndogmo J.C. *J. Phys. A: Math. Gen.*, 2000, V.33, 2273–2287.
- the six-dimensional real Lie algebras with five-dimensional nilradicals  
Campoamor-Stursberg R. *Algebra Colloq.*, 2005, V.12, 497–518.
- subgroups of the Poincaré group  
Patera J., Sharp R.T., Winternitz P. and Zassenhaus H. *J. Math. Phys.*, 1976, V.17, 977–985.
- solvable Lie algebras with the Heisenberg nilradicals  
Rubin J.L. and Winternitz P. *J. Phys. A: Math. Gen.*, 1993, V.26, 1123–1138.
- solvable Lie algebras with Abelian nilradicals  
Ndogmo J.C. and Winternitz P. *J. Phys. A: Math. Gen.*, 1994, V.27, 2787–2800.
- solvable Lie algebras with nilradicals containing Abelian ideals of codimension 1  
Snobl L. and Winternitz P. *J. Phys. A: Math. Gen.*, 2005, V.38, 2687–2700
- solvable triangular algebras  
Tremblay S. and Winternitz P. *J. Phys. A: Math. Gen.*, 2001, V.34, 9085–9099.

- some solvable rigid Lie algebras  
Campoamor-Stursberg R. *J. Phys. A: Math. Gen.*, 2002, V.35, 6293–6306;  
Campoamor-Stursberg R. *J. Math. Phys.*, 2003, V.44, 771–784.
- solvable Lie algebras with graded nilradical of maximal nilindex and a Heisenberg subalgebra  
Ancochea J.M., Campoamor-Stursberg R. and Garcia Vergnolle L. *J. Phys. A: Math. Gen.*, 2006, V.39, 1339–1355.
- properties of Casimir operators of some perfect Lie algebras and estimations for their number  
Campoamor-Stursberg R. *J. Phys. A: Math. Gen.*, 2003, V.36, 6709–6723; 2004, V.37, 3627–3643.  
Ndogmo J.C. *J. Phys. A: Math. Gen.*, 2004, V.37, 5635–5647.

Main advantage of proposed method is in that it is purely algebraic. Unlike the conventional method, it eliminates the need to solve systems of differential equations, replacing in our approach by construction of the matrix  $B(\theta)$  of inner automorphisms and by excluding the parameters  $\theta$  from the fundamental lifted invariant  $\mathcal{I} = \check{x} \cdot B(\theta)$  in some way.

Let us note, that efficient exploitation of the method imposes certain constraints on the choice of bases of the Lie algebras. That then automatically yields simpler expressions for the invariants. In some cases the simplification is considerable.