COMPUTATION OF NONLINEAR NORMAL MODES, PART II: NUMERICAL CONTINUATION IN AUTO

G. Sérandour, M. Peeters, G. Kerschen, J.C. Golinval
Structural Dynamics Research Group
Aerospace and Mechanical Engineering Department
University of Liège, Liège, Belgium
g.serandour, m.peeters,g.kerschen,jc.golinval@ulg.ac.be

Abstract

The concept of nonlinear normal modes (NNMs) is discussed in the present paper and its companion, Part I. Because there is virtually no application of the NNMs to large-scale engineering structures, these papers are an attempt to highlight one aspect that might drive their development in the future. Specifically, we support that numerical methods for the continuation of periodic solutions pave the way for an effective and practical computation of NNMs. In this context, we show that the NNMs computation is possible using an existing continuation software, namely AUTO07. This is demonstrated using a 2DOF nonlinear system. The results are compared with those obtained with the Matlab algorithm described in Part I.

Key words

Nonlinear normal modes, periodic solution, numerical computation, orthogonal collocation, continuation techniques, AUTO software.

1 Introduction

The concept of NNMs is encountered when one tries to interpret nonlinear dynamic phenomena during the free response of a discrete conservative mechanical system with n degrees of freedom (DOFs) whose motion is described by the following governing equation:

$$\mathbf{M}\,\ddot{\mathbf{x}}(t) + \mathbf{K}\,\mathbf{x}(t) + \mathbf{f}_{\text{nl}}\left\{\mathbf{x}(t), \dot{\mathbf{x}}(t)\right\} = 0 \qquad (1)$$

where $\mathbf{M}$ is the mass matrix; $\mathbf{K}$ is the stiffness matrix; $\mathbf{x}$, $\dot{\mathbf{x}}$ and $\ddot{\mathbf{x}}$ are the displacement, velocity and acceleration vectors, respectively; $\mathbf{f}_{\text{nl}}$ is the nonlinear restoring force vector.

The two main definitions of an NNM are given in the companion paper, Part I, and their detailed description and fundamental propeties can be found in [Vakakis et al., 1996; Vakakis, 1997; Kerschen et al., 2008]. An NNM can be viewed as an extension of the well-established concept of linear normal modes (LNMs) to nonlinear systems. In this paper, we use the extended definition of an NNM given in [Kerschen et al., 2008] and also used in the companion paper, that is the NNM motions are *(non-necessarily synchronous) periodic motions* of the conservative mechanical system (1).

In this context NNMs represent families of periodic orbits of the conservative system. In view of this, it is surprising that there have only been few attempts to compute NNMs using numerical methods [Slater, 1996; Pesheck, 2000; Lee et al., 2005; Arquier, 2007]. Indeed, methods to compute periodic orbits in nonlinear systems based on numerical continuation are well-established (see, e.g., [Allgower, 1997; Seydel, 1994]). Moreover, softwares that implement these methods have been developed for a number of years and are readily available (see, e.g., [Doedel, 2007a; Govaerts, 2007]).

In the following, it is shown that the AUTO software can be used to efficiently compute NNMs. It confirms the view of the authors that numerical methods can provide an efficient and practical alternative to analytical asymptotic methods that too often rely on complex mathematical developments.

2 Computing NNMs in AUTO

In this part details are given on which strategy to adopt to compute NNMs using AUTO software. First, the software AUTO and its main capabilities are presented. Then, the problem of NNM computation is described. Finally, the strategy used to compute NNMs is given.

2.1 AUTO07, software for continuation and bifurcation analysis

AUTO is a software for continuation and bifurcation problems. Although it can perform limited continuation and bifurcation analysis on algebraic problems and partial differential equations it is primarily suited to the

study of ordinary differential equation of the form:

$$\mathbf{g}(\mathbf{z}, \boldsymbol{\lambda}) = 0 \tag{2}$$

where $\mathbf{g}$ and $\mathbf{z}$ are n-dimensional vectors and $\boldsymbol{\lambda}$ denotes one or more parameter. This software has proven successful in dealing with a wide variety of problems, from chemical reactions to population evolution and much more (for a more complete list of AUTO applications see [Doedel, 2007b]). In structural dynamics, it has been extensively used for the computation of forced response and limit cycle arising in nonlinear dynamical systems (see, e.g., [Touzé et al., 2007; Sérandour, 2005]).

AUTO relies on the orthogonal collocation method together with the pseudo-arclength continuation for the continuation of periodic solutions.

Considering the fact that AUTO is well-known within the engineering community and that it has been used for structural dynamics problems, it is surprising that few attempts have been made (to the authors' knowledge) to compute NNMs using this software. We will show in the following section that although the NNM computation problem is not represented by an equation of the form of (2), this can be easily circumvented to allow AUTO to be used in that particular case.

2.2 Computation of periodic orbits in conservative systems

As briefly described in the introductory section, NNMs can be viewed as families of periodic orbits of a conservative system. This poses a problem when one wants to use continuation schemes for their computation. Indeed, continuation codes such as AUTO allow for the computation of periodic orbits when those are isolated and depend on an external parameter $\boldsymbol{\lambda}$ as in (2). The problem is then solved as a two-point boundary value problem with $\boldsymbol{\lambda}$ as the primary continuation parameter and the period of the periodic solution as a secondary continuation parameter. For instance, when computing the forced response of a dissipative system, one can use the forcing frequency as an explicitly available primary parameter and the periodic orbits are indeed isolated due to the dissipative character of the system. This ceases to be valid when considering a conservative system, and an alternative formulation needs to be devised. Such a problem has been described in depth in [Muñoz-Almaraz et al., 2003]. A way to overcome the absence of primary continuation parameter is to introduce the so-called unfolding parameter together with its unfolding function in the equations describing the conservative system. This has for effect to introduce an additional parameter that will subsequently be used as the primary parameter for the continuation. It is worth noting the peculiarity of this strategy is that the unfolding parameter remains null at all time for a periodic solution to exist. This concept is explained in more details in [Sepulchre, 1997]. This strategy has been used in [Doedel et al., 2003] for the computation in AUTO of families of periodic orbits in conservative systems and can be seen as a blueprint for the computation of NNMs in AUTO.

Let us consider the problem described by (1). Its periodic orbits are solutions of the following system of equation:

$$\mathbf{g}(\mathbf{z}) = 0 \tag{3}$$

and the following periodicity condition:

$$\mathbf{z}(0) = \mathbf{z}(\mathrm{T}) \tag{4}$$

where

$$\mathbf{g}(\mathbf{z}) = \begin{pmatrix} \dot{\mathbf{x}} \\ -\mathbf{M}^{-1}\left[\mathbf{K}\mathbf{x} + \mathbf{f}_{\text{nl}}(\mathbf{x}, \dot{\mathbf{x}})\right] \end{pmatrix} \tag{5}$$

$$\mathbf{z} = \begin{bmatrix} \mathbf{x}^* & \dot{\mathbf{x}}^* \end{bmatrix}^* \tag{6}$$

and T is the period of the solution and star denotes the transpose operation. This is usually reformulated in the following adimensional form:

$$\mathrm{T}\,\mathbf{g}(\mathbf{z}) = 0 \tag{7a}$$
$$\mathbf{z}(0) = \mathbf{z}(1) \tag{7b}$$

We now introduce the unfolding parameter $\boldsymbol{\lambda}$. Its associated unfolding function $\mathbf{G}_{\boldsymbol{\lambda}}$ is obtained from the first integral of the system described by (1), that is the total energy E of the system, and is given by:

$$\mathbf{G}_{\boldsymbol{\lambda}}(\mathbf{z}_{\text{i}}) = \frac{\partial E(\mathbf{z})}{\partial \mathbf{z}_{\text{i}}} \qquad \text{i} = 1 \ldots 2n \tag{8}$$

It is now possible for AUTO to continue periodic solutions of the following boundary value problem:

$$\mathrm{T}\,\mathbf{g}(\mathbf{z}) + \boldsymbol{\lambda}\,\mathbf{G}_{\boldsymbol{\lambda}}(\mathbf{z}) = 0 \tag{9a}$$
$$\mathbf{z}(0) = \mathbf{z}(1) \tag{9b}$$

where T is the unknown period and $\boldsymbol{\lambda}$ is the primary continuation parameter. When computing periodic solution, AUTO automatically adds to the system described by (9) an integral phase constraint to ensure the unicity of the solution.

It is worth mentioning that, although the first integral of the system is the most appropriate choice for deriving the unfolding function, it is perfectly acceptable to choose an unfolding function corresponding to damping instead, as shown in [Doedel et al., 2003; Arquier, 2007]. This can be extremely convenient when an expression of the gradient of the first integral becomes difficult to obtain (i.e., systems with a large number of DOFs).

2.3 Strategy for the computation of NNMs with AUTO

Now that it has been established that periodic orbits of conservative systems can be computed in AUTO, it remains to define a strategy to apply this method to the calculation of NNMs. Indeed, continuation with AUTO can only be achieved if an initial solution is known a priori and is used as starting point for the continuation. In the case of NNMs, we do know that at low energies (i.e., small amplitude of motion), the nonlinear effects remain negligible for certain types of nonlinearity, and the NNMs are equivalent to the LNMs of the underlying linear system. One way of computing the starting point for the AUTO continuation would be to solve the underlying linear system for low level of energy and use the results of this analysis as starting point for the study of the nonlinear system. There are two main drawbacks to this approach, namely we do not know a priori what constitutes a low level of energy at which nonlinear effects remain negligible and we increase the computational burden by looking for a complete solution to the linear problem. An alternative strategy has therefore been used in the present study.

We first compute independently the natural frequencies of the underlying linear system by solving the simpler eigenvalue problem. We then compute a branch of trivial solutions (i.e. $E = 0$) using the period as continuation parameter while setting the unfolding parameter to zero. Computing solutions for $E = 0$ ensures that NNMs and LNMs are indeed equivalent and have similar resonant frequencies at which non-trivial solutions will coexist with trivial solutions. Indeed, at these particular frequencies, degenerate bifurcations occur. Branch switching is then performed in order to switch to the branches of non-trivial solutions and points on these branches are used as starting points for the continuation of periodic orbits, using this time the unfolding parameter as primary continuation parameter.

3 Numerical Example

The NNM computation with AUTO is now demonstrated using a 2DOF system similar to that described in the companion paper. The governing equations of the system are

$$\begin{aligned} \ddot{x}_1 + (2x_1 - x_2) + 0.5\,x_1^3 &= 0 \\ \ddot{x}_2 + (2x_2 - x_1) &= 0 \end{aligned} \qquad (10)$$

Solving the eigenvalue problem of the underlying linear system leads to two natural eigenfrequencies that are $f_1 = 1/2\pi \simeq 0.159$ Hz ($\mathrm{T}_1 = 1$s) and $f_2 = \sqrt{3}/2\pi \simeq 0.276$ Hz ($\mathrm{T}_1 = \sqrt{3}$s). Following the strategy described in the previous section we can continue two branches of non-trivial solutions starting from the two aforementioned frequencies. These branches represent synchronous NNMs ($S11+$ and $S11-$) that are described in Part I. At low energy level they are identical to their linear counterparts. As the level of energy increases, the frequency-energy plot (FEP) in Figure 1 clearly shows that both the modal curves and frequencies of oscillation are dependent on the system total energy. The continuation of NNMs with AUTO also confirms the main features that have been described in the companion paper and in [Peeters et al., 2008]. Following the $S11-$ branch at high energy level shows that the 1:1 out-of-phase motion persists and S11- extends to infinity. The $S11+$ branch has a rather more complex behaviour, where internal resonance appears at higher level of energy as seen in Figure 2. Indeed, the symmetric motion $S11+$ becomes $S31$ where a 3:1 internal resonance appears between the in and out-of-phase NNMs. The motion, however, remains symmetric as seen in Figure 3. This process is then repeated when $S31$ becomes $S51$ and so on... In addition to that, unsymmetric motions ($U21$, $U41$...) also arise at high level of energy. These differ from the aforementioned internally resonant motions as they are not direct continuation of the symmetric solution but appear at branch point bifurcations. Their computation requires branch switching at the bifurcation points. Figure 4 shows that, between the two branch point bifurcations (BP), the branch of symmetric solutions $S11+$ coexists with a branch of unsymmetric solutions $U21$. The results from AUTO and the method described in the companion paper are in perfect agreement, as shown in Figure 2.

4 Comments and Conclusion

This paper has shown that NNMs can be computed with the readily and freely available continuation software AUTO. All the features that are typical to NNMs and that have no linear conterparts have been observed and the results obtained match those described in the companion paper, Part I. The computation of NNMs with AUTO requires a careful choice of AUTO computation constants, most importantly the mesh interval and the number of collocation points. This is particularly true when the internal resonance ratio increases (i.e. $S71$, $U81$...). In addition, the fact that unsymmetric solutions occur at branch switching bifurcations imposes to monitor all special solutions (i.e. branch points). This paper has also shown that the apparent difficulty in computing NNMs using analytical asymptotic methods can be conveniently overcome using numerical methods, either with a limited implementation effort, as shown in the companion paper, or using existing tools, as presented in this paper. This paves the way for an effective and practical computation of NNMs.

References

Allgower, E.J., Georg, K. (1997) *Numerical Path Following*, Hanbook of Numerical Analysis, Vol. **5**, North Holland Publishing.

Arquier, R. (2007) Une méthode de calcul des modes de vibrations non-linéaires de structures. *PhD Thesis.* Université de la Méditerrané, Marseille.

Doedel, E. J., Paffenroth, R. C., Keller, H. B., Dichmann, D. J., Galán-Vioque, J., Vanderbauwhede, A. (2003) Computation of Periodic Solutions of Conservative Systems with application to the 3-Body Problem *International Journal of Bifurcation and Chaos*, **13-6**, pp. 1353-1381

Doedel, E. (2007a). *AUTO, Software for Continuation and Bifurcation Problems in Ordinary Differential Equations*, http://indy.cs.concordia.ca/auto/.

Doedel, E. (2007b). *Publications by E.J. Doedel*, http://cmvl.cs.concordia.ca/publications.html.

Govaerts, W., Kuznetsov, Y.I. (2007) *Matcont, CL Mathcont and CL Matcont for maps continuation software in Matlab*, http://www.matcont.ugent.be.

Kerschen, G., Peeters, M., Golinval, J.C., Vakakis, A.F. (2008) Nonlinear normal modes, Part I: A useful framework for the structural dynamicist. *Mechanical Systems and Signal Processing*, submitted.

Lee, Y.S., Kerschen, G., Vakakis, A.F., Panagopoulos, P.N., Bergman, L.A., McFarland, D.M. (2005) Complicated dynamics of a linear oscillator with a light, essentially nonlinear attachment. *Physica D*, **204**, pp. 41-69.

Muñoz-Almaraz, F.J., Freire, E. , Galn, J., Doedel, E., Vanderbauwhede, A. (2003) Continuation of Periodic Orbits in Conservative and Hamiltonian Systems. *Physica D*, **181**, pp. 1-38

Peeters, M., Viguié, R., Sérandour, G., Kerschen, G., Golinval, J.C. (2008) Nonlinear normal modes, Practical computation using numerical continuation techniques. *Mechanical Systems and Signal Processing*, submitted.

Pesheck, E. (2000) Reduced-order modeling of nonlinear structural systems using nonlinear normal modes and invariant manifolds. *PhD Thesis*, University of Michigan, Ann Arbor.

Sepulchre, J.A., MacKay, R. (1997) Localized oscillations in conservative or dissipative networks of weakly coupled autonomous oscillators.*Nonlinearity*, **10**, pp. 679-713

Sérandour, G. (2005) Non-linear Dynamics of Vibratory Rate Sensors. *PhD Thesis*. The University of Nottingham, UK.

Seydel, R. (1994). *Practical Bifurcation and Stability Analysis, from Equilibirum to Chaos*. Springer-Verlag, 2nd Edition.

Slater, J.C. (1996) A numerical method for determining nonlinear normal modes. *Nonlinear Dynamics*, **10**, pp. 19-30.

Touzé, C. Amabili, A., Thomas, O. (2007) - Reduced-order models for large-amplitude vibrations of shells including in-plane inertia. In *Proceedings of the EUROMECH Colloquium on Geometrically Nonlinear Vibrations*. Porto, Portugal, July 2007.

Vakakis, A.F., Manevitch, L.I., Mikhlin, Y.V., Pilipchuk, V.N., Zevin, A.A. (1996). *Normal Modes and Localization in Nonlinear Systems*. John Wiley & Sons. New York.

Vakakis, A.F. (2008) Nonlinear normal modes and their applications in vibration theory: an overview. *Mechanical Systems and Signal Processing*, textbf11, pp. 3-22.

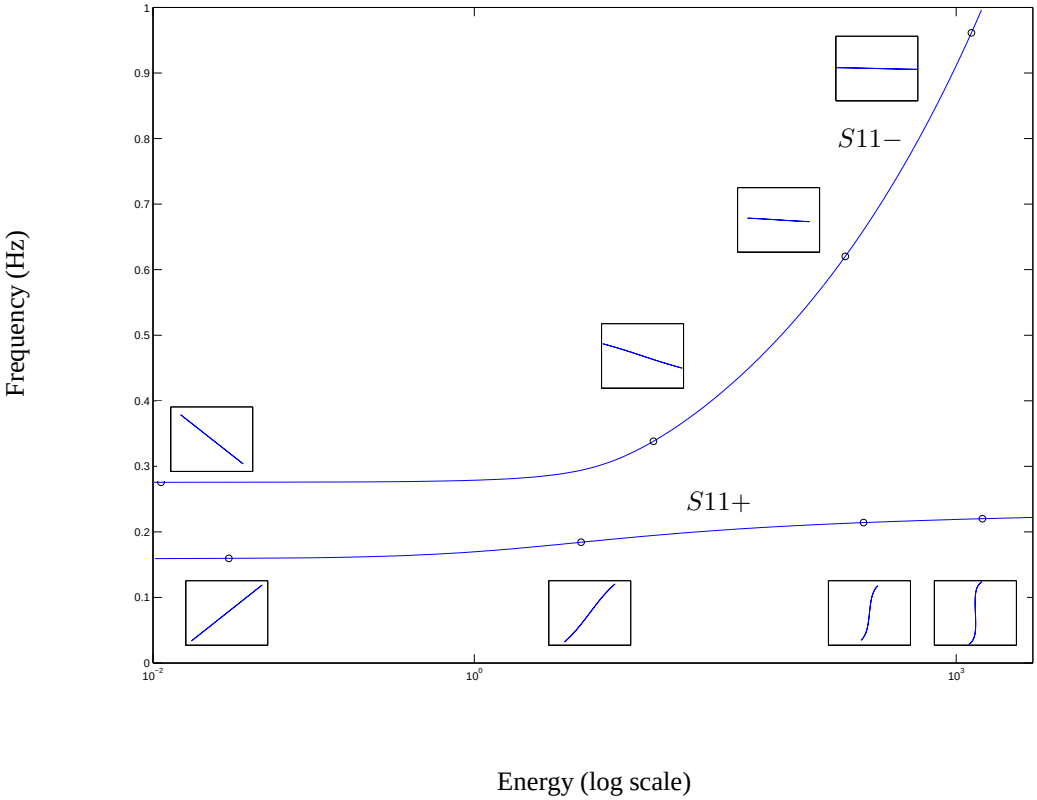

Figure 1. Frequency-energy plot of the 2DOF system computed with the proposed numerical method. NNM motions depicted in the configuration space are inset. The horizontal and vertical axes in these plots are the displacements of the first and second DOFs, respectively; the aspect ratio is set so that increments on the horizontal and vertical axes are equal in size to indicate whether or not the motion is localized to a particular DOF.

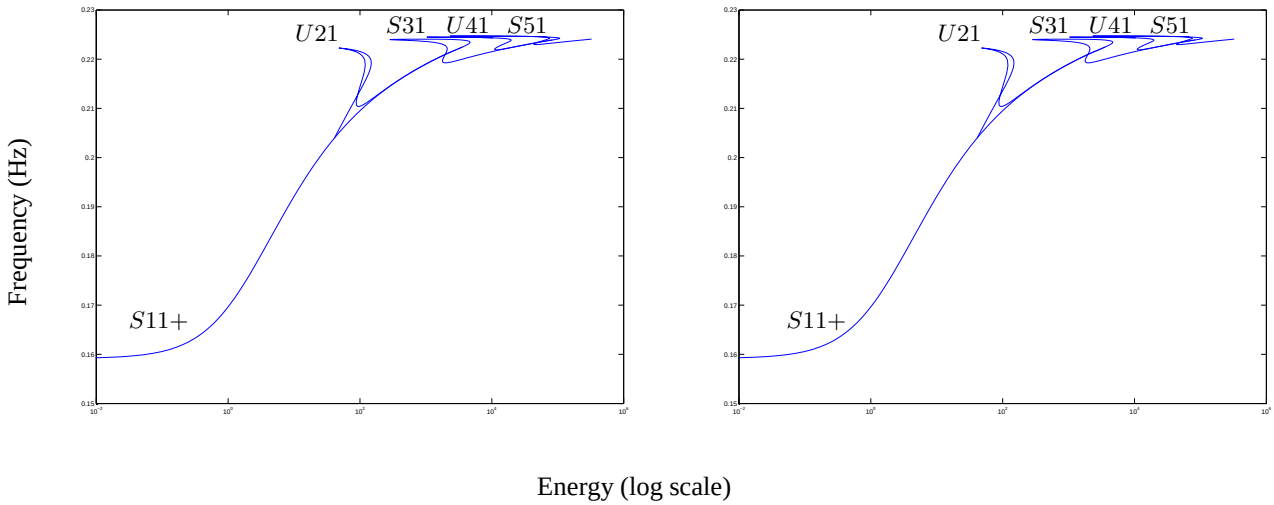

Figure 2. $S11+$ at higher energy levels and internally resonant NNMs ($U21$, $S31$, $U41$, $S51$). Results from both AUTO (left) and the method presented in Part I (right) show perfect agreement.

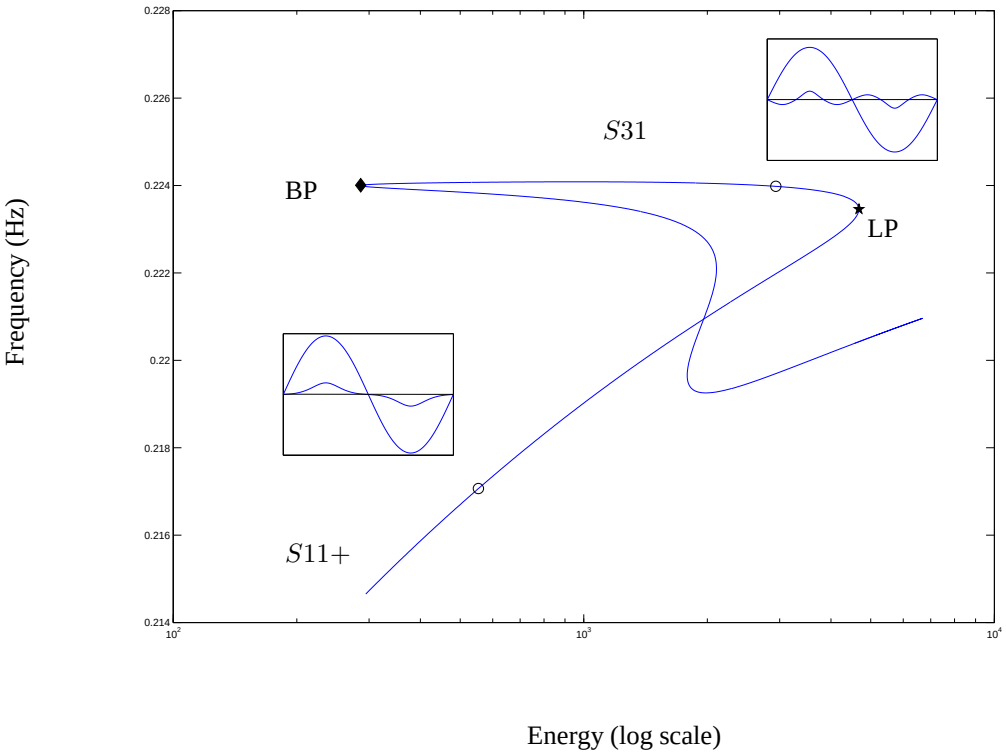

Figure 3. Close-up of the $S11$-$S31$ transition occuring at a Fold bifurcation (LP). The NNM time response over one period are inset, showing that the solution remains symmetric.

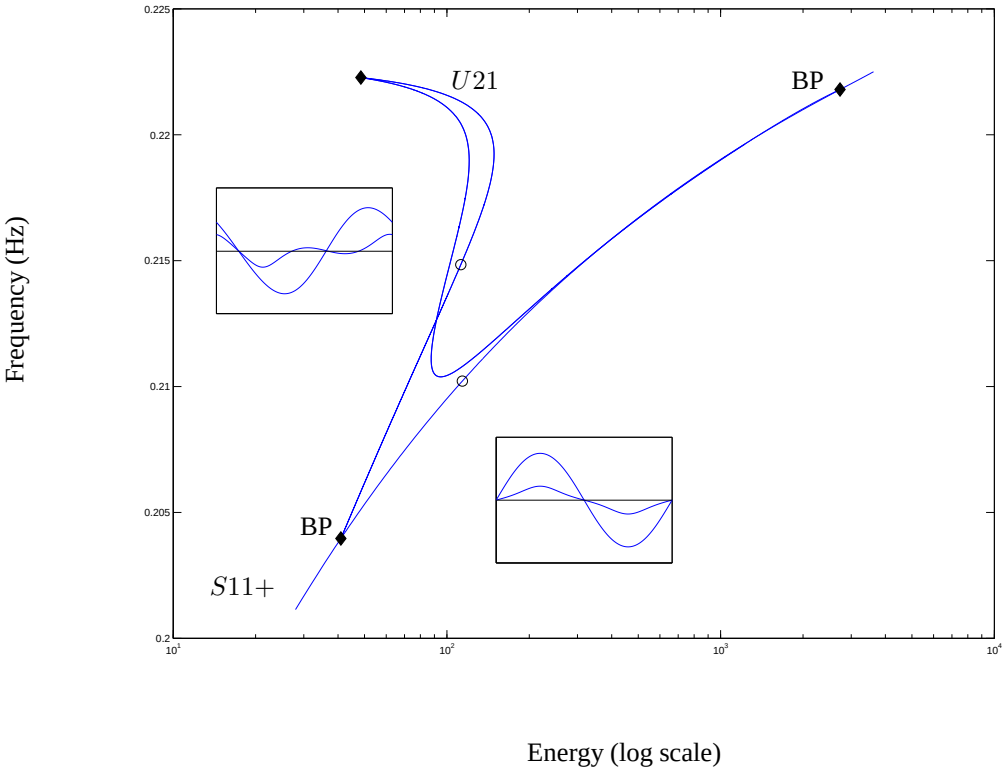

Figure 4. Close-up of the $S11$-$U21$ transition. The NNM time response over one period are inset, illustrating the symmetry breaking phenomena occuring between the two branch point bifurcations (BP).