COLLOQUIUM MATHEMATICUM

VOL. 90 2001 NO. 2

COMPUTATION OF SOME EXAMPLES OF BROWN'S SPECTRAL MEASURE IN FREE PROBABILITY

BY

PHILIPPE BIANE (Paris) and FRANZ LEHNER (Linz and Paris)

Abstract. We use free probability techniques to compute spectra and Brown measures of some non-hermitian operators in finite von Neumann algebras. Examples include $u_n + u_\infty$ where u_n and u_∞ are the generators of $\mathbb{Z}_n$ and $\mathbb{Z}$ respectively, in the free product $\mathbb{Z}_n * \mathbb{Z}$, or elliptic elements of the form $S_\alpha + iS_\beta$ where S_α and S_β are free semicircular elements of variance α and β.

1. Introduction. Recently Haagerup and Larsen [9] have computed the spectrum and the Brown measure of R-diagonal elements in a finite von Neumann algebra, in terms of the distribution of its radial part. (See also [19] for a combinatorial approach.) The purpose of this paper is to apply free probability techniques for computing spectra and Brown measures of some non-hermitian and non-R-diagonal elements in finite von Neumann algebras, which can be written as a free sum of an R-diagonal element and an element with arbitrary $*$-distribution.

Motivations for this study are twofold. On the one hand some of these elements appear as transition operators of random walks on groups or semigroups (see e.g. [10], [11], [2]); here we shall for example treat linear combinations of u_n and u_∞, the generators of $\mathbb{Z}_n$ and $\mathbb{Z}$ in $\mathbb{Z}_n * \mathbb{Z}$ and $u_2 + v_2 + u_\infty$. On the other hand random matrix theory has a close connection with free probability (see [21]), but for the moment very little has been done for understanding limit distributions of spectra of non-normal matrices in terms of free probability. For example, the empirical distribution on the eigenvalues of a random matrix with independent identically distributed complex entries, suitably rescaled, converges, with probability one, as its size grows to infinity, to the circular law (the uniform distribution on the unit disk; see [6], [7], [1]), which is the Brown measure for a circular element, in the sense of Voiculescu. It is known that the circular element is the limit in $*$-distribution

2000 *Mathematics Subject Classification*: Primary 22D25, 46L54; Secondary 15A52, 43A05, 60J15.

Key words and phrases: convolution operator, free probability, free product group, random matrix, random walk, spectral measure.

of the above random matrices, but it is not possible to deduce from this the convergence of the empirical distribution on the spectrum (see Lemma 2.1 below).

Another example that we shall consider in this paper is the free sum of an arbitrary element with a circular element. Hopefully, the corresponding Brown measures should represent the limit of eigenvalue distributions of random matrices of the form $A + W$ where A is a matrix with some limit $*$-distribution, and W is a matrix with independent entries. In addition to the circular element discussed above, this is known to be true for the so-called *elliptic element*, which can be written as $S_\alpha + iS_\beta$ and whose Brown measure was first computed in [15] by ad-hoc methods. It turns out to be treatable by our method as well. The empirical eigenvalue distribution of its matrix model with Gaussian random matrices is computed in [12] and shown to converge to the uniform measure on its spectrum, an ellipse.

However in this paper we shall stick to the purely free probabilistic aspects of the subject, and not touch upon the random matrix problem. We hope to deal with this elsewhere.

This paper is organized as follows. In Section 2 we recall preliminary facts about Brown measures and free probability theory. In Section 3 we give a general approach towards the computation of the Brown measure for the sum of an R-diagonal element with an arbitrary element. We specialize in Sections 4 and 5 to the cases where the R-diagonal element is a Haar unitary or a circular element, respectively. We close with some final remarks in Section 6. The pictures of random matrix spectra appearing in various sections of this papers were computed with GNU octave and plotted with gnuplot; the plots of densities of various Brown measures, which accompany or replace the rather unwieldy density formulae, were computed with Mathematica.

Acknowledgements. We thank the referee for pointing out a mistake in formula 3.2. This work was started during a special semester at the Erwin Schrödinger Institute in Vienna in spring 1999, organized by the Institut für Funktionalanalysis of the University of Linz and its head J. B. Cooper. The second author was supported by the EU-network "Non-commutative geometry" ERB FMRX CT960073.

2. Preliminaries

2.1. *The Fuglede–Kadison determinant and Brown's spectral measure.* Let $\mathcal{M}$ be a finite von Neumann algebra with faithful tracial state τ and denote, for invertible $a \in \mathcal{M}$, by $\Delta(a) = e^{\tau(\log|a|)}$ its Fuglede–Kadison determinant (cf. [5]). Denoting by μ_x the spectral measure for the self-adjoint element $x \in \mathcal{M}$, i.e. the unique probability measure on the real line satisfying

$\tau(x^n) = \int t^n \, d\mu(t)$, we have the following formula for the logarithm of the determinant, which serves as a definition of the determinant in the case where a is not invertible:

$$\log \Delta(a) = \int_{\mathbb{R}} \log t \, d\mu_{|a|}(t).$$

The function $\Delta(\lambda - a)$ is a subharmonic function of the complex variable λ, and there is a unique probability measure μ_a on $\mathbb{C}$, with support on the spectrum of a, called the *Brown measure* of a, such that

$$\log \Delta(\lambda - a) = \int \log |\lambda - z| \, \mu_a(dz);$$

it is given by

$$\mu_a = \frac{1}{2\pi} \nabla^2 \log \Delta(\lambda - a)$$

where ∇^2 is the Laplace operator in the complex plane, in the sense of distributions (see [4]). If a is normal, then μ_a is just the spectral measure of a. When $\mathcal{M}$ is $M_n(\mathbb{C})$, with the canonical normalized trace, then μ_a is the empirical distribution on the spectrum of a (counting multiplicities). Although the Brown measure of a can be computed from its $*$-distribution, i.e. the collection of all its $*$-moments $\tau(a^{\varepsilon_1} \ldots a^{\varepsilon_n})$, where a^{ε_j} is either a or a^*, it does not depend continuously on these $*$-moments. Indeed let for example a_n be the $n \times n$ nilpotent matrix with ones on the first upper diagonal and zeros elsewhere; then as n goes to infinity the $*$-moments of a_n converge towards those of a Haar unitary (a unitary element u with $\tau(u^n) = 0$ for $n \neq 0$) whose Brown measure is the Haar measure on the unit circle, whereas the Brown measure of a_n is δ_0 for all n.

LEMMA 2.1. *Let $(a_n; n \geq 0)$ be a uniformly bounded sequence whose $*$-distributions converge towards that of a, and suppose the Brown measure of a_n converges weakly towards some measure μ. Then*:

(i) $\int \log |\lambda - z| \, \mu(dz) \leq \Delta(\lambda - a) = \int \log |\lambda - z| \, \mu_a(dz)$ *for all* $\lambda \in \mathbb{C}$,

(ii) $\int \log |\lambda - z| \, \mu(dz) = \Delta(\lambda - a) = \int \log |\lambda - z| \, \mu_a(dz)$ *for all λ large enough.*

Proof. The distribution of $|\lambda - a_n|$ has a support which remains in a fixed compact set, and it converges weakly towards that of $|\lambda - a|$. Part (i) follows from this and the fact that the function log is a limit of a decreasing sequence of continuous functions. If λ is large enough, then the union of the supports of the distributions of the $|\lambda - a_n|$ is away from 0, hence the function log is continuous there and (ii) follows from weak convergence. ∎

The outcome of (i) of the lemma is that the measure μ_a is a balayée of μ, while from (ii) we get the following

COROLLARY 2.2. *Let U_a be the unbounded connected component of the complement of the support of μ_a. Then the support of μ is included in $\mathbb{C}\backslash U_a$.*

Proof. The functions $\int \log|\lambda - z|\,\mu_a(dz)$ and $\int \log|\lambda - z|\,\mu(dz)$ are harmonic and subharmonic in U_a, respectively, hence their difference is a non-negative superharmonic function on U_a. Since this function attains the value 0 by (ii), it is identically 0 by the minimum principle, so $\int \log|\lambda - z|\,\mu(dz)$ is harmonic on U_a, and thus the support of μ is included in $\mathbb{C} \setminus U_a$. ∎

Conversely, given two measures μ and μ_a on $\mathbb{C}$ satisfying (i) and (ii), we do not know whether there always exists a corresponding sequence $(a_n)_{n\geq 0}$, satisfying the hypotheses of Lemma 2.1.

2.2. *R- and S-transforms.* We shall refer to [21], and [20] or [13] for basic concepts of free probability theory. Let $(\mathcal{M}, \tau)$ be as in Section 2.1, and let $a \in \mathcal{M}$. The power series

$$G_a(\zeta) = \frac{1}{\zeta} \sum_{n=0}^{\infty} \frac{\tau(a^n)}{\zeta^n}$$

can be inverted (for composition of formal power series), in the form

$$K_a(z) = \frac{1}{z} + \sum_{n=0}^{\infty} c_{n+1} z^n = \frac{1}{z}(1 + R_a(z)).$$

The power series R_a is called the *R-transform* of a (note that this slightly differs from the original definition of Voiculescu) and its coefficients are called the *free cumulants* of a. Let

$$\psi_a(z) = \sum_{n=1}^{\infty} \tau(a^n) z^n = \frac{1}{z} G_a\left(\frac{1}{z}\right) - 1$$

be the generating moment series for a, and assume that the first moment is non-zero, so that $\psi_a'(0) \neq 0$. Then ψ_a has an inverse χ_a, and the S-transform of a is defined as

$$S_a(z) = \frac{1+z}{z}\,\chi_a(z).$$

Observe that the power series $zS_a(z)$ and $R_a(z)$ are then inverses of each other (when the mean is non-zero). The relevance of these series to free probability is that, if $a, b \in \mathcal{M}$ are free, then

$$R_{a+b} = R_a + R_b \quad \text{and} \quad S_{ab} = S_a S_b$$

(see e.g. [21]).

2.3. *Calculus of R-diagonal elements.* We use the same notations as in the previous section.

DEFINITION 2.3. A non-commutative random variable x is called *R-diagonal* if x has polar decomposition $x = uh$, where u is a Haar unitary free from the radial part $h = |x|$.

Recall that a unitary $u \in \mathcal{M}$ is called a *Haar unitary* if $\tau(u^n) = 0$ for all integers $n \neq 0$. One can check that the product of an arbitrary element y with a free Haar unitary is an R-diagonal element. According to [9], any R-diagonal element with polar decomposition $x = uh$ has the same distribution as a product $a\widetilde{h}$, where $\widetilde{h}$ has a symmetric distribution, and its absolute value is distributed as h, whereas a is a self-adjoint unitary, free from $\widetilde{h}$, and of zero trace. Indeed, one can assume $\widetilde{h} = a'h$, where a' is a symmetry commuting with h and aa' is a Haar unitary free from h. Let a, b be two free R-diagonal elements. Then one has equality in $*$-distribution of the pairs (a, b) and (ua, ub) where u is a Haar unitary free from $\{a, b\}$, therefore $a + b$ has the same $*$-distribution as $u(a + b)$ which is R-diagonal, and thus the sum of two free R-diagonal elements is again R-diagonal. Let $f_x(z^2) = R_{\widetilde{h}}(z)$ be the cumulant series of $\widetilde{h}$, which determines the $*$-distribution of x. Then the power series $z(1 + z)S_{x^*x}(z)$ and $f_x(z)$ are inverses of each other. Furthermore if a, b are two free R-diagonal elements, then

$$f_{a+b} = f_a + f_b. \tag{2.1}$$

See [17], [18] and [9].

2.4. *Brown measure of R-diagonal elements.* In [9] the Brown measure of an R-diagonal element is determined as follows.

THEOREM 2.4 ([9, Thm. 4.4, Prop. 4.6]). *Let u, h be $*$-free random variables in $(\mathcal{M}, \tau)$, with u a Haar unitary and h positive such that the distribution μ_h of h is not a Dirac measure. Then the Brown measure μ_{uh} of uh has the following properties.*

(i) *μ_{uh} is rotation invariant and its support is the annulus with inner radius $\|h^{-1}\|_2^{-1}$ and outer radius $\|h\|_2$.*

(ii) *The S-transform $S_{\mu_{h^2}}$ of h^2 has an analytic continuation to a neighbourhood of $]\mu_h(\{0\}) - 1, 0]$ and its derivative $S'_{\mu_{h^2}}$ is strictly negative on this interval and its range is $S_{\mu_{h^2}}(]\mu_h(\{0\}) - 1, 0]) = [\|h\|_2^{-2}, \|h^{-1}\|_2^2[$.*

(iii) *$\mu_{uh}(\{0\}) = \mu_h(\{0\})$ and for $t \in]\mu_h(\{0\}), 1]$,*

$$\mu_{uh}\left(B\left(0, \frac{1}{\sqrt{S_{\mu_{h^2}}(t-1)}}\right)\right) = t.$$

(iv) *μ_{uh} is the only rotationally symmetric probability measure satisfying* (iii).

(v) *If* h *is invertible then* $\sigma(uh) = \operatorname{supp}\mu_{uh}$, *i.e., the annulus discussed above.*

(vi) *If* h *is not invertible then* $\sigma(uh) = B(0, \|h\|_2)$.

The proof involves a formula for the spectral radius of products of free elements.

PROPOSITION 2.5 ([9, Prop. 4.1]). *Let* a, b *be* $*$*-free centred elements in* $\mathcal{M}$. *Then the spectral radius of* ab *is*

$$\varrho(ab) = \|a\|_2\|b\|_2.$$

In particular, an R-diagonal element $a = uh$ can be written as u_1u_2h, with free Haar unitaries u_1, u_2 and therefore its spectral radius is $\varrho(a) = \|u_1\|_2\,\|u_2h\|_2 = \|a\|_2$.

3. Adding an R-diagonal element. In this section we give a general approach to computing the Brown measure of the sum of a random variable with an arbitrary distribution and a free R-diagonal element. So we let a be an arbitrary element, h be self-adjoint and u a Haar unitary, with $\{a, u, h\}$ forming a free family.

3.1. *The spectrum of* $a + uh$. The spectrum of $a + uh$ is determined as follows. For $\lambda \notin \sigma(a)$, $\lambda - a - uh$ is invertible if and only if $1 - uh(\lambda - a)^{-1}$ is invertible. If h is not invertible, then by the result of Haagerup and Larsen on R-diagonal elements, the latter is the case if and only if

$$\|h(\lambda - a)^{-1}\|_2 = \|h\|_2\,\|(\lambda - a)^{-1}\|_2 < 1; \tag{3.1}$$

if h is invertible, we get the additional possibility that $1 < \|h^{-1}\|_2\,\|\lambda - a\|_2$. In this case we can look at $(uh)^{-1}(\lambda - a) - 1$.

The case where $\lambda \in \sigma(a)$ must be considered individually. Complications arise for those λ for which $\lambda \in \sigma(a)$ but $\|(\lambda - a)^{-1}\|_2 < \infty$. Otherwise condition (3.1) will be satisfied when approaching λ from the outside of $\sigma(a)$, so that λ lies in the closure of the spectrum of $a + uh$, hence in the spectrum.

3.2. *The Brown measure of* $a + uh$. We can assume that $u = u_1^*u_2$ with u_1 and u_2 Haar unitaries, where $\{u_1, u_2, a, h\}$ is a free family, to get

$$\begin{aligned} \log\Delta(\lambda - a - uh) &= \tau(\log|u_1^*(u_1(\lambda - a) - u_2h)|) \\ &= \tau(\log|u_1(\lambda - a) - u_2h|) \\ &= \int \log|z|\,d\mu_{u_1(\lambda-a)-u_2h}(z) \end{aligned}$$

and this is the Fuglede–Kadison determinant of $x_\lambda = u_1(\lambda - a) - u_2h$, which is an R-diagonal element whose $*$-distribution can be computed according to (2.1), i.e. $f_{x_\lambda} = f_{u_1|\lambda-a|} + f_{u_2h}$. This in turn will yield the S-transform of $x_\lambda^*x_\lambda$, and then by Theorem 2.4, we can compute $\log\Delta(\lambda - a - uh)$.

To proceed further, we need the determining series f_{x_λ}.

LEMMA 3.1. *Let x be an R-diagonal element. Then*

$$\left(\frac{1}{z}(1+f_{x_\lambda}(z))\right)^{\langle -1\rangle}(\zeta)=\frac{1}{\zeta}\left(1+R_{x_\lambda^* x_\lambda}\left(\frac{1}{\zeta}\right)\right). \tag{3.2}$$

Proof. Let $x=uh$ be the polar decomposition of x and denote by $\widetilde{h}$ the symmetrization of the positive part h. Recall that $f_x(z^2)=R_{\widetilde{h}}(z)$ and let $\zeta=\frac{1}{z}(1+f_x(z))=K_{\widetilde{h}}(\sqrt{z})/\sqrt{z}$. We need to show that

$$\frac{1}{\zeta}\left(1+R_{xx^*}\left(\frac{1}{\zeta}\right)\right)=z.$$

To see this, note that

$$G_{xx^*}(v^2)=G_{h^2}(v^2)=\frac{G_{\widetilde{h}}(v)}{v}$$

and therefore with $v=K_{\widetilde{h}}(\sqrt{z})=\zeta\sqrt{z}$ we have

$$G_{xx^*}(K_{\widetilde{h}}(\sqrt{z})^2)=\frac{\sqrt{z}}{K_{\widetilde{h}}(\sqrt{z})}=\frac{1}{\zeta};$$

applying K_{xx^*} to both sides of this equation yields

$$z\zeta^2=K_{xx^*}\left(\frac{1}{\zeta}\right)=\zeta\left(1+R_{xx^*}\left(\frac{1}{\zeta}\right)\right),$$

which is the claimed formula. ∎

To be more specific, assume that a is self-adjoint. Then the computation of the distribution of $(\lambda-a)^*(\lambda-a)$ is conveniently accomplished by using the Cauchy transform of a, namely factorizing $\zeta-|\lambda-x|^2=(x-x_+)(x-x_-)$ with

$$x_\pm=\frac{1}{2}(\lambda+\overline{\lambda}\pm\sqrt{(\lambda-\overline{\lambda})^2+4\zeta})=\operatorname{Re}\lambda\pm i\sqrt{(\operatorname{Im}\lambda)^2-\zeta} \tag{3.3}$$

and expanding into partial fractions

$$\frac{1}{\zeta-|\lambda-x|^2}=\frac{1}{x_+-x_-}\left(\frac{1}{x_+-x}-\frac{1}{x_--x}\right)$$

we get

$$G_{|\lambda-a|^2}(\zeta)=\int\frac{d\mu_a(x)}{\zeta-|\lambda-x|^2}=\frac{G_a(x_+)-G_a(x_-)}{x_+-x_-}. \tag{3.4}$$

Using the same technique one can compute the 2-norm of the inverse of $\lambda-a$. Indeed, as a is self-adjoint we have

$$\|(\lambda-a)^{-1}\|_2^2 = \int \frac{d\mu_a(x)}{|\lambda-x|^2} = \int \frac{d\mu_a(x)}{(\lambda-x)(\overline{\lambda}-x)} = \frac{1}{\lambda-\overline{\lambda}} \int \left(\frac{1}{\overline{\lambda}-x} - \frac{1}{\lambda-x} \right) d\mu_a(x) = -\frac{G_a(\lambda)-G_a(\overline{\lambda})}{\lambda-\overline{\lambda}}. \tag{3.5}$$

Consider the simplest non-trivial random variable, namely $a = u_2 = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, having 2-point spectrum, so that $|\lambda - u_2|^2$ has a Bernoulli distribution. The R-transform of $|\lambda-u_2|^2 = 1+|\lambda|^2 + (\lambda+\overline{\lambda})u_2$ is easily computed to be

$$R_{x_\lambda^* x_\lambda}(z) = (1+|\lambda|^2)\, z + \frac{1}{2}(\sqrt{1+4(\lambda+\overline{\lambda})^2 z^2} - 1)$$

and inverting it according to (3.2) leads to an equation of fourth degree, which is apparently unsuitable for further computations. So even this simple case seems to be intractable by this method. In fact, so far we have no concrete example where the general method above can be carried out to the end. We shall develop other methods, in the next two sections, in order to treat the cases where the R-diagonal element is a Haar unitary or a circular element.

4. Haar unitary case. Now a is an element with an arbitrary distribution, free from a Haar unitary u.

4.1. *The spectrum.* The spectrum of $a+u$ is determined as follows: one has $\lambda \in \sigma(a+u)$ if and only if $1 \in \sigma(u^*(\lambda-a))$ and since the latter is R-diagonal, we infer from Theorem 2.4 that a necessary and sufficient condition is

$$\|(\lambda-a)^{-1}\|_2^{-1} \le 1 \le \|(\lambda-a)\|_2 \tag{4.1}$$

if $\lambda \notin \sigma(a)$; otherwise the condition is simply $1 \le \|\lambda - a\|_2$.

4.2. *First approach to the Fuglede–Kadison determinant.* We get the following formula for the Fuglede–Kadison determinant:

$$\log \Delta(\lambda-a-u) = \tau(\log|\lambda-a-u|) = \tau(\log|u^*(\lambda-a)-1|) = \int \log|z-1|\, d\mu_{u^*(\lambda-a)}(z). \tag{4.2}$$

Observe that $u^*(\lambda-a)$ is an R-diagonal element, and we can evaluate the integral as follows. The Brown measure of an R-diagonal element uh is rotationally symmetric with radial distribution $\nu(dr)$ and one has

$$\int \log|z-1|\, d\mu_{uh}(z) = \int_{\|h^{-1}\|_2^{-1}}^{\|h\|_2} \int_0^{2\pi} \log|re^{i\theta}-1|\, d\theta\, \nu(dr)$$

where the inner integral reduces to

$$\frac{1}{2\pi}\int_0^{2\pi}\log|re^{i\theta}-1|\,d\theta=\begin{cases}0, & r<1,\\ \log r, & r\geq 1.\end{cases}$$

Introduce the radial distribution function

$$F_{uh}(r)=\mu_{uh}(B(0,r))=2\pi\int_{\|h^{-1}\|_2^{-1}}^{r}\nu(d\varrho),$$

which according to Theorem 2.4 is related to the moment generating function ψ_{h^2} by

$$\psi_{h^2}\left(\frac{F_{uh}(r)-1}{F_{uh}(r)r^2}\right)=F_{uh}(r)-1$$

(for $\|h^{-1}\|_2^{-1}\leq r\leq\|h\|_2$). By partial integration (note that $F(\|h\|_2)=1$),

$$\begin{aligned}\tau(\log|uh-1|)&=\int_{\max(1,\|h^{-1}\|_2^{-1})}^{\|h\|_2}2\pi\log(r)\,\nu(dr)\\&=\log rF_{uh}(r)\Big|_{\max(1,\|h^{-1}\|_2^{-1})}^{\|h\|_2}-\int_{\max(1,\|h^{-1}\|_2^{-1})}^{\|h\|_2}\frac{F_{uh}(\varrho)}{\varrho}\,d\varrho\\&=\int_{\max(1,\|h^{-1}\|_2^{-1})}^{\|h\|_2}\frac{1-F_{uh}(\varrho)}{\varrho}\,d\varrho.\end{aligned}$$

EXAMPLE 4.1 (2×2 matrix). Let a have the $*$-distribution of a 2×2 matrix, and consider $a+u$, u a Haar unitary. Let $\mu_\pm=\mu_\pm(\lambda)$ be the eigenvalues of $|\lambda-a|^2$ and let

$$G_{|\lambda-a|^2}(\zeta)=\frac{1}{2}\left(\frac{1}{\zeta-\mu_+}+\frac{1}{\zeta-\mu_-}\right)$$

be its Cauchy transform. Then

$$\psi(z)=\frac{1}{z}G\left(\frac{1}{z}\right)-1=\frac{1}{2}\left(\frac{1}{1-\mu_+z}+\frac{1}{1-\mu_-z}\right)-1$$

and we get $F(r)$ by solving the equation $\psi\left(\frac{t-1}{tr^2}\right)=t-1$ for t:

$$\frac{1}{1-\mu_+\frac{t-1}{tr^2}}+\frac{1}{1-\mu_-\frac{t-1}{tr^2}}=2t.$$

The obvious solution $t=1$ is not interesting for us, and dividing it out leads to the other solution

$$F(r)=\frac{2\mu_+\mu_--r^2(\mu_+\mu_-)}{2(r^2-\mu_+)(r^2-\mu_-)}=\frac{\det|\lambda-a|^2-r^2\tau(|\lambda-a|^2)}{\det(r^2-|\lambda-a|^2)}.$$

The logarithm of the Fuglede–Kadison determinant is, for $\lambda \in \sigma(a+u)$,

$$\begin{aligned}\tau(\log|\lambda - a - u|) &= \int_1^{\|\lambda-a\|_2} \frac{1-F(r)}{r}\,dr \\ &= \int_1^{\|\lambda-a\|_2} \frac{1}{2}\left(\frac{r}{r^2-\mu_+} + \frac{r}{r^2-\mu_-}\right)dr \\ &= \frac{1}{4}(\log|r^2-\mu_+| + \log|r^2-\mu_-|)\Big|_1^{\|\lambda-a\|_2} \\ &= \frac{1}{4}\big(\log\big|\|\lambda-a\|_2^4 - \det|\lambda-a|^2\big| \\ &\quad - \log\big|1-2\|\lambda-a\|_2^2 + \det|\lambda-a|^2\big|\big) \\ &= \frac{1}{2}\log\left|\frac{\mu_+-\mu_-}{2}\right| - \frac{1}{4}(\log|1-\mu_+| + \log|1-\mu_-|).\end{aligned}$$

It is now convenient to use the representation of the Laplacian in terms of

$$\partial_\lambda = \frac{1}{2}\left(\frac{\partial}{\partial\,\mathrm{Re}\,\lambda} - i\frac{\partial}{\partial\,\mathrm{Im}\,\lambda}\right)$$

and its adjoint, namely

$$\nabla^2 = \frac{\partial^2}{\partial(\mathrm{Re}\,\lambda)^2} + \frac{\partial^2}{\partial(\mathrm{Im}\,\lambda)^2} = 4\partial_{\bar\lambda}\partial_\lambda.$$

Then we have the formulae

$$\begin{aligned}\partial_\lambda\|\lambda-a\|_2^2 &= \partial_\lambda\tau((\lambda-a)^*(\lambda-a)) = \tau(\overline{\lambda}-a^*), \\ \partial_\lambda\det(\lambda-a) &= \partial_\lambda((\lambda-\lambda_1(a))(\lambda-\lambda_2(a))) \\ &= 2\lambda-\lambda_1(a)-\lambda_2(a) \\ &= 2\tau(\lambda-a)\end{aligned}$$

and the density of the Brown measure of $a+u$ is

$$\begin{aligned}p_{a+u}(\lambda) &= \frac{2}{\pi}\partial_{\bar\lambda}\partial_\lambda\big(\log\big|\|\lambda-a\|_2^4 - \det|\lambda-a|^2\big| \\ &\quad - \log\big|1-2\|\lambda-a\|_2^2 + \det|\lambda-a|^2\big|\big) \\ &= \frac{2}{\pi}\partial_{\bar\lambda}\left(\frac{2\|\lambda-a\|_2^2\tau(\overline{\lambda}-a^*) - 2\tau(\lambda-a)\det(\overline{\lambda}-a^*)}{\|\lambda-a\|_2^4 - \det|\lambda-a|^2}\right. \\ &\quad \left. - \frac{-2\tau(\overline{\lambda}-a^*) + 2\tau(\lambda-a)\det(\overline{\lambda}-a^*)}{1-2\|\lambda-a\|_2^2+\det|\lambda-a|^2}\right)\end{aligned} \tag{4.3}$$

$$= \frac{4}{\pi}\Bigg(\frac{\|\lambda - a\|_2^2 - |\tau(\lambda - a)|^2}{\|\lambda - a\|_2^4 - \det|\lambda - a|^2}$$

$$- 2\frac{\big|\|\lambda - a\|_2^2\, \tau(\overline{\lambda} - a^*) - \det(\overline{\lambda} - a^*)\, \tau(\lambda - a)\big|^2}{(\|\lambda - a\|_2^4 - \det|\lambda - a|^2)^2}$$

$$- \frac{2|\tau(\lambda - a)|^2 - 1}{1 - 2\|\lambda - a\|_2^2 + \det|\lambda - a|^2}$$

$$+ 2\frac{\big|\tau(\overline{\lambda} - a^*) - \tau(\lambda - a)\det(\overline{\lambda} - a^*)\big|^2}{(1 - 2\|\lambda - a\|_2^2 + \det|\lambda - a|^2)^2}\Bigg)$$

and in terms of eigenvalues

$$\begin{aligned}(4.4)\quad & p_{a+u}(\lambda) \\ &= \frac{2}{\pi}\partial_{\bar{\lambda}}\partial_\lambda\left(\frac{1}{2}\log\left|\frac{\mu_+ - \mu_-}{2}\right| - \frac{1}{4}(\log|1-\mu_+| + \log|1-\mu_-|)\right) \\ &= \frac{1}{\pi}\partial_{\bar{\lambda}}\left(\frac{1}{\mu_+ - \mu_-}\partial_\lambda(\mu_+ - \mu_-) + \frac{1}{2}\left(\frac{1}{1-\mu_+}\partial_\lambda\mu_+ + \frac{1}{1-\mu_-}\partial_\lambda\mu_-\right)\right) \\ &= \frac{1}{\pi}\Bigg(\frac{\partial_{\bar{\lambda}}\partial_\lambda(\mu_+ - \mu_-)}{\mu_+ - \mu_-} - \left|\frac{\partial_\lambda(\mu_+ - \mu_-)}{\mu_+ - \mu_-}\right|^2 \\ &\quad + \frac{1}{2}\left(\frac{\partial_{\bar{\lambda}}\partial_\lambda\mu_+}{1-\mu_+} + \frac{\partial_{\bar{\lambda}}\partial_\lambda\mu_-}{1-\mu_-} + \left|\frac{\partial_\lambda\mu_+}{1-\mu_+}\right|^2 + \left|\frac{\partial_\lambda\mu_-}{1-\mu_-}\right|^2\right)\Bigg)\end{aligned}$$

In particular, if $a = \left[\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}\right]$ (Bernoulli distribution) one gets $\mu_\pm = \{|\lambda-\alpha|^2|, |\lambda - \beta|^2\}$ and consequently the density is

$$p_{a+u}(\lambda) = -\frac{|\beta - \alpha|^2}{\pi(|\lambda-\alpha|^2 - |\lambda-\beta|^2)^2} + \frac{1}{2\pi}\left(\frac{1}{(1-|\lambda-\alpha|^2)^2} + \frac{1}{(1-|\lambda-\beta|^2)^2}\right)$$

on the spectrum, which is determined by the inequalities

$$(4.5)\qquad \frac{1}{\mu_+} + \frac{1}{\mu_-} \geq 2, \qquad \mu_+ + \mu_- \geq 2.$$

Specifying further $\alpha = 1$, $\beta = -1$, so that a is a symmetry, the spectrum is the region bounded by the lemniscate-like curve in the complex plane with the equation

$$|\lambda|^2 + 1 = |\lambda^2 - 1|^2$$

and we get the picture shown in Figure 1.

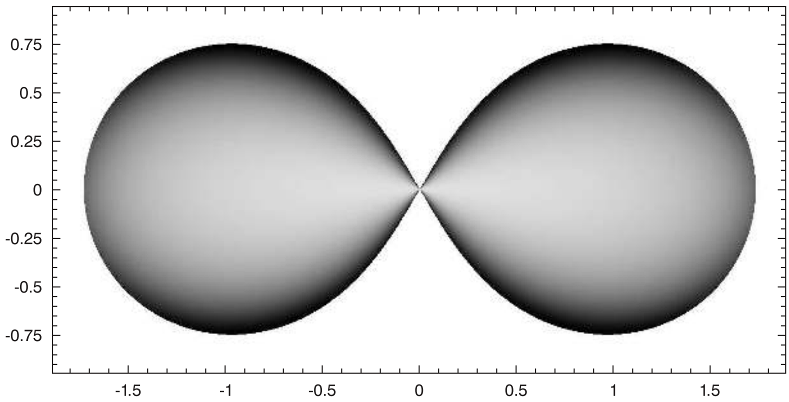

Fig. 1. Density of $\mu_{u_2+u_\infty}$

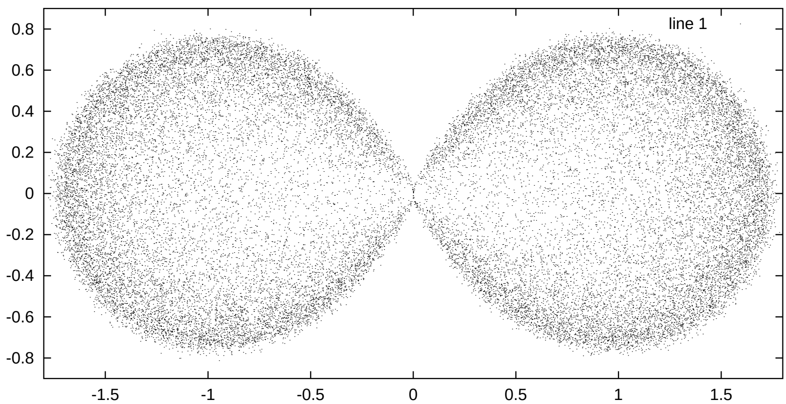

Fig. 2. 200 samples of eigenvalues of 150×150 random matrices $U_2 + U_\infty$

This should be compared with the sample Figure 2 of eigenvalues of random $2N\times 2N$ matrices of the form $X = U_2+U_\infty$, where U_∞ is chosen with the Haar measure on $U(2N)$, and $U_2 = V\Lambda V^*$, with V a Haar distributed unitary independent of U_∞, and Λ a fixed symmetry of trace zero.

As another example, set $a = \left[\begin{smallmatrix} 0 & t \\ 0 & 0 \end{smallmatrix}\right]$. As we will see, the spectrum and Brown measure are radially symmetric. The eigenvalues of $|\lambda - a|^2$ are

$$\mu_\pm = \frac{t^2 + 2|\lambda|^2 \pm t\sqrt{t^2+4|\lambda|^2}}{2} \tag{4.6}$$

and hence, substituting this into (4.5), we get

$$\sigma(a+u) = \{\lambda : 1 - t^2/2 \le |\lambda|^2 \le \sqrt{t^2/2 + 1/4} + 1/2\},$$

which is a full disk for $t \ge \sqrt{2}$ and an annulus otherwise. For the density we substitute the parameters

$$\|\lambda - a\|_2^2 = |\lambda|^2 + t^2/2, \qquad \det|\lambda - a|^2 = |\lambda|^4$$

into formula (4.3) to get the radially symmetric density function

$$p_{a+u}(\lambda) = \frac{4}{\pi}\left(\frac{2t^2}{(4|\lambda|^2 + t^2)^2} + \frac{(1-|\lambda|^2)^2 - (1-2|\lambda|^2)t^2}{((1-|\lambda|^2)^2 - t^2)^2}\right).$$

4.3. *An alternative expression for the Fuglede–Kadison determinant.* In order to treat more complicated examples, instead of the integral (4.2) it will be more convenient to use a more direct formula for the Kadison–Fuglede determinant, which we state as a lemma.

LEMMA 4.2 ([9, Proof of Theorem 4.4]). *Let uh be an R-diagonal element and define functions on $\mathbb{R}_+ \setminus \{0\}$ by*

$$f(v) = \tau((1 + vh^2)^{-1}), \qquad g(v) = \frac{1 - f(v)}{vf(v)}.$$

Then $g(v)$ is strictly decreasing with $g(]0,\infty[) =]\|h^{-1}\|_2^{-2}, \|h\|_2^2[$ and for every $z \in]\|h^{-1}\|_2^{-2}, \|h\|_2^2[$ there is a unique $v > 0$ such that $z^2 = g(v)$. With this v we have

$$\log \Delta(uh - z) = \frac{1}{2}\int \log(1 + vt)\, d\mu_{h^2}(t) + \frac{1}{2}\log\frac{z^2}{1 + vz^2}.$$

For our problem of computing $\log \Delta(\lambda - a - u) = \log \Delta(u^*(\lambda - a) - 1)$ this translates as follows. Put $f(v,\lambda) = \tau((1 + v|a - \lambda|^2)^{-1})$ and denote by $v(\lambda)$ the unique positive solution of the equation $(1 + v)f(v,\lambda) = 1$. Then

$$\begin{aligned}\log \Delta(\lambda - a - u) &= \log \Delta(u^*(\lambda - a) - 1)\\ &= \frac{1}{2}\tau(\log(1 + v|a - \lambda|^2)) - \frac{1}{2}\log(1 + v).\end{aligned}$$

Note that this approach cannot be used in the general setting of Section 3.2, as it does not tell how to evaluate the Kadison–Fuglede determinant at $z = 0$.

For the rest of this section we assume that a is normal with spectral measure μ_a, so that we can write

$$f(v,\lambda) = \int \frac{d\mu_a(t)}{1 + v|\lambda - t|^2} \tag{4.7}$$

and again with $(1 + v)f(v,\lambda) = 1$,

$$\log \Delta(a + u - \lambda) = \frac{1}{2}\int \log(1 + v|\lambda - t|^2)\, d\mu_a(t) - \frac{1}{2}\log(1 + v).$$

For the density of the Brown measure we obtain

$$
\begin{aligned}
p(\lambda) &= \frac{2}{\pi}\,\partial_{\bar\lambda}\partial_\lambda \log \mathit{\Delta}(a+u-\lambda)\\
&= \frac{1}{\pi}\,\partial_{\bar\lambda}\left(\int \frac{|\lambda-t|^2\partial_\lambda v + v(\overline{\lambda}-\overline{t})}{1+v\,|\lambda-t|^2}\,d\mu(t) - \frac{1}{1+v}\,\partial_\lambda v\right)\\
&= \frac{1}{\pi}\,\partial_{\bar\lambda}\Bigg(\partial_\lambda v\,\underbrace{\left(\int \frac{|\lambda-t|^2}{1+v\,|\lambda-t|^2}\,d\mu(t) - \frac{1}{1+v}\right)}_{=0}\\
&\qquad + v\int \frac{\overline{\lambda}-\overline{t}}{1+v\,|\lambda-t|^2}\,d\mu(t)\Bigg)\\
&= \frac{1}{\pi}\,\partial_{\bar\lambda}\int \frac{1}{\lambda-t}\;\frac{v\,|\lambda-t|^2}{1+v\,|\lambda-t|^2}\,d\mu(t)\\
&= \frac{1}{\pi}\,\partial_{\bar\lambda}\int \frac{1}{\lambda-t}\left(1-\frac{1}{1+v\,|\lambda-t|^2}\,d\mu(t)\right)\\
&= \frac{1}{\pi}\int \frac{1}{\lambda-t}\;\frac{|\lambda-t|^2\,\partial_{\bar\lambda}v + v(\lambda-t)}{(1+v\,|\lambda-t|^2)^2}\,d\mu(t)\\
&= \frac{1}{\pi}\left(\partial_{\bar\lambda}v\int \frac{\overline{\lambda}-\overline{t}}{(1+v\,|\lambda-t|^2)^2}\,d\mu(t) + \int \frac{v}{(1+v\,|\lambda-t|^2)^2}\,d\mu(t)\right).
\end{aligned}
$$

Now by implicit differentiation,

$$
\begin{aligned}
1 &= (1+v)f(v,\lambda),\\
0 &= \partial_{\bar\lambda}v\,f(v,\lambda) + (1+v)(\partial_v f(v,\lambda)\,\partial_{\bar\lambda}v + \partial_{\bar\lambda}f(v,\lambda)),\\
\partial_{\bar\lambda}v &= -\frac{(1+v)\partial_{\bar\lambda}f}{f+(1+v)\partial_v f},\\
\partial_{\bar\lambda}f(v,\lambda) &= -\int \frac{v(\lambda-t)}{(1+v|\lambda-t|^2)^2}\,d\mu(t),\\
\partial_v f(v,\lambda) &= -\int \frac{|\lambda-t|^2}{(1+v|\lambda-t|^2)^2}\,d\mu(t),
\end{aligned}
$$

and thus

$$
\begin{aligned}
p(\lambda) = \frac{1}{\pi}\Bigg(&\frac{1+v}{v(f(v,\lambda)+(1+v)\partial_v f(v,\lambda))}|\partial_\lambda f(v,\lambda)|^2\\
&+ vf(v,\lambda) + v^2\,\partial_v f(v,\lambda)\Bigg).
\end{aligned}
\tag{4.8}
$$

We will apply this in three situations here. First consider a finite-dimensional normal operator a, like e.g. $a = u_n$, the generator of the von Neumann algebra of $\mathbb{Z}_n$. Then the integrals become finite sums and can be evaluated numerically. As an example see Figure 3, which should again be compared to the corresponding samples of spectra of random matrices in Figure 4.

There U_3 is a fixed 150×150 permutation matrix with the same spectral distribution as u_3 and U_∞ is again a 150×150 standard unitary random matrix.

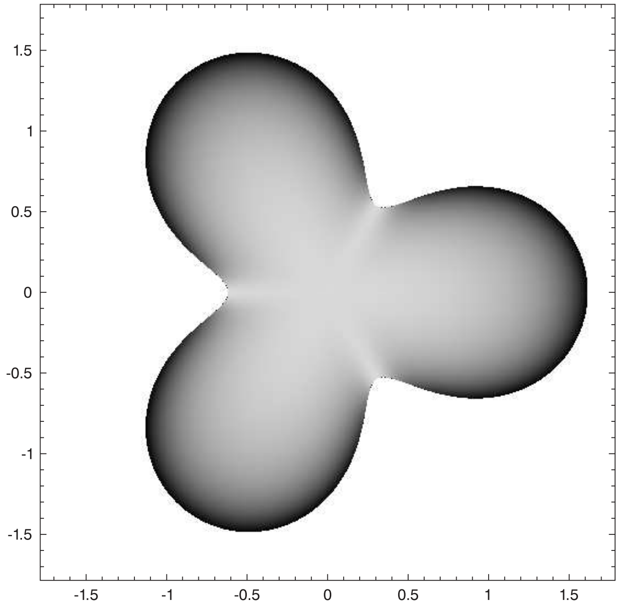

Fig. 3. Density of $\mu_{u_3+u_\infty}$

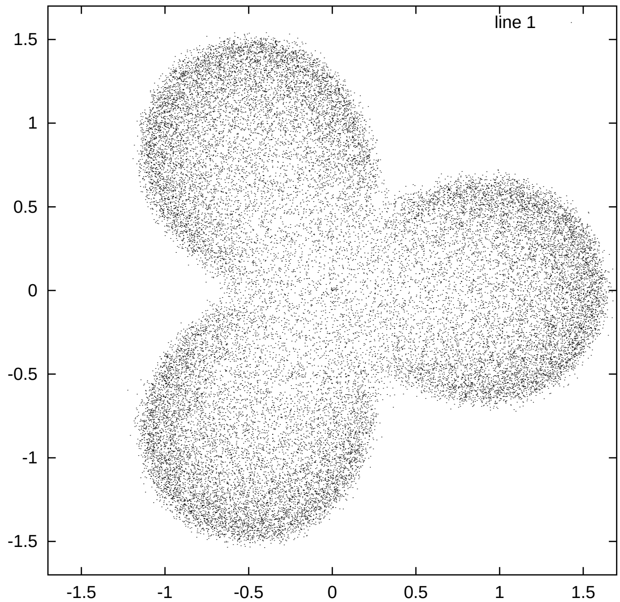

Fig. 4. 200 samples of eigenvalues of 150×150 random matrices $U_3 + U_\infty$

Secondly, assume that a is self-adjoint. Then we can factorize the denominator in the integral (4.7) as $1+v|\lambda-t|^2 = vt^2 - v(\lambda+\overline{\lambda})t + 1 + v|\lambda|^2 = v(t-z_0)(t-\overline{z}_0)$ where

$$z_0 = \operatorname{Re}\lambda + \frac{i}{v}\sqrt{v^2(\operatorname{Im}\lambda)^2 + v}.$$

From this we can express $f(v,\lambda)$ and therefore $p(\lambda)$ in terms of the Cauchy transform $G(\zeta)$ of a as follows:

$$\begin{aligned} f(v,\lambda) &= \int \frac{d\mu(t)}{v\,(t-z_0)(t-\overline{z}_0)} = \frac{1}{v}\int \frac{1}{z_0-\overline{z}_0}\left(\frac{1}{t-z_0} - \frac{1}{t-\overline{z}_0}\right) d\mu(t) \\ &= -\frac{\operatorname{Im} G(z_0)}{\sqrt{v^2(\operatorname{Im}\lambda)^2 + v}}. \end{aligned}$$

As an example consider $a = u_2 + v_2$, where u_2 and v_2 are the generators of two free copies of $\mathbb{Z}_2$. Then a is self-adjoint and distributed according to the arcsine law (or Kesten measure) and has Cauchy transform $G(\zeta) = 1/(\zeta\sqrt{1-4/\zeta^2})$. A picture of the density of the Brown measure of u_2+v_2+u is presented in Figure 5.

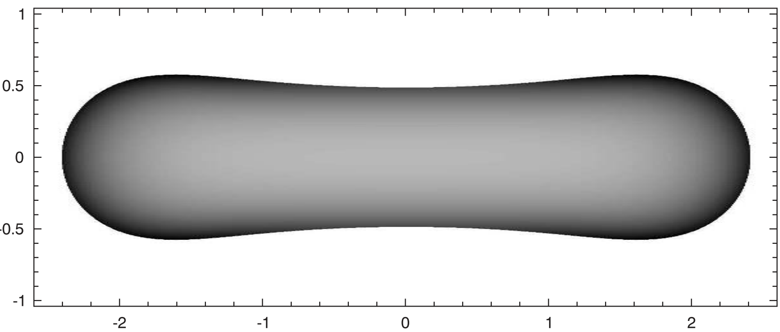

Fig. 5. Density of $\mu_{u_2+v_2+u_\infty}$

Finally, consider the free sum of an arbitrary unitary v and a Haar unitary u. Let $d\mu(\theta)$ be the spectral measure of v on the unit circle. For the evaluation of the integral (4.7) we factorize the denominator again, this time writing

$$\begin{aligned} f(v,\lambda) &= \int \frac{d\mu(\theta)}{1+v\,|\lambda-e^{i\theta}|^2} = \int \frac{d\mu(\theta)}{1+v(|\lambda|^2+1) - v(\lambda e^{-i\theta} + \overline{\lambda}e^{i\theta})} \\ &= -\int \frac{e^{i\theta}}{v\overline{\lambda}e^{2i\theta} - (1+v(|\lambda|^2+1))e^{i\theta} + v\lambda}\, d\mu(\theta) \\ &= -\frac{1}{v\overline{\lambda}}\int \frac{e^{i\theta}}{(e^{i\theta}-z_+)(e^{i\theta}-z_-)}\, d\mu(\theta) \end{aligned}$$

where

$$z_{\pm} = \frac{1}{2v\overline{\lambda}}(1 + v(|\lambda|^2 + 1) \pm \sqrt{(1 + v(|\lambda|^2 + 1)^2)\,(1 + v(|\lambda|^2 - 1)^2)}).$$

Note that $|z_+ z_-| = |\lambda/\overline{\lambda}|$ and $|z_+| > |z_-|$, and thus $|z_+| > 1 > |z_-|$. Hence

$$\begin{aligned} f(v,\lambda) &= \frac{1}{v\overline{\lambda}} \int \frac{e^{i\theta}}{z_+ z_-} \left(\frac{1}{z_+ - e^{i\theta}} - \frac{1}{z_- - e^{i\theta}} \right) d\mu(\theta) \\ &= \frac{1}{v\overline{\lambda}} \int \frac{1}{z_+ z_-} \left(\frac{z_+}{z_+ - e^{i\theta}} - \frac{z_-}{z_- - e^{i\theta}} \right) d\mu(\theta) \\ &= \frac{z_+ G(z_+) - z_- G(z_-)}{v\overline{\lambda}(z_+ - z_-)} \\ &= \frac{z_+ G(z_+) - z_- G(z_-)}{\sqrt{(1 + v(|\lambda|^2 + 1)^2)\,(1 + v(|\lambda|^2 - 1)^2)}}. \end{aligned}$$

For the determination of the spectrum (4.1) we need

$$\begin{aligned} \|(\lambda - v)^{-1}\|_2^2 &= \int \frac{d\mu(\theta)}{|\lambda - e^{i\theta}|^2} \\ &= \frac{1}{|\lambda|^2 - 1} \int \left(\frac{\lambda}{\lambda - e^{i\theta}} + \frac{\overline{\lambda}}{\overline{\lambda} - e^{-i\theta}} - 1 \right) d\mu(\theta) \\ &= \frac{\lambda G(\lambda) + \overline{\lambda} G(\overline{\lambda}) - 1}{|\lambda|^2 - 1}. \end{aligned}$$

As an example consider for $q \in [-1, 1]$ the unitary u_q with Poisson distribution, i.e. whose moments are $\tau(u_q^n) = q^{|n|}$. For $q = 0$ this is the Haar distribution, while for $q = 1$ it is the Dirac measure at 1. By Fourier transform, the density of the spectral measure is

$$d\mu_q(\theta) = \frac{1}{2\pi} \frac{1 - q^2}{|1 - qe^{i\theta}|^2}.$$

The Cauchy transform is

$$G_q(\zeta) = \begin{cases} \dfrac{1}{\zeta - q}, & |\zeta| > 1, \\ \dfrac{1}{\zeta - q^{-1}}, & |\zeta| < 1, \end{cases}$$

and from this we get the other relevant functions

$$\|(\lambda - u_q)^{-1}\|_2^2 = \begin{cases} \dfrac{|\lambda|^2 - q^2}{(|\lambda|^2 - 1)\,|\lambda - q|^2}, & |\lambda| > 1, \\ \dfrac{q^{-2} - |\lambda|^2}{(1 - |\lambda|^2)\,|\lambda - q^{-1}|^2}, & |\lambda| < 1, \end{cases}$$

$$f(v,\lambda)=\frac{qz_+-q^{-1}\,z_-}{(z_+-q)(z_--q^{-1})}\cdot\frac{1}{\sqrt{(1+v(|\lambda|^2+1)^2)\,(1+v(|\lambda|^2-1)^2)}}.$$

Substituting this into (4.8), we get pictures like Figure 6, where $q=0.7$.

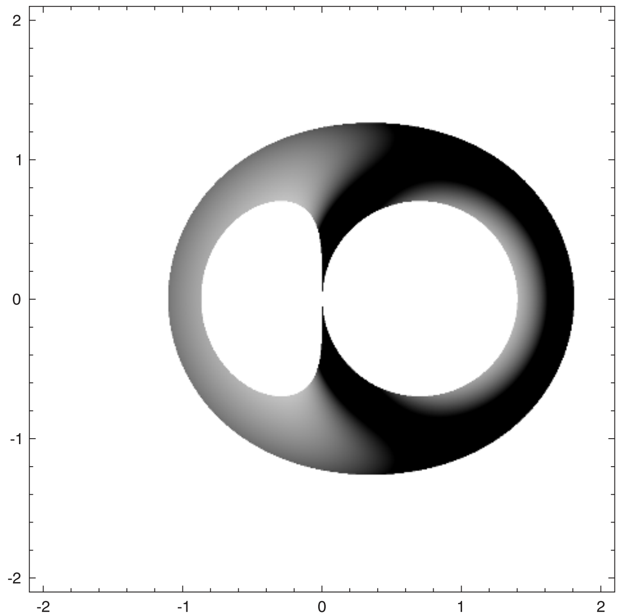

Fig. 6. Density of $\mu_{u_q+u_\infty}$ at $q=0.7$

5. Adding a circular element. A standard circular element has the $*$-distribution of $C = S_1 + iS_2$ where S_1, S_2 are free standard semicircular elements, i.e., self-adjoints whose distribution is the semicircle law $\frac{1}{2\pi}\sqrt{4-x^2}\,dx$ on $[-2,2]$. Its polar decomposition is $C=uh$ with u a Haar unitary free from h (hence C is R-diagonal), and h has the quarter circular distribution $\frac{1}{\sqrt{2}\pi}\sqrt{8-x^2}\,dx$ on $[0,\sqrt{8}]$. The symmetrized $\widetilde{h}$ in Haagerup–Larsen's decomposition $C=a\widetilde{h}$ has a semicircular distribution of variance 2. In this section we consider the Brown measure of $X_t = X_0 + C_t$, where X_0 has arbitrary $*$-distribution, it is free from C_t and C_t is a circular element of variance t, i.e. $C_t \cong \sqrt{t/2}\,C$ where C is a standard circular element. It will be convenient to assume that the C_t form a circular process, i.e., for each $s<t$, $C_t - C_s$ is $*$-free from C_s. We shall use a heat equation like approach, by differentiating in t. One has

$$\log\Delta(\lambda-X_t)=\frac{1}{2}\log\Delta(|\lambda-X_t|^2)=\frac{1}{2}\lim_{\varepsilon\to0}\log\Delta(|\lambda-X_t|^2+\varepsilon^2).$$

Define $H_{t,\varepsilon}=|\lambda-X_t|^2+\varepsilon^2$ and compute the derivative $\frac{\partial}{\partial t}\log\Delta(H_{t,\varepsilon})$. To this end let dt be small, $dC_t = C_{t+dt} - C_t$ (so that $\tau(dC_t^* dC_t) = dt$). Then

$$\begin{aligned} H_{t+dt,\varepsilon} &= |\lambda - X_{t+dt}|^2 + \varepsilon^2 = |\lambda - X_t - dC_t|^2 + \varepsilon^2 \\ &= |\lambda - X_t|^2 - (\lambda - X_t)^* dC_t - dC_t^*(\lambda - X_t) + |dC_t|^2 + \varepsilon^2 \\ &= H_{t,\varepsilon} - (\lambda - X_t)^* dC_t - dC_t^*(\lambda - X_t) + dC_t^* dC_t \\ &= H_{t,\varepsilon}[1 - H_{t,\varepsilon}^{-1}((\lambda - X_t)^* dC_t + dC_t^*(\lambda - X_t) - dC_t^* dC_t)] \end{aligned}$$

and hence

$$\begin{aligned} &\log \Delta(H_{t+dt,\varepsilon}) \\ &\quad = \log \Delta(H_{t,\varepsilon}) + \log \Delta(1 - H_{t,\varepsilon}^{-1}((\lambda - X_t)^* dC_t + dC_t^*(\lambda - X_t) - dC_t^* dC_t)) \\ &\quad = \log \Delta(H_{t,\varepsilon}) + \tau(\log |1 - H_{t,\varepsilon}^{-1}((\lambda - X_t)^* dC_t + dC_t^*(\lambda - X_t) - dC_t^* dC_t)|). \end{aligned}$$

Now observe that

$$\begin{aligned} &\tau(\log |1 + a\sqrt{dt} + b\,dt|) \\ &\qquad = \frac{1}{2}\tau(\log |1 + a\sqrt{dt} + b\,dt|^2) \\ &\qquad = \frac{1}{2}\tau(\log(1 + (a + a^*)\sqrt{dt} + (b + b^*)dt + a^* a\,dt + \mathcal{O}((dt)^{3/2})) \\ &\qquad = \frac{1}{2}\tau\left((a + a^*)\sqrt{dt} + (b + b^* + a^* a)dt - \frac{1}{2}(a + a^*)^2 dt\right) + \mathcal{O}((dt)^{3/2}) \\ &\qquad = \frac{1}{2}\tau\left((a + a^*)\sqrt{dt} + \left(b + b^* - \frac{a^2 + a^{*2}}{2}\right)dt\right) + \mathcal{O}((dt)^{3/2}). \end{aligned}$$

In our situation we have

$$\begin{aligned} a &= -H_{t,\varepsilon}^{-1}\left((\lambda - X_t)^* \frac{dC_t}{\sqrt{dt}} + \frac{dC_t^*}{\sqrt{dt}}(\lambda - X_t)\right), \\ b &= H_{t,\varepsilon}^{-1}\frac{dC_t^* dC_t}{dt}, \end{aligned}$$

so that $\tau(a) = 0$ and $\tau(b) = \tau(H_{t,\varepsilon}^{-1})$ by freeness of dC_t and $\{H_{t,\varepsilon}, \lambda - X_t\}$. Further we have

$$\begin{aligned} \tau(a^2) = \tau\Bigg(&\left(H_{t,\varepsilon}^{-1}(\lambda - X_t)^* \frac{dC_t}{\sqrt{dt}}\right)^2 + \left(H_{t,\varepsilon}^{-1} \frac{dC_t^*}{\sqrt{dt}} (\lambda - X_t)\right)^2 \\ &+ 2H_{t,\varepsilon}^{-1}(\lambda - X_t)^* \frac{dC_t}{\sqrt{dt}} H_{t,\varepsilon}^{-1} \frac{dC_t^*}{\sqrt{dt}} (\lambda - X_t)\Bigg) \end{aligned}$$

and using the formula $\tau(a_1 b_1 a_2 b_2) = \tau(a_1)\tau(a_2)\tau(b_1 b_2)$ if $\{a_1, a_2\}$ is free from $\{b_1, b_2\}$ and $\tau(b_1) = \tau(b_2) = 0$, we see that only the last term is non-zero and equal to

$$2\tau\Big((\lambda - X_t)H_{t,\varepsilon}^{-1}(\lambda - X_t)^* \frac{dC_t}{\sqrt{dt}} H_{t,\varepsilon}^{-1} \frac{dC_t^*}{\sqrt{dt}}\Big)$$
$$= 2\tau((\lambda - X_t)H_{t,\varepsilon}^{-1}(\lambda - X_t)^*)\tau(H_{t,\varepsilon}^{-1})$$
$$= 2\tau(H_{t,\varepsilon}^{-1}(H_{t,\varepsilon} - \varepsilon^2)\tau(H_{t,\varepsilon}^{-1})$$
$$= 2\tau(H_{t,\varepsilon}^{-1}) + 2\varepsilon^2\tau(H_{t,\varepsilon}^{-1})^2$$

so that

$$\frac{\log \varDelta(H_{t+dt,\varepsilon}) - \log \varDelta(H_{t,\varepsilon})}{dt}$$
$$= \frac{1}{2}\,(2\tau(H_{t,\varepsilon}^{-1}) - 2\tau(H_{t,\varepsilon}^{-1}) + 2\varepsilon^2\tau(H_{t,\varepsilon}^{-1})^2) + \mathcal{O}((dt)^{1/2})$$
$$= \varepsilon^2\tau(H_{t,\varepsilon}^{-1})^2 + \mathcal{O}((dt)^{1/2})$$

hence

$$\frac{\partial}{\partial t}\log \varDelta(H_{t,\varepsilon}) = \varepsilon^2\tau(H_{t,\varepsilon}^{-1})^2$$

and

$$\log \varDelta(H_{t,\varepsilon}^{-1}) = \log \varDelta(H_{0,\varepsilon}^{-1}) + \int_0^t \varepsilon^2\tau(H_{s,\varepsilon}^{-1})^2\, ds.$$

Let $a_{\lambda,s}$ be a self-adjoint element with symmetric distribution, whose absolute value is distributed as $|\lambda - X_s|$. Now note that by the Stieltjes inversion formula

$$\varepsilon\tau(H_{s,\varepsilon}^{-1}) = \tau(\varepsilon(|\lambda - X_s|^2 + \varepsilon^2)^{-1}) = -\tau(\mathrm{Im}[(i\varepsilon - a_{\lambda,s})^{-1}])$$
$$\underset{\varepsilon\to 0}{\longrightarrow} \pi\frac{d\mu_{a_{\lambda,s}}(x)}{dx}\bigg|_{x=0},$$

i.e., the density at 0 of the distribution of $a_{\lambda,s}$. Now we need the following

Lemma 5.1. *Let a be a self-adjoint symmetrically distributed element, free from S and C, where S and C are a semicircular and a circular element of same variance respectively. Then $|a + S|$ and $\big||a| + C\big|$ have the same distribution.*

Proof. Let b be a symmetry free from $\{a, S, C\}$. Then by [9, Prop. 4.2], ba and bS are $*$-free, thus $|a+S| = |ba+bS|$ is distributed as $|ba+C|$. Now using the fact that multiplying with a free Haar unitary u does not change the $*$-distribution of C, we can replace the latter according to $C \cong u^*C$, and get the following equalities of $*$-distributions:

$$|ba + C| \cong |ba + u^*C| \cong |uba + C| \cong \big|u|a| + C\big| \cong \big||a| + C\big|. \ \blacksquare$$

Using the lemma we get

$$|\lambda - X_s| = |\lambda - X_0 - C_s| \cong |a_\lambda + S_s|$$

where a_λ is the symmetrization of $|\lambda - X_0|$, free from the semicircular S_s and therefore

$$|a_{\lambda,s}| \cong |a_\lambda + S_s|.$$

It follows from Corollary 3 of [3, p. 711] that the distribution of $a_{\lambda,s}$ has a density at 0 which is $p_s(0) = v(s)/(\pi s)$, with

$$v(s) = \inf\left\{ v \geq 0 : \int \frac{d\mu_{|\lambda - X_0|}(x)}{x^2 + v^2} \leq \frac{1}{s} \right\}. \tag{5.1}$$

If $\lambda \notin \sigma(X_0)$, then by e.g. [3],

$$\varepsilon\tau(H_{s,\varepsilon}^{-1}) \leq \sup_{x \in \mathbb{R}} \frac{d\mu_{a_{\lambda,s}}(x)}{dx} \leq \frac{1}{\pi\sqrt{s}}.$$

Furthermore for s small enough, $\lambda \notin \sigma(X_s)$ and $\tau(|\lambda - X_s\lambda|^{-2})$ is bounded above, hence $\varepsilon\tau(H_{s,\varepsilon}^{-1})$ also, therefore we can apply the dominated convergence theorem to get

$$\begin{aligned} \log \Delta(\lambda - X_t) &= \frac{1}{2}\lim_{\varepsilon\to 0} \log \Delta(H_{0,\varepsilon}) + \frac{1}{2}\int_0^t \varepsilon^2 \tau(H_{s,\varepsilon}^{-1})^2\, ds \\ &= \log \Delta(\lambda - X_0) + \frac{1}{2}\int_0^t \frac{v(s)^2}{s^2}\, ds \\ &= \log \Delta(\lambda - X_0) + \frac{1}{2}\int_{t_\lambda}^t \frac{v(s)^2}{s^2}\, ds \end{aligned} \tag{5.2}$$

where

$$t_\lambda = \inf\{t : v(t) > 0\} = \left(\int \frac{d\mu_{|\lambda - X_0|}(x)}{x^2} \right)^{-1}.$$

So whenever $\lambda \notin \sigma(X_0)$, the density of the Brown measure is

$$p_{\lambda - X_t}(\lambda) = \frac{1}{\pi}\partial_{\bar\lambda}\partial_\lambda \int_{t_\lambda}^t \frac{v(s)^2}{s^2}\, ds = \frac{1}{\pi}\partial_{\bar\lambda}\left(\int_{t_\lambda}^t \frac{\partial_\lambda v(s)^2}{s^2}\, ds - \frac{v(t_\lambda)^2}{t_\lambda^2}\partial_\lambda t_\lambda \right)$$

and the second summand will be zero if $v(t)$ is continuous at t_λ.

Example 5.2 (2×2 matrix). Let $X_0 = a$ be as in Example 4.1, and consider $X_t = a + C_t$. Let again $\mu_\pm$ be the eigenvalues of $(\lambda - a)^*(\lambda - a)$. Then the relevant parameters are

$$\|\lambda - a\|_2^2 = \frac{\mu_+ + \mu_-}{2},$$

$$\|(\lambda - a)^{-1}\|_2^2 = \frac{1}{2}\left(\frac{1}{\mu_+} + \frac{1}{\mu_-}\right) = \frac{\|\lambda - a\|_2^2}{\det|\lambda - a|^2},$$

$$t_\lambda = \left(\int \frac{d\mu_{|\lambda-a|}(x)}{x^2}\, dx\right)^{-1} = \|(\lambda - a)^{-1}\|_2^{-2} = \frac{\det|\lambda - a|^2}{\|\lambda - a\|_2^2}.$$

The function $v(s)^2$ is the solution of the quadratic equation

$$\begin{aligned}\frac{1}{s} &= \frac{1}{2}\left(\frac{1}{\mu_+ + v^2} + \frac{1}{\mu_- + v^2}\right)\\ &= \frac{1}{2}\frac{\mu_+ + \mu_- + 2v^2}{\mu_+\mu_- + (\mu_+ + \mu_-)v^2 + v^4}\\ &= \frac{\|\lambda - a\|_2^2 + v^2}{\det|\lambda - a|^2 + 2\|\lambda - a\|_2^2\, v^2 + v^4},\end{aligned}$$

which is explicitly

$$\begin{aligned}v(s)^2 &= \frac{1}{2}(s - 2\|\lambda - a\|_2^2 \pm \sqrt{(s - 2\|\lambda - a\|_2^2)^2 - 4(\det|\lambda - a|^2 - s\|\lambda - a\|_2^2)})\\ &= \frac{1}{2}(s - 2\|\lambda - a\|_2^2 \pm \sqrt{s^2 + 4(\|\lambda - a\|_2^4 - \det|\lambda - a|^2)}).\end{aligned}$$

Now we have to choose the right branch of the square root. To this end, let us compute the spectrum of X_t: Assume $\lambda \notin \sigma(a)$; then $\lambda \in \sigma(a + C_t)$ if and only if $1 - C_t(\lambda - a)^{-1}$ is not invertible. Now $C_t(\lambda - a)^{-1}$ is R-diagonal and not invertible, so by Theorem 2.4(vi), 1 is in its spectrum if and only if its spectral radius is at least 1 and using Proposition 2.5 we get the inequality

$$1 \le \varrho(C_t(\lambda - a)^{-1}) = \|C_t\|_2\|(\lambda - a)^{-1}\|_2;$$

in other words,

$$\det|\lambda - a|^2 \le t\|\lambda - a\|_2^2$$

and hence for $s < t$, $\det|\lambda - a|^2 - s\|\lambda - a\|_2^2 < 0$, and only the "+" branch gives a non-negative solution. Consequently,

$$\begin{aligned}&\log\Delta(\lambda - X_t) - \log\Delta(\lambda - X_0)\\ &\quad= \frac{1}{2}\int_{t_\lambda}^{t}\left(\frac{1}{2s} - \frac{\|\lambda - a\|_2^2}{s^2} + \frac{\sqrt{s^2 + 4(\|\lambda - a\|_2^4 - \det|\lambda - a|^2)}}{2s^2}\right) ds\\ &\quad= \frac{1}{4}\log s + \frac{\|\lambda - a\|_2^2}{2s}\\ &\qquad+ \frac{1}{4}\log(s + \sqrt{s^2 + 4(\|\lambda - a\|_2^4 - \det|\lambda - a|^2)})\\ &\qquad- \frac{1}{4s}\sqrt{s^2 + 4(\|\lambda - a\|_2^4 - \det|\lambda - a|^2)}\Big|_{s=t_\lambda}^{t}.\end{aligned}$$

Now observe that

$$\sqrt{t_\lambda^2 + 4(\|\lambda - a\|_2^4 - \det|\lambda - a|^2)} = \frac{2\|\lambda - a\|_2^4 - \det|\lambda - a|^2}{\|\lambda - a\|_2^2}$$

and hence, setting

$$R(\lambda) = 4(\|\lambda - a\|_2^4 - \det|\lambda - a|^2) = (\mu_+ - \mu_-)^2 \tag{5.3}$$

we get

$$\begin{aligned}
&\log \Delta(\lambda - X_t) - \log \Delta(\lambda - X_0)\\
&\quad= \frac{1}{4}\log t + \frac{\|\lambda - a\|_2^2}{2t} + \frac{1}{4}\log(t + \sqrt{t^2 + R(\lambda)}) - \frac{1}{4t}\sqrt{t^2 + R(\lambda)}\\
&\qquad - \frac{1}{4}\log\frac{\det|\lambda - a|^2}{\|\lambda - a\|_2^2} - \frac{\|\lambda - a\|_2^4}{2\det|\lambda - a|^2}\\
&\qquad - \frac{1}{4}\log\frac{\det|\lambda - a|^2 + 2\|\lambda - a\|_2^4 - \det|\lambda - a|^2}{\|\lambda - a\|_2^2}\\
&\qquad + \frac{\|\lambda - a\|_2^2}{4\det|\lambda - a|^2}\cdot\frac{2\|\lambda - a\|_2^4 - \det|\lambda - a|^2}{\|\lambda - a\|_2^2}\\
&\quad= \frac{1}{4}\log t + \frac{\|\lambda - a\|_2^2}{2t} + \frac{1}{4}\log(t + \sqrt{t^2 + R(\lambda)}) - \frac{1}{4t}\sqrt{t^2 + R(\lambda)}\\
&\qquad - \frac{1}{4}\log\det|\lambda - a|^2 - \frac{1}{4}\log 2 - \frac{1}{4}
\end{aligned}$$

and finally the density is (note that $\partial_{\bar\lambda}\partial_\lambda \log\det|\lambda - a|^2 = 0$ and $\partial_{\bar\lambda}\partial_\lambda\|\lambda - a\|_2^2 = 1$)

$$\begin{aligned}
p_{a+C_t}(\lambda) &= \frac{2}{\pi}\,\partial_{\bar\lambda}\partial_\lambda \log\Delta(\lambda - X_t)\\
&= \frac{1}{\pi t} + \frac{1}{2\pi}\,\partial_{\bar\lambda}\partial_\lambda\left(\log(t + \sqrt{t^2 + R(\lambda)}) - \frac{\sqrt{t^2 + R(\lambda)}}{t}\right)\\
&= \frac{1}{\pi t} + \frac{1}{2\pi}\,\partial_{\bar\lambda}\left(\frac{1}{t + \sqrt{t^2 + R(\lambda)}} - \frac{1}{t}\right)\frac{\partial_\lambda R(\lambda)}{2\sqrt{t^2 + R(\lambda)}}\\
&= \frac{1}{\pi t} + \frac{1}{4\pi}\,\partial_{\bar\lambda}\left(\frac{t - \sqrt{t^2 + R(\lambda)}}{tR(\lambda)}\partial_\lambda R(\lambda)\right)\\
&= \frac{1}{\pi t} + \frac{1}{4\pi t}\Bigg(-\frac{|\partial_\lambda R(\lambda)|^2}{2R(\lambda)\sqrt{t^2 + R(\lambda)}}\\
&\qquad + (t - \sqrt{t^2 + R(\lambda)})\frac{R(\lambda)\partial_{\bar\lambda}\partial_\lambda R(\lambda) - |\partial_\lambda R(\lambda)|^2}{R(\lambda)^2}\Bigg).
\end{aligned}$$

Again we can specify to $a = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, and we get the spectrum

$$\sigma(a + C_t) = \{\lambda : |\lambda^2 - 1|^2 \leq t(|\lambda|^2 + 1)\}.$$

Note that for $t = 1$ this is the same as $\sigma(u_2 + u)$ from Example 4.1. However this time the density is a function of the real part alone, namely substituting $\mu_\pm = |\lambda \pm 1|^2$ into (5.3), we get $R(\lambda) = 4(\lambda + \overline{\lambda})^2$ and consequently the density depends only on the real part:

$$p_{a+C_t}(x + iy) = \frac{1}{\pi t} + \frac{1}{8\pi x^2}\left(\frac{t}{\sqrt{t^2 + 16x^2}} - 1\right).$$

The situation for the nilpotent 2×2 matrix $a = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ is as follows. We have computed the eigenvalues of $|\lambda - a|^2$ in (4.6), and thus

$$\sigma(a + C_t) = t\{\lambda : 2|\lambda|^4 \leq t(1 + 2|\lambda|^2)\},$$

which is the disk with radius $\sqrt{\sqrt{t^2/4 + 1/2} + t/2}$. This is the same as $\sigma(a + \sqrt{t}\, u)$, but with the possible hole removed. Furthermore we get $R(\lambda) = (\mu_+ - \mu_-)^2 = 1 + 4|\lambda|^2$ and the density function is again rotationally symmetric:

$$p_{a+C_t}(\lambda) = \frac{1}{\pi t}\left(1 - \frac{2|\lambda|^2}{(1 + 4|\lambda|^2)\sqrt{t^2 + 1 + 4|\lambda|^2}} + \frac{t - \sqrt{t^2 + R}}{(1 + 4|\lambda|^2)^2}\right).$$

EXAMPLE 5.3 (Elliptic law). An interesting example is given by the so-called *elliptic random variable* $S_\alpha + iS_\beta$, where S_α and S_β are free semicircular variables of variances α and β. Note that for $\alpha = \beta$ this is a circular variable $C_{2\alpha}$. The Brown measure has been computed by Haagerup (unpublished) by another method. The name *elliptic* stems from the shape of its spectrum, which is an ellipse. This can be seen as follows. Assuming that $\alpha > \beta$ let $\gamma = \alpha - \beta$. Then for $\lambda \notin \sigma(S_\gamma) = [-2\sqrt{\gamma}, 2\sqrt{\gamma}]$ we have $\lambda \in \sigma(S_\gamma + C_{2\beta})$ if and only if $1 - C_{2\beta}(\lambda - S_\gamma)^{-1}$ is not invertible. From Theorem 2.4 we infer that the spectrum of $C_{2\beta}(\lambda - S_\gamma)^{-1}$ is the disk centred at zero with radius $\|C_{2\beta}(\lambda - S_\gamma)^{-1}\|_2$, so that we get

$$\sigma(S_\gamma + C_{2\beta}) = \{\lambda : 1 \leq 2\beta\, \|(\lambda - S_\gamma)^{-1}\|_2^2\}.$$

We use formula (3.5) for the Cauchy transform

$$G_{S_\gamma}(\zeta) = \frac{\zeta - \sqrt{\zeta^2 - 4\gamma}}{2\gamma}$$

to get

$$\|(\lambda - S_\gamma)^{-1}\|_2^2 = \frac{1}{2\gamma}\left(\frac{\sqrt{\lambda^2 - 4\gamma} - \sqrt{\overline{\lambda}^2 - 4\gamma}}{\lambda - \overline{\lambda}} - 1\right) \tag{5.4}$$

and hence the spectrum is

$$\left\{\lambda : \frac{\sqrt{\lambda^2-4\gamma}-\sqrt{\overline{\lambda}^2-4\gamma}}{\lambda-\overline{\lambda}} \geq \frac{\gamma+\beta}{\beta} = \frac{\alpha}{\beta}\right\}.$$

Now consider the Zhukowski transformation $f : \xi \mapsto 1/\xi + \gamma\xi$, which maps the circles $\{e^{i\theta}/t : 0 \leq \theta < 2\pi\}$ to the ellipses

$$\{(\gamma/t + t)\cos\theta + i(\gamma/t - t)\sin\theta : 0 \leq \theta < 2\pi\}$$

and hence the open disk $\{\xi : |\xi| < 1/\sqrt{\gamma}\}$ bijectively onto $\mathbb{C} \setminus [-2\sqrt{\gamma}, 2\sqrt{\gamma}]$. Note that the excluded interval is exactly the spectrum of S_γ. So assume that $\lambda = f(\xi)$ with $|\xi| < 1/\sqrt{\gamma}$ is not in the spectrum of S_γ. Then observe that

$$\lambda^2 - 4\gamma = 1/\xi^2 + 2\gamma + \gamma^2\xi^2 - 4\gamma = (1/\xi - \gamma\xi)^2$$

and hence $\lambda \in \sigma(S_\gamma + C_{2\beta})$ if and only if

$$\frac{\alpha}{\beta} \leq \frac{1/\xi - 1/\overline{\xi} - \gamma\xi + \gamma\overline{\xi}}{1/\xi - 1/\overline{\xi} + \gamma\xi - \gamma\overline{\xi}} = \frac{1+\gamma|\xi|^2}{1-\gamma|\xi|^2}.$$

This inequality reduces to

$$|\xi|^2 \geq \frac{1}{\alpha+\beta},$$

thus

$$\sigma(S_\gamma + S_{2\beta}) \setminus [-2\sqrt{\gamma}, 2\sqrt{\gamma}] = \left\{f(\xi) : \frac{1}{\sqrt{\alpha+\beta}} \leq |\xi| < \frac{1}{\sqrt{\gamma}}\right\}$$

and taking the closure of this set we obtain $\sigma(S_\gamma + C_{2\beta})$ as the interior of the ellipse

$$\left\{\frac{2\alpha}{\sqrt{\alpha+\beta}}\cos\theta + \frac{2\beta}{\sqrt{\alpha+\beta}}\, i\sin\theta : 0 \leq \theta < 2\pi\right\}. \tag{5.5}$$

Now let us turn to the Brown measure. As already noted, the method from Section 3.2 will not work on $a = S_\gamma$. Indeed the R-transform of $|\lambda - S_\gamma|^2$ can be computed from the inverse of

$$G_{|\lambda-S_\gamma|^2}(\zeta) = \frac{1}{2\gamma}\left(1 - \frac{\sqrt{x_+^2-4\gamma}-\sqrt{x_-^2-4\gamma}}{x_+-x_-}\right)$$

where $x_\pm$ are as in (3.3). Let $\lambda = \xi + i\eta$. Then we can rewrite $x_\pm = \xi \pm \sqrt{\zeta - \eta^2}$ and abbreviating $y = \sqrt{\zeta-\eta^2}$, solve the equation $G_{|\lambda-S_\gamma|^2}(\zeta) = z$ for y, which gives

$$y^2 = \frac{\xi^2}{(1-2\gamma z)^2} + \frac{1}{z(1-\gamma z)}.$$

It follows that $K(z) = y^2 + \eta^2$ and

$$R_{|\lambda - S_\gamma|^2}(z) = z\,K(z) - 1 = \frac{\gamma z}{1-\gamma z} + \frac{\xi^2 z}{(1-2\gamma z)^2} + \eta^2 z;$$

for real λ this has been used in [14] to characterize the semicircular distributions. In order to get the determining series $f_{u|\lambda - S_\gamma|}$ according to (3.2) one has to solve a fourth order equation, which is not suitable for further computations. So we have to use formula (5.2), for which we need $v(s)$ from (5.1) first. We have done most of the work already, since

$$\int \frac{d\mu(x)}{|\lambda - x|^2 + v^2} = -G_{|\lambda - S_\gamma|^2}(-v^2),$$

thus

$$v(s)^2 = -K_{|\lambda - S_\gamma|^2}\left(-\frac{1}{s}\right) = -\left(\frac{\xi^2 s^2}{(s+2\gamma)^2} - \frac{s^2}{s+\gamma} + \eta^2\right)$$

and

$$\frac{v(s)^2}{s^2} = -\frac{(\lambda+\overline{\lambda})^2}{4(s+2\gamma)^2} + \frac{1}{s+\gamma} - \frac{(\lambda-\overline{\lambda})^2}{4s^2}.$$

and the density becomes

$$\begin{aligned} (5.6) \qquad p_{S_\alpha + iS_\beta}(\lambda) &= \frac{1}{\pi}\,\partial_{\overline{\lambda}} \int_{t_\lambda}^{2\beta} \frac{\partial_\lambda v(s)^2}{s^2}\,ds \\ &= \frac{1}{\pi}\,\partial_{\overline{\lambda}} \int_{t_\lambda}^{2\beta} \left(-\frac{2(\lambda+\overline{\lambda})}{4(s+2\gamma)^2} + \frac{2(\lambda-\overline{\lambda})}{4s^2}\right) ds \\ &= \frac{1}{2\pi}\,\partial_{\overline{\lambda}} \left(\frac{\lambda+\overline{\lambda}}{s+2\gamma} - \frac{\lambda-\overline{\lambda}}{s}\right)\Bigg|_{t_\lambda}^{2\beta} \\ &= \frac{1}{4\pi}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) - \frac{1}{2\pi}\partial_{\overline{\lambda}}\left(\frac{\lambda+\overline{\lambda}}{t_\lambda+2\gamma} - \frac{\lambda-\overline{\lambda}}{t_\lambda}\right). \end{aligned}$$

Now $t_\lambda = \|(\lambda - S_\gamma)^{-1}\|_2^{-2}$ has been computed above in (5.4), and if we set $\omega = \sqrt{\lambda^2 - 4\gamma}$, it is

$$t_\lambda = 2\gamma\left(\frac{\omega - \overline{\omega}}{\lambda - \overline{\lambda}} - 1\right)^{-1}.$$

We now claim that the second summand in (5.6) is zero. For this note that $\omega^2 - \overline{\omega}^2 = \lambda^2 - \overline{\lambda}^2$ and hence

$$
\begin{aligned}
-\frac{1}{2\pi}\,\partial_{\bar\lambda}&\biggl(\frac{\lambda+\overline{\lambda}}{t_\lambda+2\gamma}-\frac{\lambda-\overline{\lambda}}{t_\lambda}\biggr)\\
&=-\frac{1}{4\pi\gamma}\,\partial_{\bar\lambda}\biggl((\lambda+\overline{\lambda})\biggl(1-\frac{\lambda-\overline{\lambda}}{\omega-\overline{\omega}}\biggr)-(\lambda-\overline{\lambda})\biggl(\frac{\omega-\overline{\omega}}{\lambda-\overline{\lambda}}-1\biggr)\biggr)\\
&=-\frac{1}{4\pi\gamma}\,\partial_{\bar\lambda}((\lambda+\overline{\lambda})-(\omega+\overline{\omega})-(\omega-\overline{\omega})+(\lambda-\overline{\lambda}))\\
&=-\frac{1}{4\pi\gamma}\,\partial_{\bar\lambda}(2\lambda-2\omega)=0.
\end{aligned}
$$

Thus we see that the density is constant $(4\pi)^{-1}(1/\alpha+1/\beta)$ on the interior of the ellipse (5.5).

The elliptic law appears in the random matrix literature in [8].

6. Other examples. There are some other examples that can be done by ad-hoc methods.

EXAMPLE 6.1. Consider two freely independent symmetries u_2 and v_2 of trace zero, for example the generators of the left regular representation of $\mathbb{Z}_2 * \mathbb{Z}_2$. Here we compute the Brown measure of $T=\alpha u_2+\beta v_2$. To get its spectrum, look at its square

$$(\alpha u_2+\beta v_2)^2=\alpha^2+\beta^2+\alpha\beta(u_2v_2+v_2u_2).$$

Since $u_2v_2=(v_2u_2)^*$ is a Haar unitary, we see that T^2 is a normal element with spectrum $\sigma(T^2)=\alpha^2+\beta^2+\alpha\beta[-2,2]$. Since T and $-T$ have the same distribution, it follows that

$$\sigma(\alpha u_2+\beta v_2)=\{\pm\sqrt{\alpha^2+\beta^2+\alpha\beta t}:t\in[-2,2]\}.$$

The Brown measure can be deduced by the same symmetry considerations, but for the sake of simplicity let us consider the special case $\alpha=1$, $\beta=i$ only. Here the spectrum is the union of the complex intervals $[-1-i,1+i]$ and $[-1+i,1-i]$. The Brown measure of $(u_2+iv_2)^2=i(u_2v_2+v_2u_2)$ is the arcsine law (we are taking the real part of a Haar unitary)

$$d\nu(t)=\frac{dt}{\pi\sqrt{4-t^2}}$$

on the imaginary axis. By symmetry considerations we must have the same measure on each of the four "legs" of the spectrum, call it μ_0, which must satisfy

$$\int_0^{\sqrt2} f(t^2)\,d\mu_0(t)=\frac12\int_0^2 f(t)\frac{dt}{\pi\sqrt{4-t^2}}=\int_0^{\sqrt2} f(u^2)\frac{u}{\pi\sqrt{4-u^4}}\,du$$

and it follows that the density of the Brown measure is

$$d\mu\left(\frac{1\pm i}{\sqrt{2}}\,t\right) = d\mu_0(|t|) = \frac{|t|}{\pi\sqrt{4-t^4}}\,dt.$$

EXAMPLE 6.2. Other examples that are perhaps attackable arise from the following matrix models. Consider U_2+A, where $U_2 \in U(2N)$ is a unitary matrix such that $U_2 = U_2^*$ and $\operatorname{tr} U_2 = 0$, while A is an arbitrary $2N \times 2N$ matrix. The spectrum of $U_2 + A$ can be bounded as follows. Assume x is a unit eigenvector of $U_2 + A$ with eigenvalue λ. Then it can be decomposed along the spectral projections of U_2: $x = x_+ + x_-$ so that $U_2 x = x_+ - x_-$. By assumption we also have $(U_2 + A)(x_+ + x_-) = \lambda(x_+ + x_-)$, and thus

$$x_+ = \frac{1}{2}(1+\lambda-A)x, \qquad x_- = \frac{1}{2}(1-\lambda-A)x;$$

now by orthogonality $\langle x_+, x_-\rangle = 0$ we get

$$\begin{aligned}
0 &= \langle (1+\lambda-A)x, (1-\lambda+A)x\rangle \\
&= (1+\lambda)(1-\overline{\lambda})\|x\|^2 + (1+\lambda)\langle x, Ax\rangle - (1-\overline{\lambda})\langle Ax, x\rangle - \|Ax\|^2 \\
&= (1+\lambda-\overline{\lambda}-|\lambda|^2)\|x\|^2 + (\lambda+1)\overline{\langle Ax, x\rangle} + (\overline{\lambda}-1)\langle Ax, x\rangle - \|Ax\|^2.
\end{aligned}$$

Separate real and imaginary part results in two equations:

$$1 - |\lambda|^2 - \|Ax\|^2 + \lambda\,\overline{\langle Ax, x\rangle} + \overline{\lambda}\langle Ax, x\rangle = 0,$$
$$\lambda - \overline{\lambda} + \overline{\langle Ax, x\rangle} - \langle Ax, x\rangle = 0.$$

Let us now consider two specific cases.

- *A is unitary*: In this case $\|Ax\| = 1$ and $\varrho = \langle Ax, x\rangle$ satisfies

$$-|\lambda|^2 + \lambda\overline{\varrho} + \overline{\lambda}\varrho = 0, \qquad \lambda - \overline{\lambda} = \varrho - \overline{\varrho},$$

or in other words

$$|\lambda - \varrho|^2 = |\varrho|^2, \qquad \operatorname{Im}\lambda = \operatorname{Im}\varrho.$$

Thus $\lambda - \varrho$ is real and we have $\lambda - \varrho = \pm|\varrho|$, i.e.,

$$\lambda \in \{\varrho \pm |\varrho| : \varrho = \langle Ax, x\rangle \in \operatorname{co}\sigma(A)\}.$$

- *$A = iB$ is purely imaginary*: Here we assume $A + A^* = 0$ and the equations are

$$1 - |\lambda|^2 - \|Bx\|^2 + i(\overline{\lambda}-\lambda)\langle Bx, x\rangle = 0, \qquad \lambda - \overline{\lambda} = 2i\,\langle Bx, x\rangle$$

Hence

$$\operatorname{Im}\lambda = \langle Bx, x\rangle, \qquad (\operatorname{Re}\lambda)^2 = 1 - \|Bx\|^2 + \langle Bx, x\rangle^2.$$

If one puts $A = UU_3U^*$, where U_3 is a $6N \times 6N$ model of the generator of $\mathbb{Z}_3$, and U is a random unitary $6N\times 6N$ matrix, then possible eigenvalues are enclosed in the region shown in Figure 7. And indeed, samples of small

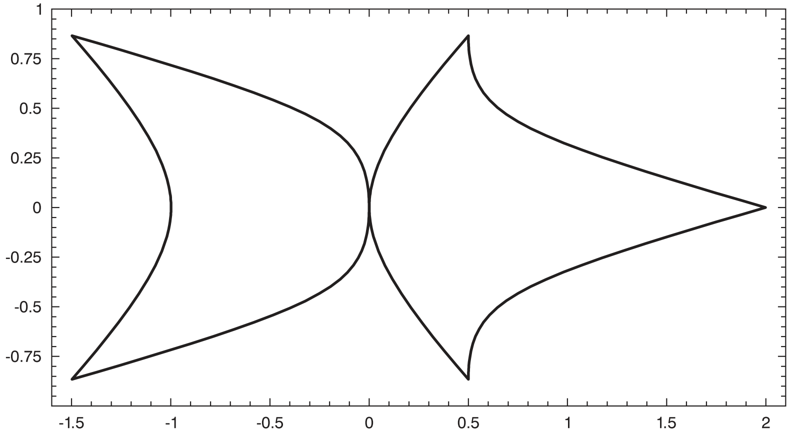

Fig. 7. Possible spectra of random $U_2 + U_3$

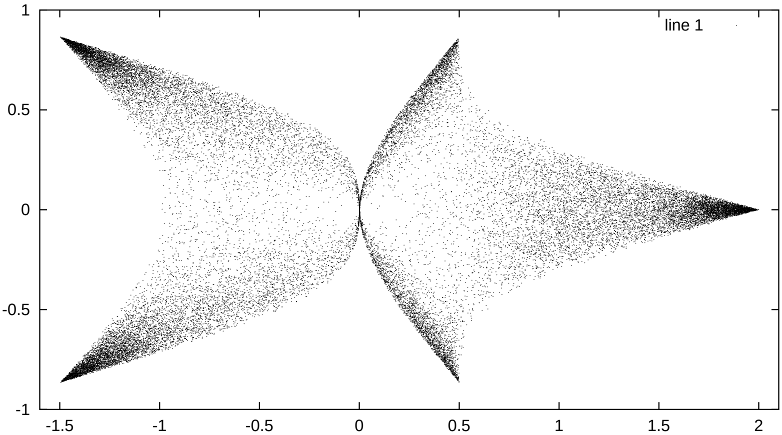

Fig. 8. 5000 samples of eigenvalues of 6×6 random matrices $U_2 + U_3$

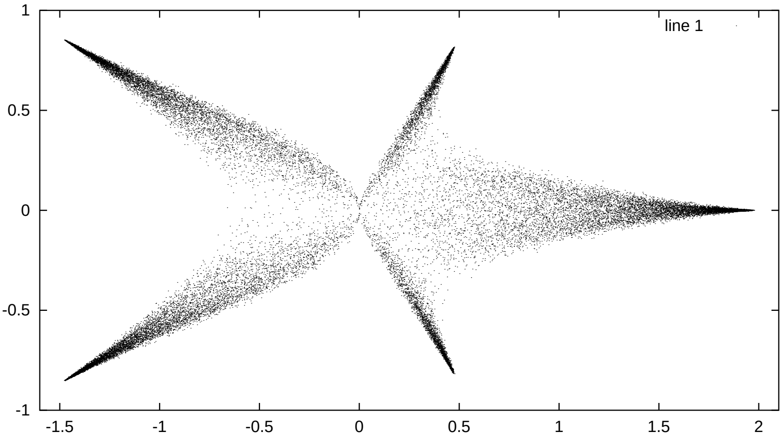

Fig. 9. 200 samples of eigenvalues of 150×150 random matrices $U_2 + U_3$

numeric random unitary matrices $U_2 + UU_3U^*$ have an eigenvalue density as shown in Figure 8, while in higher dimensions the eigenvalues concentrate (cf. Figure 9). We have been able to compute the border of the spectrum of free sums like $u_2 + u_3$ recently [16] and will investigate this topic further in future work.

REFERENCES

[1] Z. D. Bai, *Circular law*, Ann. Probab. 25 (1997), 494–529.

[2] C. Béguin, A. Valette, and A. Żuk, *On the spectrum of a random walk on the discrete Heisenberg group and the norm of Harper's operator*, J. Geom. Phys. 21 (1997), 337–356.

[3] P. Biane, *On the free convolution with a semi-circular distribution*, Indiana Univ. Math. J. 46 (1997), 705–718.

[4] L. G. Brown, *Lidskiĭ's theorem in the type* II *case*, in: Geometric Methods in Operator Algebras (Kyoto, 1983), Longman Sci. Tech., Harlow, 1986, 1–35.

[5] B. Fuglede and R. V. Kadison, *Determinant theory in finite factors*, Ann. of Math. 55 (1952), 520–530.

[6] V. L. Girko, *The circular law*, Teor. Veroyatnost. i Primenen. 29 (1984), 669–679 (in Russian).

[7] —, *Strong circular law*, Random Oper. Stochastic Equations 5 (1997), 173–196.

[8] —, *Strong elliptic law*, ibid., 269–306.

[9] U. Haagerup and F. Larsen, *Brown's spectral distribution measure for R-diagonal elements in finite von Neumann algebras*, J. Funct. Anal. 176 (2000), 331–367.

[10] P. de la Harpe, A. G. Robertson, and A. Valette, *On the spectrum of the sum of generators for a finitely generated group*, Israel J. Math. 81 (1993), 65–96.

[11] —, —, —, *On the spectrum of the sum of generators of a finitely generated group. II*, Colloq. Math. 65 (1993), 87–102.

[12] F. Hiai and D. Petz, *Logarithmic energy as an entropy functional*, in: Advances in Differential Equations and Mathematical Physics (Atlanta, GA, 1997), Amer. Math. Soc., Providence, RI, 1998, 205–221.

[13] —, —, *The Semicircle Law, Free Random Variables and Entropy*, Amer. Math. Soc., Providence, RI, 2000.

[14] O. Hiwatashi, T. Kuroda, M. Nagisa, and H. Yoshida, *The free analogue of noncentral chi-square distributions and symmetric quadratic forms in free random variables*, Math. Z. 230 (1999), 63–77.

[15] F. Larsen, *Brown measures and R-diagonal elements in finite von Neumann algebras*, Ph.D. thesis, University of Southern Denmark, 1999.

[16] F. Lehner, *On the computation of spectra in free probability*, J. Funct. Anal., 2001, to appear.

[17] A. Nica and R. Speicher, *R-diagonal pairs—a common approach to Haar unitaries and circular elements*, in: Free Probability Theory (Waterloo, ON, 1995), Fields Inst. Commun. 12, Amer. Math. Soc., Providence, RI, 1997, 149–188.

[18] —, —, *Commutators of free random variables*, Duke Math. J. 92 (1998), 553–592.

[19] P. Śniady and R. Speicher, *Continuous family of invariant subspaces for R-diagonal operators*, preprint, 2000.

[20] D. Voiculescu, *Lectures on free probability theory*, in: Lectures on Probability Theory and Statistics (Saint-Flour, 1998), Springer, Berlin, 2000, 279–349.
[21] D. V. Voiculescu, K. J. Dykema, and A. Nica, *Free Random Variables*, Amer. Math. Soc., Providence, RI, 1992.

CNRS
DMA
45 rue d'Ulm
75005 Paris, France
E-mail: Philippe.Biane@ens.fr

Institut für Analysis und Numerik
Johannes Kepler Universität Linz
4040 Linz, Austria
E-mail: lehner@bayou.uni-linz.ac.at

Current address:
Équipe d'Analyse
Université Paris 6
Boîte 186, Tour 46-0
4, place Jussieu
75252 Paris Cedex 05, France

Received 25 January 2000;
revised 18 October 2000 (3873)