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Computation of realizations of discrete-time cone-systems

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Abstract. A new notion of a realization of transfer matrix of $(\mathcal{P}, \mathcal{Q}, \mathcal{V})$-cone-system for discrete-time linear systems is proposed. Necessary and sufficient conditions for the existence of the realizations are established. A procedure is proposed for computation of a realization of a given proper transfer matrix $T(z)$ of $(\mathcal{P}, \mathcal{Q}, \mathcal{V})$-cone-system. It is shown that there exists a realization of $T(z)$ of $(\mathcal{P}, \mathcal{Q}, \mathcal{V})$-cone-system if and only if there exists a positive realization of $\overline{T}(z) = VT(z)Q^{-1}$, where V, Q and P are non-singular matrices generating the cones $\mathcal{V}$, $\mathcal{Q}$ and $\mathcal{P}$ respectively.

Key words: computation, cone, positive realization, discrete-time system, procedure.

1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear systems behaviour can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in positive systems theory is given in the monographs [1,2]. Recent developments in positive systems theory and some new results are given in [3]. Explicit solution of equations of the discrete-time systems with delays has been given in [4]. Realizations problem of positive linear systems without time-delays has been considered in many papers and books [1,2,5].

Recently, the reachability, controllability and minimum energy control of positive linear discrete-time systems with time-delays have been considered in [6,7]. The realization problem for positive multivariable discrete-time systems with delays was formulated and solved in [8,9].

The main purpose of this paper is to present a method for computation of a realization for a given proper transfer matrix of $(\mathcal{P}, \mathcal{Q}, \mathcal{V})$-cone-system. Necessary and sufficient conditions will be established for the existence of the realization. A procedure will be proposed for computation of the realizations for given proper transfer matrices. To the best knowledge of the author the positive realization problem of the cone systems for linear system has not been considered yet.

2. Positive systems in cones

Let $\Re^{m\times n}$ be the set of $m \times n$ real matrices and $\Re^n = \Re^{n\times 1}$. The set of nonnegative integers will be denoted by Z_+.

Consider the discrete-time linear system

$$x_{i+1} = Ax_i + Bu_i \tag{1a}$$

$$y_i = Cx_i + Du_i \quad i \in Z_+ = \{0, 1, ...\} \tag{1b}$$

where $x_i \in \Re^n$, $u_i \in \Re^m$, $y_i \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$, $D \in \Re^{p\times m}$.

DEFINITION 1. Let $P = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} \in \Re^{n\times n}$ be nonsingular and p_k be the kth $(k = 1, ..., n)$ its row. The set

$$\boldsymbol{\mathcal{P}} := \left\{ x_i \in \Re^n : \bigcap_{k=1}^{n} p_k x_i \geqslant 0 \right\} \tag{2}$$

is called a linear cone of the state variable generated by the matrix P. In a similar way we may define for inputs u_i the linear cone of the inputs

$$\boldsymbol{\mathcal{Q}} := \left\{ u_i \in \Re^m : \bigcap_{k=1}^{m} q_k u_i \geqslant 0 \right\} \tag{3}$$

generated by the nonsingular matrix $Q = \begin{bmatrix} q_1 \\ \vdots \\ q_m \end{bmatrix} \in \Re^{m\times m}$ and for outputs y_i, the linear cone of the outputs

$$\boldsymbol{\mathcal{V}} := \left\{ y_i \in \Re^p : \bigcap_{k=1}^{p} v_k y_i \geqslant 0 \right\} \tag{4}$$

generated by the nonsingular matrix $V = \begin{bmatrix} v_1 \\ \vdots \\ v_p \end{bmatrix} \in \Re^{p\times p}$.

DEFINITION 2. The linear system (1) is called $(\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{V}})$-cone-system if $x_i \in \boldsymbol{\mathcal{P}}$ and $y_i \in \boldsymbol{\mathcal{V}}$, $i \in Z_+$ for every $x_0 \in \boldsymbol{\mathcal{P}}$ and all $u_i \in \boldsymbol{\mathcal{Q}}$, $i \in Z_+$.

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Let $\Re_+^{m\times n}$ be the set of $m \times n$ real matrices with nonnegative entries. Note that for $\boldsymbol{\mathcal{P}} = \Re_+^n$, $\boldsymbol{\mathcal{Q}} = \Re_+^m$, $\boldsymbol{\mathcal{V}} = \boldsymbol{\Re_+^p}$ we obtain ($\Re_+^n$, $\Re_+^m$, $\Re_+^p$)-cone-system(shortly positive system) which is equivalent to the classical positive system [1,2].

THEOREM 1. The linear system (1) is $(\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{V}})$-cone-system if and only if

$$\begin{aligned} &\overline{A} = PAP^{-1} \in \Re_+^{n\times n}, \quad \overline{B} = PBQ^{-1} \in \Re_+^{n\times m}, \\ &\overline{C} = VCP^{-1} \in \Re_+^{p\times n}, \quad \overline{D} = VDQ^{-1} \in \Re_+^{p\times m} \end{aligned} \tag{5}$$

PROOF. Let

$$\bar{x}_i = Px_i, \quad \bar{u}_i = Qu_i \quad \text{and} \quad \bar{y}_i = Vy_i \tag{6}$$

From definition 1 it follows that if $x_i \in \boldsymbol{\mathcal{P}}$ then $\bar{x}_i \in \Re_+^n$, if $u_i \in \boldsymbol{\mathcal{Q}}$ then $\bar{u}_i \in \Re_+^m$ and if $y_i \in \boldsymbol{\mathcal{V}}$ then $\bar{y}_i \in \Re_+^p$.

From (1) and (6) we have

$$\begin{aligned} \bar{x}_{i+1} &= Px_{i+1} = PAx_i + PBu_i \\ &= PAP^{-1}\bar{x}_i + PBQ^{-1}\bar{u}_i = \overline{A}\bar{x}_i + \overline{B}\bar{u}_i \end{aligned} \tag{7a}$$

and

$$\begin{aligned} \bar{y}_i &= Vy_i = VCx_i + VDu_i \\ &= VCP^{-1}\bar{x}_i + VDQ^{-1}\bar{u}_i = \overline{C}\bar{x}_i + \overline{D}\bar{u}_i \end{aligned} \tag{7b}$$

It is well-known [1,4,5] that the system (7) is positive if and only if the conditions (5) are satisfied.

LEMMA. The transfer matrix

$$T(z) = C[I_n z - A]^{-1}B + D \tag{8}$$

of the $(\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{V}})$-cone-system (1) and the transfer matrix

$$\overline{T}(z) = \overline{C}\left[I_n z - \overline{A}\right]^{-1}\overline{B} + \overline{D} \tag{9}$$

of the positive system(1) are related by the equality

$$\overline{T}(z) = VT(z)Q^{-1} \tag{10}$$

PROOF. Using (9), (5) and (8) we obtain

$$\begin{aligned} \overline{T}(z) &= \overline{C}\left[I_n z - \overline{A}\right]^{-1}\overline{B} + \overline{D} \\ &= VCP^{-1}\left[I_n z - PAP^{-1}\right]^{-1}PBQ^{-1} + VDQ^{-1} \\ &= VCP^{-1}\left[P(I_n z - A)P^{-1}\right]^{-1}PBQ^{-1} + VDQ^{-1} \\ &= VC[I_n z - A]^{-1}BQ^{-1} + VDQ^{-1} = VT(z)Q^{-1}. \end{aligned}$$

3. Problem formulation

Consider the linear system (1) with its transfer matrix (8). Let $\Re^{p\times m}(z)$ be the set of $p \times m$ rational proper matrices.

DEFINITION 3. Matrices $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$, $D \in \Re^{p\times m}$ are called a realization of a given proper transfer matrix $T(z) \in \Re^{p\times m}(z)$ of the $(\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{V}})$-cone-system if they satisfy the equality (8) and the conditions

$$\begin{aligned} &PAP^{-1} \in \Re_+^{n\times n}, \quad PBQ^{-1} \in \Re_+^{n\times m}, \\ &VCP^{-1} \in \Re_+^{p\times n}, \quad VDQ^{-1} \in \Re_+^{p\times m} \end{aligned} \tag{11}$$

where P, Q and V are nonsingular matrices generating the cones $\boldsymbol{\mathcal{V}}$, $\boldsymbol{\mathcal{Q}}$ and $\boldsymbol{\mathcal{P}}$ respectively.

The realization problem of $(\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{V}})$-cone-system can be stated as follows: Given a proper transfer matrix $T(z) \in \Re^{p\times m}(z)$ and non-singular matrices P, Q, V generating the cones $\boldsymbol{\mathcal{V}}$, $\boldsymbol{\mathcal{Q}}$ and $\boldsymbol{\mathcal{P}}$ find a realization of $T(z)$ of the $(\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{V}})$-cone-system. A procedure for computation of a realization of $T(z)$ of the $(\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{V}})$-cone-system will be proposed and solvability conditions of the problem will be established.

4. Problem solution

A realization for a given $T(z) \in \Re^{p\times m}(z)$ of the $(\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{V}})$-cone-system and non-singular matrices P, Q, V can be computed by the use of the following procedure.

PROCEDURE.

Step 1. Knowing $T(z)$ and the matrices V, Q and using (10) compute the transfer matrix $\overline{T}(z)$.

Step 2. Using the known procedures [1,4,5] find a positive realization $\overline{A}, \overline{B}, \overline{C}, \overline{D}$ of the transfer matrix $\overline{T}(z)$.

Step 3. Using the relations

$$\begin{aligned} &A = P^{-1}\overline{A}P, \quad B = P^{-1}\overline{B}Q, \\ &C = V^{-1}\overline{C}P, \quad D = V^{-1}\overline{D}Q \end{aligned} \tag{12}$$

compute the desired realization.

Note that the procedure follows from Lemma and the relations (5).

THEOREM 2. There exist a realization of $T(z)$ of the $(\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{V}})$-cone-system if and only if there exist a positive realization of $\overline{T}(z)$.

The proof follows immediately from the procedure and Lemma. From Theorem 2 for single-input single-output system ($m = p = 1$) we have the following important corollary.

COROLLARY. There exist a realization A, B, C, D of the transfer function $T(z)$ of the $(\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{V}})$-cone-system if and only if there exist a positive realization $\overline{A}, \overline{B}, \overline{C}, \overline{D}$ of $\overline{T}(z)$ and the realizations are related by

$$\begin{aligned} &A = P^{-1}\overline{A}P, \quad B = P^{-1}\overline{B}Q, \\ &C = V^{-1}\overline{C}P \quad \text{and} \quad D = k\overline{D}, \end{aligned} \tag{13}$$

where $k = QV^{-1}$. For $m = p = 1$ $k = QV^{-1}$ is a scalar and the transfer functions $\overline{T}(z)$ and $T(z)$ related by $\overline{T}(z) = kT(z)$.

5. Examples

5.1. Example 1. Given

$$T(z) = \frac{2z + 1}{z^2 - 2z - 3} \tag{14}$$

and

$$P = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}, \quad Q = V = 1 \tag{15}$$

find realization of (14) of the $(\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{V}})$-cone-system. The $\boldsymbol{\mathcal{P}}$-cone generated by the matrix P is shown in Fig. 1.

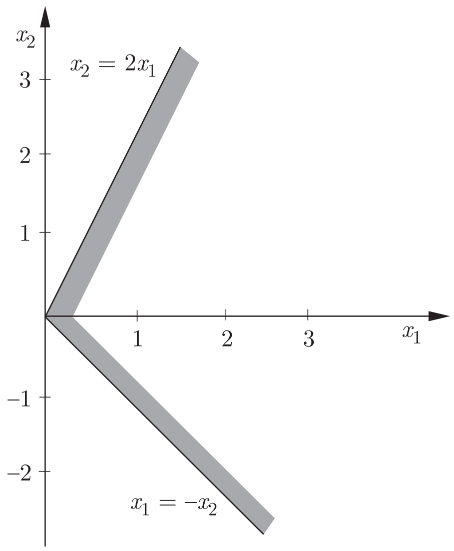

Fig. 1. $\mathcal{P}$-cone generated by the matrix P

Using the procedure we obtain

Step 1. In this case $\overline{T}(z) = T(z)$ since $Q = V = 1$.

Step 2. A-positive realization of (14) has the form [5, p. 181]

$$\overline{A} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \overline{C} = \begin{bmatrix} 1 & 2 \end{bmatrix}, \quad \overline{D} = 0 \qquad (16)$$

Step 3. Using (12), (16) and (15) we obtain the desired realization in the form

$$A = P^{-1}\overline{A}P = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 5 & -1 \end{bmatrix},$$
$$B = P^{-1}\overline{B}Q = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$
$$C = V^{-1}\overline{C}P = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \end{bmatrix},$$
$$D = V^{-1}\overline{D}Q = 0. \qquad (17)$$

5.2. Example 2. Given the transfer matrix

$$T(z) = \frac{1}{2(z-1)(z-2)(z-3)} \times \begin{bmatrix} 3z^2 - 8z + 5 & z - 3 & 2z^2 - 11z + 15 \\ -z^2 + 8z - 11 & -2z^2 + 5z - 1 & 6z^2 - 19z + 11 \end{bmatrix} \qquad (18)$$

and the non-singular matrices

$$P = \begin{bmatrix} 1 & -2 & 0 & 1 & -1 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 2 & 1 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 2 & 0 & 2 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}, \qquad (19)$$

$$V = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

find a realization of(18) of the ($\mathcal{P}$, $\mathcal{Q}$, $\mathcal{V}$)-cone-system. In this case $m = 3$ and $p = 2$. Using the procedure we obtain

Step 1. From (10) and (18) we have

$$\overline{T}(z) = VT(z)Q^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \frac{1}{2(z-1)(z-2)(z-3)}$$
$$\times \begin{bmatrix} 3z^2 - 8z + 5 & z - 3 & 2z^2 - 11z + 15 \\ -z^2 + 8z - 11 & -2z^2 + 5z - 1 & 6z^2 - 19z + 11 \end{bmatrix}$$
$$\times \begin{bmatrix} 1 & -1 & 2 \\ -1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} \frac{2z-4}{(z-1)(z-3)} & 0 & \frac{3z-7}{(z-2)(z-3)} \\ \frac{3}{z-3} & \frac{2z-3}{(z-1)(z-2)} & \frac{2}{z-3} \end{bmatrix} \qquad (20)$$

Step 2. A positive realization of (20) has the form [2]

$$\overline{A} = \text{diag}\,[1,\ 1,\ 2,\ 2,\ 3,\ 3], \quad \overline{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \\ 3 & 0 & 2 \end{bmatrix}, \qquad (21)$$
$$\overline{C} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}, \quad \overline{D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Step 3. Using (12) and (21) we obtain the desired realization in the form

$$A = P^{-1}\overline{A}P = \begin{bmatrix} 1 & -2 & 0 & 1 & -1 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 2 & 1 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 2 & 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$
$$\times \begin{bmatrix} 1 & -2 & 0 & 1 & -1 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 2 & 1 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 2 & 0 & 2 \end{bmatrix}$$
$$= \frac{1}{6} \begin{bmatrix} 12 & 11 & 6 & -3 & 6 & 1 \\ 6 & 7 & 12 & 9 & 0 & 5 \\ -6 & 15 & -18 & -21 & 0 & -9 \\ 18 & -29 & 60 & 57 & 0 & 23 \\ 12 & -20 & 42 & 30 & 12 & 14 \\ -12 & 16 & -36 & -24 & 0 & 2 \end{bmatrix}$$

$$B = P^{-1}\overline{B}Q = \begin{bmatrix} 1 & -2 & 0 & 1 & -1 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 2 & 1 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 2 & 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \\ 3 & 0 & 2 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & -1 & 2 \\ -1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1 & 2 & -4 \\ -1 & 4 & -8 \\ 3 & -6 & 12 \\ -7 & 10 & -14 \\ -10 & 10 & -14 \\ 14 & -8 & 10 \end{bmatrix}$$

$$C = V^{-1}\overline{C}P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & -2 & 0 & 1 & -1 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 2 & 1 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 2 & 0 & 2 \end{bmatrix} \tag{22}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 0 & 5 & 2 & 4 & 4 \\ -3 & -2 & 3 & 2 & 0 & 0 \end{bmatrix}$$

$$D = V^{-1}\overline{D}Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

6. Concluding remarks

The notions of a $\mathcal{P}$-cone generated by a non-singular matrix P (Definition 1) and of a ($\mathcal{P}, \mathcal{Q}, \mathcal{V}$)-cone-system have been introduced. Necessary and sufficient condition for existence of realization of the transfer matrix of ($\mathcal{P}, \mathcal{Q}, \mathcal{V}$)-cone-system have been established. A procedure has been proposed for computation realization of a given proper rational matrix $T(z)$ of a ($\mathcal{P}, \mathcal{Q}, \mathcal{V}$)-cone-system. It has been shown (Theorem 2) that there exists a realization $T(z)$ of a ($\mathcal{P}, \mathcal{Q}, \mathcal{V}$)-cone-system if and only if there exists a positive realization of $\overline{T}(z) = VT(z)Q^{-1}$, where V, Q are non-singular matrices generating the cones $\mathcal{V}$ and $\mathcal{Q}$ respectively. The procedure has been illustrated by two numerical examples. The considerations can be extended for continuous-time linear systems and for linear systems with delays. Using the notions of the cones $\mathcal{P}, \mathcal{Q}, \mathcal{V}$ we may introduce the ($\mathcal{P}, \mathcal{Q}$)-cone reachability, ($\mathcal{P}, \mathcal{Q}$)-cone controllability and ($\mathcal{P}, \mathcal{V}$)-cone observability of discrete-time and continuous-time linear systems and other notions for ($\mathcal{P}, \mathcal{Q}, \mathcal{V}$)-cone-systems.

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