Computation of Sample Mean Range of the Generalized Laplace Distribution

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Abstract

A generalization of Laplace distribution with location parameter θ, $-\infty < \theta < \infty$, and scale parameter $\phi > 0$, is defined by introducing a third parameter $\alpha > 0$ as a shape parameter. One tractable class of this generalization arises when α is chosen such that $1/\alpha$ is a positive integer. In this article, we derive explicit forms for the moments of order statistics, and the mean values of the range, quasi-ranges, and spacings of a random sample corresponded to any member of this class. For values of the shape parameter α equals $1/i$, $i = 1,...,8$, and sample sizes equal 2,3,…,15 short tables are computed for the exact mean values of the range, quasi-ranges, and spacings. Means and variances of all order statistics are also tabulated.

Keywords: Generalized Laplace distribution, Order statistics, Range, Quasi-ranges, Spacings, Multinomial expansion.

1. Introduction

The probability density function of Laplace distribution with location parameter θ, $-\infty < \theta < \infty$, and scale parameter $\phi > 0$, takes the form

$$f(x) = \frac{1}{2\phi} \exp\left\{-\frac{|x-\theta|}{\phi}\right\}, \qquad -\infty < x < \infty. \tag{1.1}$$

This distribution is also known in the literature as the double exponential distribution. Laplace distribution has been extensively studied since its early introduction by Laplace in (1774). For extensive reviews of this distribution and its statistical properties we refer the reader to Johnson *et al.* (1995) and Kotz *et al.* (2001).

One possible generalization of Laplace distribution is to introduce a shape parameter α, $\alpha > 0$. In such case, the probability density function (1.1) takes the form (Kotz *et al.* 2001)

$$f(x) = \frac{1}{2\phi\alpha^{1/\alpha-1}\Gamma(1/\alpha)} \exp\left\{-\frac{1}{\alpha}\left(\frac{|x-\theta|}{\phi}\right)^{\alpha}\right\}, \qquad -\infty < x < \infty, \tag{1.2}$$

And the cumulative distribution function $F(x)$, can be easily derived as follows. For $x \le \theta,$

$$F_{x\le\theta}(x) = \int_{-\infty}^{x} f(y)\, dy = \frac{1}{2\phi\alpha^{1/\alpha-1}\Gamma(1/\alpha)} \int_{-\infty}^{x} e^{\frac{-1}{\alpha}\left(\frac{\theta-y}{\phi}\right)^{\alpha}} dy.$$

On setting $\left((\theta-y)/\phi\right)^{\alpha}/\alpha = t$, and with simple manipulation we get

$$F_{x\le\theta}(x) = \frac{1}{2\Gamma(1/\alpha)}\Gamma\left[\left(\frac{1}{\alpha}\right),\left(\frac{1}{\alpha}\left(\frac{\theta-x}{\phi}\right)^{\alpha}\right)\right], \tag{1.3}$$

where $\Gamma(\nu,\beta) = \int_{\beta}^{\infty} t^{\nu-1} e^{-t} dt$ is the complementary incomplete gamma function. Similarly, for $x > \theta,$ the cumulative distribution function takes the form

$$F_{x>\theta}(x) = 1 - \int_{x}^{\infty} f(y)\, dy = 1 - \frac{1}{2\Gamma(1/\alpha)}\Gamma\left[\left(\frac{1}{\alpha}\right),\left(\frac{1}{\alpha}\left(\frac{x-\theta}{\phi}\right)^{\alpha}\right)\right]. \tag{1.4}$$

Evidently, when $\alpha = 1$, the p.d.f. (1.2) yields the classical Laplace distribution (1.1), and by varying the value of α, the model can better fit data with sharper peaks and heavier tails. The generalized Laplace distribution can therefore be viewed as a flexible model able to cope with empirical deviations from the Laplace model.

One interesting class of (1.2) is when the shape parameter α is assumed to be equal to $1/i,\ i = 1, 2,\ \ldots$, i.e., when $1/\alpha$ is a positive integer value. As it is known that when ν is a positive integer, $\Gamma(\nu,\beta)$ can be expanded as a finite summation (Gradshteyn and Ryzhik 2007)

$$\Gamma(\nu,\beta) = \Gamma(\nu)\, e^{-\beta} \sum_{i=0}^{\nu-1} \frac{\beta^{i}}{i!}.$$

Accordingly, under the assumption that $1/\alpha$ is a positive integer, the cumulative distribution function of the generalized Laplace distribution function (1.3) and (1.4) can be expressed as

$$F(x) = \begin{cases} \frac{1}{2} e^{\frac{-1}{\alpha}\left(\frac{\theta-x}{\phi}\right)^{\alpha}} \sum_{i=0}^{(1/\alpha)-1} \frac{1}{i!}\left[\frac{1}{\alpha}\left(\frac{\theta-x}{\phi}\right)^{\alpha}\right]^{i} & \text{for } x \le \theta, \\ 1 - \frac{1}{2} e^{\frac{-1}{\alpha}\left(\frac{x-\theta}{\phi}\right)^{\alpha}} \sum_{i=0}^{(1/\alpha)-1} \frac{1}{i!}\left[\frac{1}{\alpha}\left(\frac{x-\theta}{\phi}\right)^{\alpha}\right]^{i} & \text{for } x > \theta. \end{cases} \tag{1.5}$$

Let $X'_1, X'_2, ..., X'_n$ be independent and identically distributed as generalized Laplace distribution under the previously stated assumption about the parameter α. Further, let $X_1 \le X_2 \le ... \le X_n$ be the corresponding order statistics of a random sample of size $n\ (\ge 2)$ arranged in ascending order of magnitude. For any integer r, $2 \le r \le \lfloor n/2 \rfloor$ where $\lfloor \psi \rfloor$ is the largest integer value less than or equal ψ, the difference $R_{r,n-r+1:n} = X_{n-r+1} - X_r$; namely the distance between the r^{th} values from either end of the ordered sample, is known as the r^{th} symmetric quasi-range. Conventionally, when $r = 1$, the difference $R_{1,n:n} = X_n - X_1$ is called the sample range. Also, of special interest are the successive differences of order statistics $(X_{r+1} - X_r)$, $\mathrm{r} = 1,...,n-1$ referred to as the sample spacings (David and Nagaraja 2003).

The sample range, the symmetric quasi-ranges, and the spacings and their linear combinations have many theoretical uses and applications in statistical inference. For excellent surveys of their theoretical uses we refer the reader to (David and Nagaraja 2003, Arnold 2008). Applications of order statistics, in general, and the sample range and quasi-ranges, in particular, in estimation and tests of hypotheses are extensively surveyed in (Harter and Balakrishnan 1996, 1998). (For an early survey of such applications, see also (Chu 1957). Use of sample quasi-ranges in estimating population standard deviation is studied by Masuyama (1957), Harter (1959), and Leone *et al.* (1961). David and Nagaraja (2003) indicated the uses of spacings in building nonparametric and sime-parametric confidence intervals for the population parameters, and Ahsanullah *et al.* (2013) covered their uses in quantile estimations. The generalized Laplace distribution on the other hand, is employed in Bayesian inference and modeling (Taylor 1992), and its applications in communications, economics, engineering, and finance are extensively surveyed in (Kotz *et al.* 2001).

Using the expressions of the density and distribution functions in (1.2) and (1.5), respectively, exact explicit forms for the density function of the r^{th} order statistic X_r, $\mathrm{r} = 1,...,n$, and the s^{th} moment of X_r about the location parameter θ, are derived in the next section. The means of the range, quasi-ranges, and spacings of samples taken from the distribution under consideration are the subject of section 3. Computational aspects and directions on statistical applications of the derived means are considered in section 4. The computed tables for samples of sizes 2,3,…,15 are given in the appendix.

2. Moments of order statistics

The following two propositions derive the probability density function of the r^{th} order statistic X_r, and the s^{th} moment of X_r about the location parameter θ, respectively.

Proposition 2.1

Let X_r $(r=1,...,n)$ be the r^{th} order statistic of an i.i.d. random sample of size n (≥ 1) from the generalized Laplace distribution (1.2) under the assumption that $1/\alpha$ is a positive integer. Then, the p.d.f. of X_r denoted by $f_r(x)$, is

$$f_r(x) = \begin{cases} \dfrac{r\binom{n}{r}}{\phi\alpha^{(1/\alpha)-1}\Gamma(1/\alpha)} \displaystyle\sum_{k=0}^{n-r} \dfrac{\binom{n-r}{k}(-1)^k (k+r-1)!}{2^{k+r}} e^{-(r+k)\left(\frac{1}{\alpha}\left(\frac{\theta-x}{\phi}\right)^{\alpha}\right)} \sum_{p_0+\cdots+p_{(1/\alpha)-1}=k+r-1} \dfrac{\left[\frac{1}{\alpha}\left(\frac{\theta-x}{\phi}\right)^{\alpha}\right]^{\sum_{i=0}^{(1/\alpha)-1} i\,p_i}}{\prod_{i=0}^{(1/\alpha)-1} p_i!(i!)^{p_i}}, & x \leq \theta, \\ \dfrac{r\binom{n}{r}}{\phi\alpha^{(1/\alpha)-1}\Gamma(1/\alpha)} \displaystyle\sum_{k=0}^{r-1} \dfrac{\binom{r-1}{k}(-1)^k (k+n-r)!}{2^{k+n-r+1}} e^{-(k+n-r+1)\left(\frac{1}{\alpha}\left(\frac{x-\theta}{\phi}\right)^{\alpha}\right)} \sum_{p_0+\cdots+p_{(1/\alpha)-1}=k+n-r} \dfrac{\left[\frac{1}{\alpha}\left(\frac{x-\theta}{\phi}\right)^{\alpha}\right]^{\sum_{i=0}^{(1/\alpha)-1} i\,p_i}}{\prod_{i=0}^{(1/\alpha)-1} p_i!(i!)^{p_i}}, & x > \theta, \end{cases} \quad (2.1)$$

where $p_0, p_1, ..., p_{(1/\alpha)-1}$ are non-negative integer values.

Proof: The probability density function of the r^{th} order statistic is given by

$$f_r(x) = \frac{n!}{(r-1)!(n-r)!} f(x)\left[F(x)\right]^{r-1}\left[1-F(x)\right]^{n-r}, \quad -\infty < x < \infty \quad (2.2)$$

$$= r\binom{n}{r} f(x) \sum_{k=0}^{n-r} \binom{n-r}{k} (-1)^k \left[F(x)\right]^{k+r-1}.$$

Hence, if $x \leq \theta$, then substituting from (1.2) and (1.5) for $f(x)$ and $F(x)$, respectively, we get

$$f_{r,x\leq\theta}(x) = \frac{r\binom{n}{r}}{\phi\alpha^{1/\alpha-1}\Gamma(1/\alpha)} \sum_{k=0}^{n-r} \frac{\binom{n-r}{k}(-1)^k}{2^{k+r}} e^{-(k+r)\left(\frac{1}{\alpha}\left(\frac{\theta-x}{\phi}\right)^{\alpha}\right)} \left[\sum_{i=0}^{(1/\alpha)-1} \frac{1}{i!}\left(\frac{1}{\alpha}\left(\frac{\theta-x}{\phi}\right)^{\alpha}\right)^{i}\right]^{k+r-1}. \quad (2.3)$$

Now, by the multinomial expansion of the finite power summation (Bender and Williamson 2006), it could be easily proved that

$$\left[\sum_{i=0}^{S} \frac{y^i}{i!}\right]^m = \sum_{p_0+p_1+\cdots+p_S=m} \frac{m!}{p_0!\, p_1! \cdots p_S!} \left(\frac{y^0}{0!}\right)^{p_0} \left(\frac{y^1}{1!}\right)^{p_1} \cdots \left(\frac{y^S}{S!}\right)^{p_S} = m! \sum_{p_0+p_1+\cdots+p_S=m} \left[\frac{y^{\left(\sum_{i=0}^{S} i\,p_i\right)}}{\prod_{i=0}^{S} p_i!(i!)^{p_i}}\right], \quad (2.4)$$

where m, m, $p_0, p_1, ..., p_S$ are non-negative integer values.

The application of this expansion on the last power summation of (2.3) gives the upper right hand side of (2.1).

Similarly, by expanding $\left[1-(1-F(x))\right]^{r-1}$, (2.2) could be written as

$$f_r(x) = r\binom{n}{r} f(x)\sum_{k=0}^{r-1}\binom{r-1}{k}(-1)^k \left[1-F(x)\right]^{k+n-r}. \tag{2.5}$$

Hence, for $x > \theta,$ by substituting from (1.2) and (1.5) in (2.5), and following similar steps as before, we reach the lower right hand side of (2.1).

Deriving the distribution function of X_r is a straightforward exercise involving simple variable transformations and some algebraic manipulation. The resulting distribution comprises two functions of finite series of the complementary incomplete gamma function.

Proposition 2.2

Under the same stated assumptions of proposition 2.1, the s^{th} moment about the location parameter $\theta,$ of the order statistic $X_r,$ denoted by $E_{X_r}\left[(x-\theta)^s\right]$ is given by

$$E_{X_r}\left[(x-\theta)^s\right] = \frac{r\binom{n}{r}\left(\phi\alpha^{1/\alpha}\right)^s}{\Gamma(1/\alpha)}\left[\sum_{k=0}^{n-r}\frac{\binom{n-r}{k}(-1)^{k+s}}{2^{k+r}}\, G_{k+r-1} + \sum_{k=0}^{r-1}\frac{\binom{r-1}{k}(-1)^{k}}{2^{k+n-r+1}}\, G_{k+n-r}\right], \tag{2.6}$$

Where for the non-negative integer values $p_0, p_1, ..., p_{(1/\alpha)-1}$ the term G_{k+r-1} is given by

$$G_{k+r-1} = \sum_{p_0+\cdots+p_{(1/\alpha)-1}=k+r-1} \frac{(k+r-1)!}{\prod_{i=0}^{(1/\alpha)-1} p_i!\,(i!)^{p_i}} \frac{\Gamma\left(\left(\sum_{i=0}^{(1/\alpha)-1} i\,p_i\right)+(1/\alpha)(s+1)\right)}{(k+r)^{\left(\sum_{i=0}^{(1/\alpha)-1} i\,p_i\right)+(1/\alpha)(s+1)}}, \tag{2.7}$$

and G_{k+n-r} is similarly given by (2.7) but with $(k+n-r)$ in place of $(k+r-1)$**.**

Proof: Consider the s^{th} central moment of the r^{th} order statistic,

$$E_{X_r}\left[(x-\theta)^s\right] = \int_{-\infty}^{0} x^s f_{r,x\le 0}(x)dx + \int_{0}^{\infty} x^s f_{r,x>0}(x)dx. \tag{2.8}$$

After substitution from (2.1), and making the change of variable $y = (-x/\phi)^{\alpha}/\alpha$, i.e. $x = -\phi\alpha^{1/\alpha}y^{1/\alpha}$ and $dx = -\phi\alpha^{(1/\alpha)-1}y^{(1/\alpha)-1}dy$, the first integral in (2.8) yields

$$\int_{-\infty}^{0} x^s f_{r,x\le 0}(x)dx = \frac{r\binom{n}{r}\left(\phi\alpha^{1/\alpha}\right)^s}{\Gamma(1/\alpha)}\sum_{k=0}^{n-r}\frac{\binom{n-r}{k}(-1)^{k+s}}{2^{k+r}}$$

$$\times \sum_{p_0+\cdots+p_{(1/\alpha)-1}=k+r-1} \frac{(k+r-1)!}{\prod_{i=0}^{(1/\alpha)-1} p_i!\,(i!)^{p_i}} \int_0^{\infty} e^{-(k+r)y}\, y^{\left(\sum_{i=0}^{(1/\alpha)-1} i\,p_i\right)+(1/\alpha)(s+1)-1} dy$$

$$= \frac{r\binom{n}{r}\left(\phi\alpha^{1/\alpha}\right)^s}{\Gamma(1/\alpha)} \sum_{k=0}^{n-r} \frac{\binom{n-r}{k}(-1)^{k+s}}{2^{k+r}} \, G_{k+r-1} \,, \tag{2.9}$$

where G_{k+r-1} is given by (2.7).

Similarly, by substitution for $f_{r,x>0}(x)$ from (2.1), and the variable change, the second integral in (2.8) gives

$$\int_0^\infty x^s f_{r,x>0}(x)dx = \frac{r\binom{n}{r}\left(\phi\alpha^{1/\alpha}\right)^s}{\Gamma(1/\alpha)} \sum_{k=0}^{r-1} \frac{\binom{r-1}{k}(-1)^{k}}{2^{k+n-r+1}} \, G_{k+n-r} \,, \tag{2.10}$$

where G_{k+n-r} is as given by (2.7) but with $(k+n-r)$ in place of $(k+r-1)$**.** The proof is then concluded by the substitution from (2.9) and (2.10) in (2.8).

Corollary 2.1**:** On setting $\alpha = 1,$ we get the s^{th} moment about the location parameter θ of the r^{th} order statistic connected with a sample of size $n\ (\geq 1)$ from the classical two-parameter Laplace distribution (1.1) as

$$E_{X_r}\left[(x-\theta)^s\right] = (\phi)^s \frac{n!\ s!}{(r-1)!(n-r)!} \left[\sum_{k=0}^{n-r} \frac{\binom{n-r}{k}(-1)^{k+s}(k+r)^{-(s+1)}}{2^{k+r}} + \sum_{k=0}^{r-1} \frac{\binom{r-1}{k}(-1)^{k}(k+n-r+1)^{-(s+1)}}{2^{k+n-r+1}} \right] \tag{2.11}$$

This same result appeared in Johnson *et al.* (1995, p. 168) but with an excessive multiplication by 2^{-1}.

3. Means of sample range, quasi-ranges, and spacings

In the early beginning of the last century, Pearson (1902) proved that the expected value of the sample spacings takes the form

$$E(X_{r+1} - X_r) = \binom{n}{r} \int_{-\infty}^{\infty} \left[F(x)\right]^r \left[1-F(x)\right]^{n-r} dx, \qquad r = 1,\ldots,n-1. \tag{3.1}$$

By summing both sides of (3.1) for all successive spacings from $r = r_1$ to $r = r_2 - 1$, $\left(1 \leq r_1 < r_2 \leq n\right)$, we get

$$E(X_{r_2} - X_{r_1}) = \sum_{r=r_1}^{r_2-1} E(X_{r+1} - X_r) = \sum_{r=r_1}^{r_2-1} \binom{n}{r} \int_{-\infty}^{\infty} \left[F(x)\right]^r \left[1-F(x)\right]^{n-r} dx. \tag{3.2}$$

The left hand side of (3.2) represents the mean of the difference between any two order statistics X_{r_2} and X_{r_1}. Setting $r_1 = 1$ and $r_2 = n,$ and making use of the binomial theorem, we get the known form of the sample mean range derived by Tippett (1925)

$$E\left(R_{1,n:n}\right) = E(X_n - X_1) = \int_{-\infty}^{\infty} \left\{1 - \left[F(x)\right]^n - \left[1-F(x)\right]^n\right\} dx. \tag{3.3}$$

Now, since $\left[F(x)\right]^{n}+\left[1-F(x)\right]^{n}=1 \ \ at \ \ x=\pm\infty$, (3.3) could be written as

$$E\left(R_{1,n:n}\right)=\left\{x\left(\left[F(x)\right]^{n}+\left[1-F(x)\right]^{n}\right)\right\}_{-\infty}^{\infty}-\int_{-\infty}^{\infty}\left\{\left[F(x)\right]^{n}+\left[1-F(x)\right]^{n}\right\}dx. \tag{3.4}$$

Applying the integration by parts reversely on (3.4), we get

$$E\left(R_{1,n:n}\right)=\int_{-\infty}^{\infty}xn\left\{\left[F(x)\right]^{n-1}-\left[1-F(x)\right]^{n-1}\right\}f(x)\,dx. \tag{3.5}$$

Another approach to derive (3.5) is to consider the difference between means of the two extreme order statistics X_n and X_1

$$E\left(R_{1,n:n}\right)=E(X_n)-E(X_1)=\int_{-\infty}^{\infty}xn\left[F(x)\right]^{n-1}f(x)dx-\int_{-\infty}^{\infty}xn\left[1-F(x)\right]^{n-1}f(x)dx. \tag{3.6}$$

The following theorem states the derivation of explicit forms for the exact mean values of the range, quasi-ranges, and spacings of a random sample taken from the generalized Laplace distribution.

Theorem 3.1

Let $X_1<X_2<...<X_n$ be the order statistics of a random sample of size $n\ (\geq 2)$ drawn from the generalized Laplace distribution (1.2) under the condition that $1/\alpha$ is a positive integer. Then, the mean difference $E\left(X_{r_2}-X_{r_1}\right),\ \left(1\leq r_1<r_2\leq n\right)$ is given by

$$E\left(X_{r_2}-X_{r_1}\right)=\phi\alpha^{(1/\alpha)-1}\sum_{r=r_1}^{r_2-1}\binom{n}{r}\left[\sum_{k=0}^{n-r}\frac{\binom{n-r}{k}(-1)^k}{2^{k+r}}M_{k+r}+\sum_{k=0}^{r}\frac{\binom{r}{k}(-1)^k}{2^{k+n-r}}M_{k+n-r}\right], \tag{3.7}$$

where for the non-negative integer values $p_0,p_1,...,p_{(1/\alpha)-1}$ the term M_{k+r} is given by

$$M_{k+r}=\sum_{p_0+p_1+\cdots+p_{(1/\alpha)-1}=k+r}\frac{(k+r)!\ \Gamma\left(\left(\sum_{i=0}^{(1/\alpha)-1}i\,p_i\right)+\frac{1}{\alpha}\right)}{\left(\prod_{i=0}^{(1/\alpha)-1}p_i!(i!)^{p_i}\right)(k+r)^{\left(\sum_{i=0}^{(1/\alpha)-1}i\,p_i\right)+\frac{1}{\alpha}}}, \tag{3.8}$$

and M_{k+n-r} is similarly given by (3.8) but with $(k+n-r)$ in place of $(k+r)$.

Proof: Recalling (3.2), it is true that

$$E(X_{r_2}-X_{r_1})=\sum_{r=r_1}^{r_2-1}\binom{n}{r}\left[\int_{-\infty}^{\theta}\left[F(x)\right]^{r}\left[1-F(x)\right]^{n-r}dx+\int_{\theta}^{\infty}\left[F(x)\right]^{r}\left[1-F(x)\right]^{n-r}dx\right]$$

$$=\sum_{r=r_1}^{r_2-1}\binom{n}{r}\left[\sum_{k=0}^{n-r}\binom{n-r}{k}(-1)^k\int_{-\infty}^{\theta}\left[F(x)\right]^{k+r}dx+\sum_{k=0}^{r}\binom{r}{k}(-1)^k\int_{\theta}^{\infty}\left[1-F(x)\right]^{k+n-r}dx\right]. \tag{3.9}$$

Consider the first integral in (3.9), by substituting for $\mathrm{F(x)};\ x \le \theta$ from (1.5), and making the change of variable $y = (1/\alpha)((\theta - x)/\phi)^{\alpha}$ we get

$$\int_{-\infty}^{\theta} [F(x)]^{k+r} dx = \frac{\phi \alpha^{(1/\alpha)-1}}{2^{k+r}} \int_{0}^{\infty} \left[\sum_{i=0}^{(1/\alpha)-1} \frac{y^i}{i!} \right]^{k+r} e^{-y(k+r)} y^{(1/\alpha)-1} dy. \tag{3.10}$$

Applying the multinomial expansion (2.4) on the finite power summation in (3.10) then gives

$$\int_{-\infty}^{\theta} [F(x)]^{k+r} dx = \frac{\phi \alpha^{(1/\alpha)-1}}{2^{k+r}} \sum_{p_0+p_1+\cdots+p_{(1/\alpha)-1}=k+r} \frac{(k+r)!}{\prod_{i=0}^{(1/\alpha)-1} p_i!(i!)^{p_i}} \int_{0}^{\infty} e^{-y(k+r)} y^{\left(\sum_{i=0}^{(1/\alpha)-1} i\, p_i\right)+\frac{1}{\alpha}-1} dy. \tag{3.11}$$

With the change of variable $z = y/(k+r)$, the integral in (3.11) is clearly a gamma function, and so

$$\int_{-\infty}^{\theta} [F(x)]^{k+r} dx = \frac{\phi \alpha^{(1/\alpha)-1}}{2^{k+r}} \mathrm{M}_{k+r}, \tag{3.12}$$

where M_{k+r} is as given by (3.8).

Similarly, the second integral in (3.9), after substituting for $\mathrm{F(x)};\ x > \theta$ from (1.5), making the change of variable $y = (1/\alpha)((x-\theta)/\phi)^{\alpha}$, and applying expansion (2.4) yields

$$\int_{\theta}^{\infty} [1-F(x)]^{k+n-r} dx = \frac{\phi \alpha^{(1/\alpha)-1}}{2^{k+n-r}} \sum_{p_0+p_1+\cdots+p_{(1/\alpha)-1}=k+n-r} \frac{(k+n-r)!}{\prod_{i=0}^{(1/\alpha)-1} p_i!(i!)^{p_i}} \int_{0}^{\infty} e^{-y(k+n-r)} y^{\left(\sum_{i=0}^{(1/\alpha)-1} i\, p_i\right)+\frac{1}{\alpha}-1} dy.$$

And with the change of variable $z = y/(k+n-r)$, we get

$$\int_{\theta}^{\infty} [1-F(x)]^{k+n-r} dx = \frac{\phi \alpha^{(1/\alpha)-1}}{2^{k+n-r}} \mathrm{M}_{k+n-r}, \tag{3.13}$$

where M_{k+n-r} is given by (3.8) but with $(k+n-r)$ in place of $(k+r)$.

Finally, the substitution from (3.12) and (3.13) in (3.9) completes the proof of the theorem.

Corollary 3.1: The following are straightforward consequences of theorem 3.1

1. Setting $r_1 = 1$ and $r_2 = n$ in (3.7) gives the mean of the sample range.
2. Setting $r_1 = m$ and $r_2 = n - m + 1;\ 2 \le m \le \lfloor n/2 \rfloor$, in (3.7) gives the mean of the m^{th} symmetric quasi-range.
3. Finally, setting $r_2 = r_1 + 1$ in (3.7) gives the mean of the spacing $(X_{r_1+1} - X_{r_1})$; $\mathrm{r}_1 = 1, \ldots n-1$.

Corollary 3.2: For $n \geq 3,$ and $r_1 = 1, ..., \lfloor (n+1)/2 \rfloor - 1,$ the following relation is true

$$E\left(X_{n-r_1+1} - X_{n-r_1}\right) = E\left(X_{r_1+1} - X_{r_1}\right). \quad (3.14)$$

Poof: Set $r_1 = 1, ..., \lfloor (n+1)/2 \rfloor - 1$ and $r_2 = r_1 + 1,$ and then substitute in (3.7) to get both sides of (3.14), the equality of the two sides becomes evidently true.

Corollary 3.3: On setting $\alpha = 1$ we get the mean difference $E\left(X_{r_2} - X_{r_1}\right),$ $(1 \leq r_1 < r_2 \leq n)$ connected with a sample of size n (≥ 2) from the classical two-parameter Laplace distribution (1.1) as

$$E(X_{r_2} - X_{r_1}) = \phi \sum_{r=r_1}^{r_2-1} \binom{n}{r} \left[\sum_{k=0}^{n-r} \frac{\binom{n-r}{k} (-1)^k (k+r)^{-1}}{2^{k+r}} + \sum_{k=0}^{r} \frac{\binom{r}{k} (-1)^k (k+n-r)^{-1}}{2^{k+n-r}} \right]. \quad (3.15)$$

Remark 3.1: The explicit from (3.7) does not depend on the location parameter $\theta,$ $-\infty < \theta < \infty$. Not only is this so, but the dependence on the scale parameter ϕ, $\phi > 0$ is especially simple multiplicative one. Hence, for example, if instead of the sample range $R_{1,n:n}$, we consider $W_{1,n:n} = R_{1,n:n} / \phi,$ then $E\left(R_{1,n:n}\right) = \phi E\left(W_{1,n:n}\right)$. Since, in this case the form of $W_{1,n:n}$, is independent of $\theta,$ and ϕ, a single table suffices for the means of the range, for each sample size, and each value of α**.**

4. Computation algorithm and tabulation

Apart from the precision matters of computing large factorials and gamma functions, the main difficulty in programming the derived forms (2.6) and (3.7) is the generation of all possible terms of the involved multinomial expansions. The following algorithm provides a straightforward approach to overcome this difficulty.

4.1 The compositions algorithm

Following the basic number theory concepts of partitions and compositions (See Andrews 1976, Knuth 1997, Stojmenovic 2008), by partitions of a positive integer m into z parts, we mean all possible sets of z nonnegative integers that sum up to m. Compositions are merely partitions in which the orders of the summands are taken into account. Generating all of the decompositions of a nonnegative integer m into z nonnegative integers, is the same as to generate all of the possible $z-1$ subsets of $m+z-1$ objects.

The following algorithm; in Pascal-like notation, is adopted to generate all of the compositions needed for computing summation like (3.8).

Algorithm composition (k + r);

$p_0 \leftarrow k+r;\ t \leftarrow k+r;\ h \leftarrow -1;\ p_i \leftarrow 0\ (i = 1,2,...,(1/\alpha)-1)$;

While $p_{(1/\alpha)-1} \neq k+r$ *do* $\{$

  If $t > 1$ *then* $h \leftarrow -1;$

  $h \leftarrow h + 1;\ t \leftarrow p_h;\ p_h \leftarrow 0;\ p_0 \leftarrow t - 1;\ p_{h+1} \leftarrow p_{h+1} + 1$ $\}$

A Fortran 95 code, running on a Windows-based Intel® 3.00GHz CPU is written to conduct all computations. The Fortran language provides better control over the precision and accuracy of the computations. Interfacing this Fortran code with R is now under consideration by the author besides other computational routines attributed to the Generalized Normal and the multinomial distributions.

4.2 Tabulation

Given that the sampling distribution is the generalized Laplace (1.2) with parameters $\phi = 1$ and α equals 1, $1/2$,..., $1/8$ respectively, exact mean values of the range, quasi-ranges, and spacings, as well as means and variances of all order statistics are computed for samples of sizes equal to 2,3,…,15. Table 1 in the appendix gives the means of ranges and quasi-ranges, and table 2 gives the means of the corresponding lower order spacings. It should be noted that, for any sample size $n \geq 3$, the identity relation (3.14) eliminates the need to compute the means of the $\lfloor (n-1)/2 \rfloor$ higher order spacings, i.e., the means of spacings not appeared in table 2, can be obtained by symmetry. Means and variances of higher order statistics are listed in tables 3 and 4, respectively. Again, there is no need to list the means and variances of the lower order statistics, as it can be obtained using the relation $E(X_r) = -E(X_{n-r+1})$, $r \leq \lfloor n/2 \rfloor$, and for any odd sample size, $E(X_{\lfloor n/2 \rfloor + 1}) = 0$.

Of special interest is the case when $\alpha = 1$, for which the underling distribution reduces to the classical Laplace distribution (1.1). For this case, Govindarajulu (1966), has derived closed-form expressions for the first two moments and the product (linear) moment of order statistics, in terms of the moments of order statistics of a random sample drawn from the negative exponential, as the folded distribution of the double exponential. For samples of size $n = 2,3,...,20$, Govindarajulu evaluated and tabulated the means of order statistics, accurate to six decimal places, and the covariances, accurate to five decimal places. All his computed means and variances for $n = 2,...,15$, match exactly the corresponding computed values for this case in tables 3 and 4.

5. Conclusion

In this article, a generalization of the Laplace distribution is considered. The model enhances the classical Laplace distribution by introducing a third parameter $\alpha > 0$, to further control the shape of the distribution. By varying the value of α, the model can better fit data with sharper peaks and heavier tails. The generalized Laplace distribution can therefore be viewed as a flexible model able to cope with empirical deviations from the Laplace model with two parameters only.

Under the assumption that the reciprocal of the shape parameter is a positive integer, we derive explicit forms for density functions and the moments of order statistics, and the mean values of the range, quasi-ranges, and spacings of a random sample corresponded to any member of this class. For the largest eight values of the shape parameter α, and for samples of sizes equal 2,3,…,15, exact mean and variance values of all order statistics, and exact mean values of the range, quasi-ranges, and spacings are evaluated and tabulated.

The theoretical development in this article depends on the complementary incomplete gamma function and the multinomial expansions. Conducting the computations invited for developing an algorithm for generating all compositions of any nonnegative integer number. Directions of the many possible applications of the results are briefly surveyed in the introduction.

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Appendix

Table 1: Means of the ranges and quasi-ranges of samples of size 2,3,…,15 from the Generalized Laplace distribution with parameters $\phi = 1$, and $\alpha = 1,\ 1/2,...,1/8$.

n	(r_1 , r_2)	α =1	1/2	1/3	1/4	1/5	1/6	1/7	1/8
2	(1,2)	1.50000	2.437500	3.802083	5.819092	8.807991	13.23642	19.792450	29.489545
3	(1,3)	2.25000	3.656250	5.703125	8.728638	13.21198	19.85463	29.688675	44.234317
4	(1,4)	2.77083	4.583264	7.239311	11.18493	17.05495	25.77999	38.730595	57.926895
	(2,3)	0.68750	0.875208	1.094568	1.359754	1.683085	2.078547	2.562913	3.156584
5	(1,5)	3.17708	5.364410	8.593068	13.41460	20.61740	31.35892	47.345364	71.093375
	(2,4)	1.14583	1.458680	1.824280	2.266257	2.805141	3.464246	4.271521	5.260974
6	(1,6)	3.51250	6.049422	9.820659	15.48133	23.97154	36.67337	55.626056	83.839576
	(2,5)	1.50000	1.939351	2.455116	3.080917	3.846715	4.786692	5.941904	7.362369
	(3,4)	0.43750	0.497338	0.562610	0.636938	0.721994	0.819353	0.930756	1.058184
7	(1,7)	3.79895	6.663166	10.95021	17.41698	27.15346	41.76430	63.619210	96.218407
	(2,6)	1.79375	2.366954	3.043335	3.867427	4.880005	6.127821	7.667136	9.566587
	(3,5)	0.76562	0.870342	0.984567	1.114642	1.263490	1.433868	1.628823	1.851823
8	(1,8)	4.04923	7.220867	11.99960	19.24242	30.18747	46.65993	71.357574	108.26782
	(2,7)	2.04700	2.759260	3.604494	4.638951	5.915450	7.494878	9.450657	11.872506
	(3,6)	1.03398	1.190037	1.359859	1.552857	1.773669	2.026652	2.316574	2.648830
	(4,5)	0.31835	0.337517	0.359082	0.384283	0.413191	0.445895	0.482571	0.523477
9	(1,9)	4.27156	7.732926	12.98156	20.97301	33.09194	51.38212	78.867007	120.01829
	(2,8)	2.27059	3.124395	4.143888	5.397671	6.951685	8.882365	11.282117	14.264045
	(3,7)	1.26445	1.481290	1.716615	1.983430	2.288631	2.638673	3.040547	3.502117
	(4,6)	0.57304	0.607530	0.646347	0.691709	0.743744	0.802611	0.868628	0.942258
1	(1,10	4.47161	8.206921	13.90575	22.62075	35.88154	55.94863	86.168837	131.49530
	(2,9)	2.47117	3.466978	4.663887	6.143392	7.985511	10.28353	13.150528	16.725163
	(3,8)	1.46824	1.754063	2.063891	2.414785	2.816377	3.277701	3.808472	4.419576
	(4,7)	0.78893	0.844820	0.906303	0.976938	1.057222	1.147607	1.248721	1.361379
	(5,6)	0.24921	0.251595	0.256412	0.263867	0.273528	0.285117	0.298489	0.313577
1	(1,11	4.65344	8.648601	14.77973	24.19526	38.56829	60.37424	93.281058	142.72060
	(2,10	2.65325	3.790120	5.165970	6.875636	9.014076	11.69263	15.046632	19.242340
	(3,9)	1.65184	2.012836	2.404512	2.848295	3.356970	3.942581	4.618058	5.397866
	(4,8)	0.97866	1.063999	1.155569	1.258756	1.374797	1.504688	1.649574	1.810801
	(5,7)	0.45690	0.461258	0.470089	0.483756	0.501467	0.522715	0.547229	0.574891
1	(1,12	4.82012	9.068667	15.61947	25.71521	41.17182	64.67874	100.22517	153.71897
	(2,11	2.82003	4.102370	5.661214	7.604866	10.04484	13.11249	16.969162	21.810621
	(3,10	1.81934	2.266121	2.749095	3.293998	3.917938	4.637205	5.470701	6.436995
	(4,9)	1.14931	1.277814	1.410328	1.554195	1.712537	1.887961	2.084610	2.304519
	(5,8)	0.63735	0.654994	0.675723	0.700134	0.728170	0.760080	0.797862	0.841396
	(6,7)	0.20426	0.204927	0.205940	0.206631	0.207167	0.207956	0.211031	0.216207
1	(1,13	4.97397	9.463171	16.41767	27.17328	43.68783	68.86319	107.00317	164.49563
	(2,12	2.97393	4.398067	6.138393	8.317134	11.06351	14.53005	18.900960	24.409981
	(3,11	1.97359	2.507763	3.085411	3.736799	4.484058	5.348254	6.350128	7.519576
	(4,10	1.30518	1.481799	1.660492	1.850932	2.058789	2.288627	2.543177	2.832015
	(5,9)	0.79861	0.834709	0.871800	0.911239	0.954410	1.002651	1.055758	1.120370
	(6,8)	0.37934	0.380139	0.381471	0.382129	0.382937	0.384917	0.387569	0.397212
1	(1,14	5.11682	9.832601	17.17482	28.57555	46.13095	72.95048	113.65257	175.09129
	(2,13	3.11681	4.676340	6.594848	9.013087	12.07644	15.95639	20.863358	27.059570
	(3,12	2.11664	2.736299	3.409723	4.176058	5.060503	6.086029	7.277768	8.666222
	(4,11	1.44907	1.675050	1.902972	2.149281	2.420159	2.719065	3.049580	3.417726
	(5,10	0.94546	1.002612	1.059322	1.122381	1.192663	1.269524	1.352772	1.444624
	(6,9)	0.53427	0.535635	0.538284	0.545041	0.555395	0.567871	0.581612	0.598222
	(7,8)	0.17278	0.175437	0.175741	0.176460	0.177859	0.178972	0.179245	0.180456
1	(1,15	5.25016	10.19035	17.91529	29.94395	48.51279	76.94150	120.16271	185.49634
	(2,14	3.25015	4.949506	7.052395	9.710192	13.08833	17.38239	22.831839	29.731155
	(3,13	2.25007	2.963481	3.742848	4.628013	5.650738	6.840505	8.228826	9.849506
	(4,12	1.58294	1.869378	2.158595	2.465641	2.800593	3.170199	3.580598	4.036580
	(5,11	1.08095	1.172003	1.261036	1.352343	1.449741	1.554997	1.669574	1.793498
	(6,10	0.67447	0.688915	0.704714	0.720901	0.739126	0.759820	0.783404	0.808974
	(7,9)	0.32396	0.326620	0.329323	0.329954	0.330314	0.330982	0.332455	0.333842

Table 2: Means of spacings of samples of size 2,3,…,15 from the Generalized Laplace distribution with parameters $\phi = 1$, and $\alpha = 1,\ 1/2,...,1/8$.

n	(r, r+1)	α =1	1/2	1/3	1/4	1/5	1/6	1/7	1/8
2	(1,2)	1.500000	2.437500	3.802083	5.819092	8.807991	13.236420	19.792450	29.489545
3	(1,2)	1.125000	1.828125	2.851562	4.364319	6.605993	9.927315	14.844337	22.117158
4	(1,2)	1.041667	1.854028	3.072371	4.912589	7.685934	11.850722	18.083841	27.385155
	(2,3)	0.687500	0.875208	1.094568	1.359754	1.683085	2.078547	2.562913	3.156584
5	(1,2)	1.015625	1.952865	3.384394	5.574172	8.906132	13.947341	21.536922	32.916200
	(2,3)	0.572917	0.729340	0.912140	1.133129	1.402571	1.732123	2.135761	2.630487
6	(1,2)	1.006250	2.055036	3.682771	6.200210	10.062415	15.943341	24.842076	38.238604
	(2,3)	0.531250	0.721006	0.946253	1.221989	1.562360	1.983669	2.505574	3.152092
	(3,4)	0.437500	0.497338	0.562610	0.636938	0.721994	0.819353	0.930756	1.058184
7	(1,2)	1.002604	2.148106	3.953439	6.774781	11.136731	17.818240	27.976037	43.325910
	(2,3)	0.514062	0.748306	1.029384	1.376393	1.808258	2.346977	3.019156	3.857382
	(3,4)	0.382812	0.435171	0.492284	0.557321	0.631745	0.716934	0.814411	0.925911
8	(1,2)	1.001116	2.230803	4.197554	7.301736	12.136010	19.582527	30.953459	48.197658
	(2,3)	0.506510	0.784612	1.122317	1.543047	2.070891	2.734113	3.567041	4.611838
	(3,4)	0.357812	0.426260	0.500389	0.584287	0.680239	0.790379	0.917001	1.062677
	(4,5)	0.318359	0.337517	0.359082	0.384283	0.413191	0.445895	0.482571	0.523477
	(1,2)	1.000488	2.304266	4.418839	7.787673	13.070129	21.249882	33.792445	52.877124
	(2,3)	0.503069	0.821552	1.213636	1.707120	2.331527	3.121846	4.120785	5.380964
	(3,4)	0.345703	0.436880	0.535134	0.645861	0.772443	0.918031	1.085959	1.279929
	(4,5)	0.286523	0.303765	0.323173	0.345855	0.371872	0.401306	0.434314	0.471129
10	(1,2)	1.000217	2.369971	4.620932	8.238681	13.948018	22.832554	36.509155	57.385073
	(2,3)	0.501465	0.856458	1.299998	1.864304	2.584567	3.502915	4.671028	6.152794
	(3,4)	0.339658	0.454621	0.578794	0.718923	0.879577	1.065047	1.279875	1.529098
	(4,5)	0.269857	0.296613	0.324946	0.356535	0.391847	0.431245	0.475116	0.523901
	(5,6)	0.249219	0.251595	0.256412	0.263867	0.273528	0.285117	0.298489	0.313577
11	(1,2)	1.000098	2.429240	4.806880	8.659815	14.777109	24.340805	39.117213	61.739133
	(2,3)	0.500705	0.888642	1.380729	2.013670	2.828553	3.875025	5.214287	6.922237
	(3,4)	0.336589	0.474419	0.624472	0.794770	0.991086	1.218947	1.484242	1.793532
	(4,5)	0.260882	0.301370	0.342740	0.387500	0.436665	0.490986	0.551172	0.617955
	(5,6)	0.228451	0.230629	0.235044	0.241878	0.250734	0.261358	0.273615	0.287445
12	(1,2)	1.000044	2.483149	4.979131	9.055174	15.563490	25.783124	41.628008	65.954179
	(2,3)	0.500342	0.918124	1.456060	2.155434	3.063454	4.237643	5.749231	7.686813
	(3,4)	0.335015	0.494154	0.669383	0.869902	1.102700	1.374622	1.693045	2.066238
	(4,5)	0.255981	0.311410	0.367303	0.427030	0.492184	0.563941	0.643374	0.731561
	(5,6)	0.216546	0.225033	0.234891	0.246752	0.260502	0.276062	0.293416	0.312595
	(6,7)	0.204264	0.204927	0.205940	0.206631	0.207167	0.207956	0.211031	0.216207
13	(1,2)	1.000020	2.532552	5.139643	9.428077	16.312160	27.166568	44.051107	70.042827
	(2,3)	0.500166	0.945152	1.526491	2.290167	3.289727	4.590902	6.275416	8.445202
	(3,4)	0.334204	0.512982	0.712460	0.942934	1.212634	1.529813	1.903475	2.343781
	(4,5)	0.253289	0.323545	0.394346	0.469846	0.552189	0.642988	0.743710	0.855822
	(5,6)	0.209631	0.227285	0.245164	0.264555	0.285737	0.308867	0.334094	0.361579
	(6,7)	0.189674	0.190069	0.190735	0.191065	0.191469	0.192458	0.193784	0.198606
14	(1,2)	1.000009	2.578130	5.289991	9.781234	17.027254	28.497045	46.394608	74.015863
	(2,3)	0.500081	0.970021	1.592562	2.418515	3.507972	4.935181	6.792795	9.196674
	(3,4)	0.333785	0.530625	0.753376	1.013388	1.320172	1.683482	2.114094	2.624248
	(4,5)	0.251807	0.336219	0.421825	0.513450	0.613748	0.724770	0.848404	0.986551
	(5,6)	0.205596	0.233488	0.260519	0.288670	0.318634	0.350827	0.385580	0.423201
	(6,7)	0.180745	0.180099	0.181272	0.184291	0.188768	0.194449	0.201184	0.208883
	(7,8)	0.172782	0.175437	0.175741	0.176460	0.177859	0.178972	0.179245	0.180456
15	(1,2)	1.000004	2.620423	5.431451	10.116881	17.712229	29.779557	48.665436	77.882593
	(2,3)	0.500040	0.993012	1.654773	2.541089	3.718797	5.270943	7.301506	9.940825
	(3,4)	0.333567	0.547052	0.792127	1.081186	1.425073	1.835153	2.324114	2.906463
	(4,5)	0.250990	0.348688	0.448779	0.556649	0.675426	0.807601	0.955512	1.121541
	(5,6)	0.203241	0.241544	0.278161	0.315721	0.355307	0.397588	0.443085	0.492262
	(6,7)	0.175256	0.181148	0.187696	0.195473	0.204406	0.214419	0.225475	0.237566
	(7,8)	0.161983	0.163310	0.164661	0.164977	0.165157	0.165491	0.166227	0.166921

Table 3: Means of order statistics of samples of size 2,3,…,15 from the Generalized Laplace distribution with parameters $\phi = 1$, and $\alpha = 1,\ 1/2,...,1/8$.

n	r	α =1	1/2	1/3	1/4	1/5	1/6	1/7	1/8
2	2	0.750000	1.218750	1.901042	2.909546	4.403995	6.618210	9.896225	14.744772
3	2	1.125000	1.828125	2.851562	4.364319	6.605993	9.927315	14.844337	22.117158
4	3	0.343750	0.437604	0.547284	0.679877	0.841542	1.039274	1.281456	1.578292
	4	1.385417	2.291632	3.619655	5.592466	8.527476	12.889996	19.365298	28.963447
5	4	0.572917	0.729340	0.912140	1.133129	1.402571	1.732123	2.135761	2.630487
	5	1.588542	2.682205	4.296534	6.707300	10.308703	15.679464	23.672682	35.546687
6	4	0.218750	0.248669	0.281305	0.318469	0.360997	0.409677	0.465378	0.529092
	5	0.750000	0.969675	1.227558	1.540458	1.923357	2.393346	2.970952	3.681184
	6	1.756250	3.024711	4.910329	7.740669	11.985772	18.336687	27.813028	41.919788
7	5	0.382812	0.435171	0.492284	0.557321	0.631745	0.716934	0.814411	0.925911
	6	0.896875	1.183477	1.521667	1.933714	2.440002	3.063911	3.833568	4.783293
	7	1.899479	3.331583	5.475106	8.708495	13.576734	20.882150	31.809605	48.109204
8	5	0.159180	0.168758	0.179541	0.192141	0.206596	0.222948	0.241286	0.261738
	6	0.516992	0.595018	0.679929	0.776428	0.886834	1.013326	1.158287	1.324415
	7	1.023503	1.379630	1.802247	2.319475	2.957725	3.747439	4.725328	5.936253
	8	2.024619	3.610434	5.999800	9.621212	15.093735	23.329966	35.678787	54.133911
9	6	0.286523	0.303765	0.323173	0.345855	0.371872	0.401306	0.434314	0.471129
	7	0.632227	0.740645	0.858307	0.991715	1.144315	1.319336	1.520273	1.751058
	8	1.135296	1.562197	2.071944	2.698835	3.475842	4.441182	5.641058	7.132023
	9	2.135784	3.866463	6.490783	10.486509	16.545971	25.691064	39.433503	60.009147
10	6	0.124609	0.125798	0.128206	0.131933	0.136764	0.142559	0.149244	0.156788
	7	0.394466	0.422410	0.453152	0.488469	0.528611	0.573803	0.624361	0.680690
	8	0.734124	0.877031	1.031946	1.207392	1.408189	1.638851	1.904236	2.209788
	9	1.235589	1.733489	2.331943	3.071696	3.992756	5.141765	6.575264	8.362581
	10	2.235806	4.103460	6.952876	11.310377	17.940773	27.974319	43.084419	65.747654
11	7	0.228451	0.230629	0.235044	0.241878	0.250734	0.261358	0.273615	0.287445
	8	0.489332	0.532000	0.577784	0.629378	0.687398	0.752344	0.824787	0.905401
	9	0.825921	1.006418	1.202256	1.424148	1.678485	1.971291	2.309029	2.698933
	10	1.326626	1.895060	2.582985	3.437818	4.507038	5.846315	7.523316	9.621170
	11	2.326724	4.324300	7.389865	12.097633	19.284147	30.187120	46.640529	71.360303
12	7	0.102132	0.102464	0.102970	0.103316	0.103584	0.103978	0.105516	0.108103
	8	0.318678	0.327497	0.337862	0.350067	0.364085	0.380040	0.398931	0.420698
	9	0.574659	0.638907	0.705164	0.777098	0.856269	0.943980	1.042305	1.152259
	10	0.909675	1.133061	1.374548	1.646999	1.958969	2.318602	2.735351	3.218497
	11	1.410016	2.051185	2.830607	3.802433	5.022423	6.556245	8.484581	10.905310
	12	2.410061	4.534334	7.809738	12.857607	20.585914	32.339370	50.112590	76.859489
13	8	0.189674	0.190069	0.190735	0.191065	0.191469	0.192458	0.193784	0.198606
	9	0.399305	0.417354	0.435900	0.455620	0.477205	0.501325	0.527879	0.560185
	10	0.652594	0.740899	0.830246	0.925466	1.029395	1.144313	1.271588	1.416007
	11	0.986799	1.253881	1.542705	1.868400	2.242029	2.674127	3.175064	3.759788
	12	1.486965	2.199033	3.069196	4.158567	5.531756	7.265028	9.450480	12.204990
	13	2.486985	4.731585	8.208839	13.586644	21.843917	34.431596	53.501586	82.247817
14	8	0.086391	0.087719	0.087871	0.088230	0.088930	0.089486	0.089623	0.090228
	9	0.267136	0.267818	0.269142	0.272520	0.277698	0.283935	0.290806	0.299111
	10	0.472733	0.501306	0.529661	0.561190	0.596332	0.634762	0.676386	0.722312
	11	0.724539	0.837525	0.951486	1.074640	1.210080	1.359532	1.524790	1.708863
	12	1.058324	1.368150	1.704862	2.088029	2.530252	3.043014	3.638884	4.333111
	13	1.558405	2.338170	3.297424	4.506543	6.038224	7.978196	10.431679	13.529785
	14	2.558414	4.910193	8.578006	14.276127	23.052118	36.460953	56.811977	87.531241
15	9	0.161983	0.163310	0.164661	0.164977	0.165157	0.165491	0.166227	0.166921
	10	0.337239	0.344457	0.352357	0.360450	0.369563	0.379910	0.391702	0.404487
	11	0.540480	0.586001	0.630518	0.676171	0.724870	0.777498	0.834787	0.896749
	12	0.791470	0.934689	1.079297	1.232820	1.400296	1.585099	1.790299	2.018290
	13	1.125037	1.481740	1.871424	2.314006	2.825369	3.420252	4.114413	4.924753
	14	1.625077	2.474753	3.526197	4.855096	6.544166	8.691195	11.415919	14.865577
	15	2.625081	5.095176	8.957648	14.971977	24.256395	38.470752	60.081355	92.748171

Table 4: Variances of order statistics of samples of size 2,3,…,15 from the Generalized Laplace distribution with parameters $\phi = 1$, and $\alpha = 1,\ 1/2,...,1/8$.

n	r	α =1	1/2	1/3	1/4	1/5	1/6	1/7	1/8
2	2	1.4375	6.01465	24.0403	93.04821	352.56553	1317.8705	4884.24640	18005.76893
3	2	1.4149	7.27942	32.0987	130.7022	509.17787	1933.4696	7231.34576	26801.02090
4	3	0.5207	1.06558	2.20315	4.57972	9.53990	19.89111	41.50265	86.66153
	4	1.4417	8.49135	39.7040	166.7097	660.95547	2536.2192	9546.20380	35518.32068
5	4	0.5024	1.21756	2.81457	6.30610	13.83354	29.91689	64.08329	136.40051
	5	1.4702	9.54706	46.6356	200.5963	806.87211	3123.8892	11823.9661	44147.10548
6	4	0.3033	0.45660	0.70769	1.11834	1.78881	2.88444	4.67822	7.62143
	5	0.5079	1.42476	3.56955	8.40223	19.04224	42.12147	91.69318	197.47816
	6	1.4939	10.4676	52.9883	232.6280	947.56272	3697.8777	14067.5659	52693.33247
7	5	0.2912	0.49792	0.84998	1.44515	2.44519	4.11971	6.91794	11.58862
	6	0.5191	1.63551	4.35462	10.64379	24.74683	55.74838	122.99891	267.58271
	7	1.5128	11.2804	58.8611	263.0686	1083.6472	4259.5436	16279.8520	61162.79536
8	5	0.2103	0.26480	0.34724	0.46820	0.64290	0.89400	1.25471	1.77344
	6	0.2918	0.56966	1.05773	1.90330	3.35305	5.82088	10.00055	17.05425
	7	0.5307	1.83689	5.13869	12.96203	30.80595	70.52203	157.48408	345.77494
	8	1.5278	12.0075	64.3333	292.1337	1215.6428	4810.0408	18463.2829	69560.56435
9	6	0.2020	0.28052	0.39852	0.57325	0.82972	1.20494	1.75327	2.55450
	7	0.2969	0.65061	1.29188	2.42928	4.41582	7.84793	13.73109	23.75805
	8	0.5412	2.02584	5.91046	15.32378	37.13798	86.26316	194.78283	431.34537
	9	1.5399	12.6652	69.4666	319.9963	1343.9750	5350.3400	20619.9643	77891.05114
1	6	0.1595	0.17806	0.20869	0.25303	0.31384	0.39553	0.50444	0.64922
	7	0.2012	0.31363	0.48283	0.73588	1.11223	1.67015	2.49552	3.71491
	8	0.3033	0.73302	1.53812	2.99996	5.59955	10.15525	18.05491	31.64653
	9	0.5504	2.20234	6.66555	17.70961	43.68659	102.83605	234.60111	523.69871
	1	1.5498	13.2661	74.3092	346.7961	1468.9967	5881.2634	22751.7129	86158.11897
1	7	0.1535	0.18523	0.23204	0.29754	0.38690	0.50762	0.67010	0.88854
	8	0.2036	0.35398	0.58342	0.93175	1.45736	2.24679	3.42816	5.19111
	9	0.3098	0.81378	1.78978	3.60328	6.88498	12.71569	22.93914	40.69001
	1	0.5583	2.36733	7.40243	20.10710	50.41005	120.13265	276.69369	622.32173
	1	1.5580	13.8195	78.8994	372.6475	1591.0042	6403.5152	24860.1097	94365.17865
1	7	0.1278	0.13057	0.14093	0.15811	0.18211	0.21371	0.25431	0.30592
	8	0.1523	0.20318	0.27413	0.37177	0.50491	0.68580	0.93121	1.26398
	9	0.2075	0.39706	0.69310	1.15095	1.85284	2.92128	4.53865	6.97628
	1	0.3158	0.89164	2.04332	4.23153	8.25837	15.50810	28.35603	50.86058
	1	0.5651	2.52198	8.12090	22.50790	57.27627	138.06462	320.85227	726.76611
	1	1.5649	14.3327	83.2678	397.6454	1710.2488	6917.7041	26946.5437	102515.2654
1	8	0.1233	0.13423	0.15323	0.18045	0.21674	0.26378	0.32408	0.40096
	9	0.1535	0.22640	0.32658	0.46442	0.65361	0.91287	1.26774	1.75308
	1	0.2117	0.44069	0.80815	1.38816	2.29202	3.68670	5.82220	9.07234
	1	0.3213	0.96620	2.29673	4.87936	9.70893	18.51445	34.28005	62.12937
	1	0.5709	2.66741	8.82136	24.90618	64.25990	156.55850	366.89772	836.63662
	1	1.5709	14.8115	87.4397	421.8694	1826.9464	7424.3620	29012.2449	110611.1001
1	8	0.1063	0.10132	0.10257	0.10837	0.11788	0.13095	0.14776	0.16878
	9	0.1220	0.14501	0.17714	0.21996	0.27561	0.34721	0.43897	0.55640
	1	0.1559	0.25215	0.38535	0.57046	0.82735	1.18317	1.67507	2.35395
	1	0.2159	0.48381	0.92643	1.63991	2.77026	4.53802	7.27540	11.48161
	1	0.3262	1.03742	2.54883	5.54279	11.22783	21.71889	40.68723	74.46668
	1	0.5760	2.80466	9.50446	27.29781	71.34060	175.55234	414.67441	951.58226
	1	1.5760	15.2603	91.4358	445.3873	1941.2839	7923.9569	31058.3109	118655.1370
1	9	0.1027	0.10331	0.10973	0.12094	0.13654	0.15673	0.18209	0.21349
	1	0.1226	0.15968	0.20799	0.27070	0.35161	0.45580	0.58989	0.76243
	1	0.1588	0.27894	0.44824	0.68712	1.02315	1.49416	2.15222	3.06893
	1	0.2199	0.52589	1.04660	1.90375	3.28392	5.47095	8.89491	14.20534
	1	0.3305	1.10545	2.79889	6.21875	12.80764	25.10731	47.55501	87.84251
	1	0.5804	2.93460	10.1709	29.67979	78.50174	194.99335	464.04606	1071.28930
	1	1.5804	15.6829	95.2738	468.2573	2053.4249	8416.9043	33085.7273	126649.6034