algorithm was given and in a simple case used to obtain the optimal solution.

An important step in attacking the problem was to show for the n-stage process how the initial and terminal conditions implied inequality constraints on intermediate values of the state and control variables of the problem. These inequalities restrict the domain over which the control and the value functions must be defined, and they also restrict the values which the controls may take on. This reformulation allowed the optimal control law to be found by the dynamic programming algorithm.

The computation for the example was very complex and suggests that carrying out similar computations for more realistic, more complex systems will be difficult. In more general problems, in which a network of airports is considered, the minimization step becomes a high dimensional nonlinear programming problem. The development of efficient computer algorithms for carrying out the computation in more general cases appears to be an interesting area of research.

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Technical Notes and Correspondence

Computation of Regions of Transient Stability of Multimachine Power Systems

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Abstract—**A major difficulty in applying Lyapunov theory to the problem of specifying transient stability regions of n-machine power systems is computational complexity, which increases markedly with n. This note outlines a method, requiring only a nominal amount of computation, to determine such regions.**

I. Introduction

The application of Lyapunov theory to the study of transient stability of multimachine power systems, initiated by power engineers in 1966 [1], [2], has continued to the present time. A recent paper by Willems in these Transactions [3] presents some of the more significant advances and provides an extensive bibliography.

The chief attraction of the Lyapunov method is its potential for reducing the computation time associated with investigating the transient stability of an n-machine interconnection. However, since the number and complexity of the computations increases rapidly with n, the potential for savings in overall computation may not be realized. In this note, we present a technique providing significant savings in computation; however, somewhat more conservative regions are obtained.

II. Mathematical Model

The starting point for the analysis is the swing equation model of a multimachine power system. For a detailed development of the model with the usual simplifying assumptions see [3]. For an n-machine interconnection, the dynamics are expressed in terms of the state vector $(\boldsymbol{\alpha},\boldsymbol{\omega})$ by [4].

$$\dot{\alpha} = T\omega \qquad (1)$$
$$\dot{\omega} = -M^{-1} D\,\omega - M^{-1} T^t[f(\alpha) - f(\alpha^0)]$$

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where

$$T = \begin{bmatrix} & \cdot & -1 \\ & \cdot & \\ & \cdot & \\ & \cdot & -1 \\ & \cdot & \\ & \cdot & \cdot \\ I_{n-1} & \cdot & \cdot \\ & \cdot & \cdot \\ & \cdot & \\ & \cdot & \\ & \cdot & \\ & \cdot & -1 \\ & \cdot & \end{bmatrix}$$

$M = \operatorname{diag}\{M_i, i = 1,2,\cdots,n\}$, $M_i > 0$, $D = \operatorname{diag}\{D_i, i = 1,2,\cdots,n\}$, $D_i \geq 0$, $f(\alpha) = \operatorname{col}\{f_i(\alpha)\colon i = 1,2,\cdots,n-1\}$ with

$$f_i(\alpha) = \sum_{\substack{j=1\\ j\neq i}}^{n-1} b_{ij}\sin(\alpha_i - \alpha_j) + b_{in}\sin\alpha_i, \qquad i = 1,2,\cdots,n-1 \qquad (2)$$

$b_{ij} = b_{ji} \geq 0$, and $\omega \in \mathbb{R}^n$ contains the velocity components. Here $\boldsymbol{\alpha}$ is the $n - 1$ vector of intermachine angles obtained by taking the difference of the power angle of ith machine and that of the nth machine which has been *arbitrarily* chosen as the reference machine.

III. Stability

To study the stability of the equilibrium point $(\boldsymbol{\alpha}^0,\mathbf{0})$ of (1), we pick the "total energy" as a Lyapunov function V:

$$V(\alpha,\omega) = \tfrac{1}{2}\langle\omega,M\omega\rangle + W(\alpha) \qquad (3)$$

where

$$W(\alpha) \triangleq \int_{\alpha^0}^{\alpha} \langle[f(\xi) - f(\alpha^0)],d\xi\rangle. \qquad (4)$$

Since $H(\alpha) \triangleq (\partial f/\partial\alpha)(\alpha)$ is symmetric, the integral is path independent and well-defined.

V vanishes at the equilibrium point $(\boldsymbol{\alpha}^0,\mathbf{0})$ and along trajectories

$$\dot{V}_{(1)} = -\langle\omega, D\omega\rangle \leq 0. \quad (5)$$

It remains to specify a bounded transient stability region containing $(\alpha^0, 0)$ for which $V(\alpha,\omega) = l$ along the boundary and within which $V(\alpha,\omega)$ is positive definite. Our method of finding this region is new.

Taking an exact second-order Taylor expansion of $W(\cdot)$ about α^0, we have

$$W(\alpha) = W(\alpha^0) + \left\langle \frac{\partial W}{\partial \alpha}(\alpha^0), (\alpha - \alpha^0) \right\rangle + \langle Q(\alpha,\alpha^0)(\alpha - \alpha^0), (\alpha - \alpha^0)\rangle \quad (6)$$

where

$$Q(\alpha,\alpha^0) \triangleq \int_0^1 (1-t)H[t\alpha + (1-t)\alpha^0]\, dt. \quad (7)$$

The first two terms in (6) are zero, leaving only the quadratic terms in $(\alpha - \alpha^0)$. In fact $Q(\alpha,\alpha^0)$ may be evaluated by carrying out the indicated integration.

We find $Q(\alpha,\alpha^0) = Q^d(\alpha,\alpha^0) + Q^s(\alpha,\alpha^0)$ where

$$Q^d = \mathrm{diag}\left\{ \frac{b_{in}}{(\alpha_i - \alpha_i^0)^2} h(\alpha_i,\alpha_i^0), \qquad i = 1,2,\cdots,n-1 \right\} \quad (8)$$

$$(Q^s)_{ij} = \begin{cases} \displaystyle\sum_{\substack{l=1 \\ l\neq i}}^{n-1} \frac{b_{il}}{(\alpha_{il} - \alpha_{il}^0)^2} h(\alpha_{il},\alpha_{il}^0), & i = j \\ \displaystyle -\frac{b_{ij}}{(\alpha_{ij} - \alpha_{ij}^0)^2} h(\alpha_{ij},\alpha_{ij}^0), & i \neq j. \end{cases} \quad (9)$$

Here $\alpha_{ij} \triangleq \alpha_i - \alpha_j$, and $h(\xi,\xi_0) \triangleq \int_{\xi^0}^{\xi} [\sin u - \sin \xi^0]\, du$.

We next determine a region where $W(\alpha)$ is positive definite. Consider the function h and suppose $|\xi^0| < (\pi/2)$. Then $h(\cdot,\xi^0)$ is a positive definite function over the interval (ξ^l, ξ^u) with $\xi^l \triangleq -\pi - \xi^0$ and $\xi^u \triangleq \pi - \xi^0$. We assume that $\alpha^0 \in \{\alpha \in \mathbb{R}^{n-1}: |\alpha_i| < 90°, |\alpha_i - \alpha_j| < 90°, i,j = 1,2,\cdots,n-1\}$, and consider $W(\cdot)$ on the polytope.

$$P(\alpha^0) = \{\alpha \in \mathbb{R}^{n-1}: \alpha_i \in (\alpha_i^l,\alpha_i^u), \alpha_{ij} \in (\alpha_{ij}^l,\alpha_{ij}^u), \; i,j = 1,2,\cdots,n-1\}. \quad (10)$$

Now on $P(\alpha^0)$ the symmetric matrix $Q^s(\alpha,\alpha^0)$ is positive semidefinite and $Q^d(\alpha,\alpha^0)$ is positive definite. It follows that $W(\cdot)$ is positive definite on $P(\alpha^0)$. Furthermore,

$$\begin{aligned} W(\alpha) &\geq \langle(\alpha - \alpha^0), Q^d(\alpha,\alpha^0)(\alpha - \alpha^0)\rangle \\ &= \sum_{i=1}^{n-1} b_{in} h(\alpha_i,\alpha_i^0) \\ &\triangleq U(\alpha). \end{aligned} \quad (11)$$

Since on $P(\alpha^0)$, $U(\alpha)$ increases monotonically along rays emerging from α^0 we may choose

$$l = \min\{U(\alpha): \alpha \in \partial\bar{P}(\alpha^0)\} \quad (12)^1$$

and the bounded region Ω_l of transient stability is then

$$\Omega_l = \{(\bar{\alpha},\omega): V(\alpha,\omega) < l\}. \quad (13)$$

Minimizing $U(\cdot)$ along each of the hyperplanes which defines $P(\alpha^0)$ is simple since $U(\cdot)$ is a sum of nonnegative uncoupled functions and in fact reduces to a one parameter minimization. The bound l is the smallest of the above minima. It depends on the choice of reference machine and a judicious choice can lead to an improved bound.

[1] $\bar{P}$, $\partial\bar{P}$ denote, respectively, the closure and the boundary of P.

IV. Conclusion

A new approach to specify regions of transient stability for n-machine interconnections has been presented. An important feature is that the computational complexity does not increase with n.

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Comparison of Friedland's and Lin-Sage's Bias Estimation Algorithms

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***Abstract*—Two recent sequential algorithms due to Friedland and Lin-Sage for bias estimation are shown to be essentially identical.**

I. Introduction

The implementation of a Kalman filter needs, among other things, the knowledge of the means of noises (and any other biases) which influence the dynamics and measurements of the stochastic system under consideration. The problem of identifying these means, if not known to start with, has been approached in the past by treating the noise means as additional states and estimating the augmented state vector. Friedland [1] has shown how the augmented state vector can be estimated with significant saving in computation by decoupling the bias estimation from the state estimation. Lin–Sage [2] have recently applied the invariant imbedding technique for solving this problem. There, they have also remarked that Friedland's algorithms [1] are similar to their algorithms. Here we wish to investigate a stronger conjecture that the two algorithms are, in fact, essentially identical. The basis for this conjecture is the well known fact that the same Kalman filter equations can be derived from several different considerations such as minimum variance and maximum likelihood via invariant imbedding [3]. It will be seen from the next section that the proof of the above conjecture is not trivial.

II. Lin–Sage Algorithm versus Friedland's Algorithm

Since several identical symbols are used by Lin–Sage [2], and Friedland [1] in a different sense, we will distinguish between such symbols by using boldface italic letters like $\boldsymbol{Q}$, $\boldsymbol{q}$ when referring to [1]. Also we will refer to equations in [1] by subscript F and those in [2] by subscript LS.

The system $(1)_{LS}$, $(2)_{LS}$ considered by Lin–Sage is given by

$$x(k+1) = \Phi(k)\, x(k) + \Gamma(k)\, w(k) + \Psi(k)\, u(k) \quad (1)$$

$$z(k) = H(k)\, x(k) + F(k)\, v(k) + N(k)\, w(k) \quad (2)$$

where $w(k)$ and $v(k)$ are white, uncorrelated random noise sequences with $w(k) = N[\mu_w(k), V_w(k)]$, $v(k) = N[\mu_v(k), V_v(k)]$, and $N[a, B]$ denotes a Gaussian sequence with mean 'a' and covariance B.

The above system can also be expressed in Friedland's form $[(35a)_F, (36)_F]$, viz.,

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