Computation of Tangent, Euler, and Bernoulli Numbers*

By Donald E. Knuth and Thomas J. Buckholtz

Abstract. Some elementary methods are described which may be used to calculate tangent numbers, Euler numbers, and Bernoulli numbers much more easily and rapidly on electronic computers than the traditional recurrence relations which have been used for over a century. These methods have been used to prepare an accompanying table which extends the existing tables of these numbers. Some theorems about the periodicity of the tangent numbers, which were suggested by the tables, are also proved.

1. Introduction. The tangent numbers T_n, Euler numbers E_n, and Bernoulli numbers B_n, are defined to be the coefficients in the following power series:

$$\tan z = T_0/0! + T_1 z/1! + T_2 z^2/2! + \cdots = \sum_{n \geqq 0} T_n z^n/n!\,, \tag{1}$$

$$\sec z = E_0/0! + E_1 z/1! + E_2 z^2/2! + \cdots = \sum_{n \geqq 0} E_n z^n/n!\,, \tag{2}$$

$$z/(e^z - 1) = B_0/0! + B_1 z/1! + B_2 z^2/2! + \cdots = \sum_{n \geqq 0} B_n z^n/n!\,. \tag{3}$$

Much of the older mathematical literature uses a slightly different notation for these numbers, to take account of the zero coefficients. Thus we find many papers where $\tan z$ is written $T_1 z + T_2 z^3/3! + T_3 z^5/5! + \cdots$, $\sec z$ is written $E_0 + E_1 z^2/2! + E_2 z^4/4! + \cdots$, and $z/(e^z - 1)$ is written $1 - z/2 + B_1 z^2/2! - B_2 z^4/4! + B_3 z^6/6! \cdots$. Some other authors have used essentially the notation defined above but with different signs; in particular our E_{2n} is often accompanied by the sign $(-1)^n$.

In Section 2 we present simple methods for computing T_n, E_n, and B_n which are readily adapted to electronic computers, and in Section 3 more details of the computer program are explained. A table of T_n and E_n for $n \leqq 120$, and B_n for $n \leqq 250$, is appended to this paper, thereby extending the hitherto published values of T_n for $n \leqq 60$ [6], E_n for $n \leqq 100$ [2, 3], and B_n for $n \leqq 220$ [7, 4].

Using the methods of this paper it is not difficult to extend the tables much further, and the authors have submitted a copy of the values of T_n $(n \leqq 835)$, E_n $(n \leqq 808)$, B_n $(n \leqq 836)$ to the Unpublished Mathematical Tables repository of this journal.

Section 4 shows how the formulas of Section 2 lead to some simple proofs of arithmetical properties of these numbers.

2. Formulas for Computation. The traditional method of calculating T_n and E_n is to use recurrence relations, such as the following: Let $\cos z = \sum_{n \geqq 0} C_n z^n/n$;

Received February 6, 1967.
* Supported in part by NSF Grant GP 3909.

then the coefficient of $z^n/n!$ in $(\tan z)\ (\cos z)$ is

$$\sum_k \binom{n}{k} T_k C_{n-k}$$

and in $(\sec z)\ (\cos z)$ it is

$$\sum_k \binom{n}{k} E_k C_{n-k}\ .$$

Hence, making use of the fact that $T_{2n} = E_{2n+1} = 0$, we have the recurrence relations

$$(4)\quad \binom{2n+1}{1}T_1 - \binom{2n+1}{3}T_3 + \cdots + (-1)^n \binom{2n+1}{2n+1} T_{2n+1} = 1\,, \qquad n \geqq 0\,;$$

$$(5)\quad \binom{2n}{0}E_0 - \binom{2n}{2}E_2 + \cdots + (-1)^n \binom{2n}{2n} E_{2n} = 0\,, \qquad n > 0\,.$$

The disadvantage of these formulas is that the binomial coefficients as well as the numbers T_n, E_n become very large when n is large, so a time-consuming multiplication of multiple-precision numbers is implied. As Lehmer [4] has observed, we may simplify the calculations if we remember the values of

$$\binom{2n+1}{k}T_k\,, \qquad \binom{2n}{k}E_k$$

so that when n increases by 1 we need only multiply

$$\binom{2n+1}{k}T_k$$

by

$$\frac{(2n+2)(2n+3)}{(2n+2-k)(2n+3-k)}$$

to get the next value; but the method to be described here is even simpler and has other advantages.

The tangent numbers may be evaluated by noting that $D(\tan^n z)$ is $n \tan^{n-1} z\ (1 + \tan^2 z)$; hence the nth derivative of $\tan z$ is a polynomial in $\tan z$. We have $D^n(\tan z) = P_n(\tan z)$, where the polynomials $P_n(x)$ are defined by

$$(6)\qquad P_1(x) = x\,, \qquad P_{n+1}(x) = (1 + x^2)P_n'(x)\,.$$

Thus if we write

$$D^n(\tan z) = T_{n0} + T_{n1}\tan z + T_{n2}\tan^2 z + \cdots$$

the coefficients T_{nk} satisfy the recurrence equation

$$(7)\qquad T_{0k} = \delta_{1k}\,; \qquad T_{n+1,k} = (k-1)T_{n,k-1} + (k+1)T_{n,k+1}\,.$$

Since $T_n = D^n(\tan z)|_{z=0} = T_{n0}$, and since T_{nk} is zero except for at most $(n+3)/2$ values of k, formula (7) shows that the calculation of all $T_{n+1,k}$ from the values of $T_{n,k}$ essentially requires only $(n+2)/2$ multiplications of a small number k by a iarge number $T_{n,k}$ and $n/2$ additions of large numbers. Since we are interested only ln T_{n0} for odd values of n, we might try to use the relation

$$T_{n+2,k} = (k-2)(k-1)T_{n,k-2} + 2k^2T_{n,k} + (k+1)(k+2)T_{n,k+2}$$

but a count of the operations involved shows this provides little if any improvement over (7), and so the simpler form (7) is preferable.

Similarly, we have $D(\sec z \tan^n z) = \sec z\,(n \tan^{n-1} z + (n+1)\tan^{n+1} z)$, hence if we write

$$D^n(\sec z) = (\sec z)(E_{n0} + E_{n1}\tan z + E_{n2}\tan^2 z + \cdots) \tag{8}$$

we have the recurrence

$$E_{0k} = \delta_{0k}\,; \qquad E_{n+1,k} = kE_{n,k-1} + (k+1)E_{n,k+1}\,. \tag{9}$$

Since $E_n = E_{n0}$, this relation yields an efficient method for calculating the Euler numbers. A somewhat similar recurrence relation was used by Joffe [3] to calculate Euler numbers; his method requires essentially the same amount of computation, but as explained in the next section there is a way to modify (9) to obtain a considerable advantage.

The identities $\tan(\pi/4 + z/2) = \tan z + \sec z$ and $D^n(\tan(\pi/4 + z/2)) = 2^{-n}P_n(\tan(\pi/4 + z/2))$ imply that the sums of the numbers T_{nk} have a very simple form:

$$2^{-n}P_n(1) = 2^{-n}\sum_{k\geq 0} T_{nk} = \begin{cases} E_n,\ n \text{ even}\,, \\ T_n\,,\ n \text{ odd}\,. \end{cases} \tag{10}$$

This relation can be used to advantage when both E_n and T_n are being calculated.

The definition of $\tan z$ implies

$$\tan z = \frac{\sin z}{\cos z} = \frac{(e^{iz} - e^{-iz})}{i(e^{iz} + e^{-iz})} = \frac{1}{z}\left(\frac{2iz}{e^{2iz}+1} - iz\right) = \frac{1}{z}\left(\frac{2iz}{e^{2iz}-1} - \frac{4iz}{e^{4iz}-1} - iz\right)$$
$$= \frac{1}{z}\left(-iz + \sum_{n\geq 0}((2iz)^n - (4iz)^n)B_n/n!\right);$$

and by equating coefficients we obtain the well-known identity

$$B_n = -i^{-n}nT_{n-1}/2^n(2^n-1)\,, \qquad n > 1\,. \tag{11}$$

Hence, the Bernoulli numbers may be obtained from the tangent numbers by a calculation which (on a binary computer) is especially simple.

The celebrated von Staudt-Clausen theorem [8, 1] states that

$$B_{2n} = C_{2n} - \sum_{p \text{ prime};(p-1)\backslash 2n} \frac{1}{p} \tag{12}$$

where C_{2n} is an integer. The table appended to this paper expresses B_n in this form, and, as shown below, the calculation of (11) may be carried out without any multiple-precision division.

3. Details of the Computation. By the recurrence (7) we may discard the value of $T_{n,k}$ once $T_{n+1,k+1}$ has been calculated, so only about n of the values $T_{n,k}$ need to be retained in the computer memory at any one time. A further technique can be employed when the memory size has been exceeded; for example, suppose we start with the computation of T_{nk} for $n \leqq 4$:

	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
$n = 0$	0	1				
$n = 1$	1	0	1			
$n = 2$	0	2	0	2		
$n = 3$	2	0	8	0	6	
$n = 4$	0	16	0	40	0	24

and suppose that very little memory space is available, so that we cannot completely evaluate all of the entries for $n = 5$; we might obtain

$n = 5$	16	0	136	0	240	0	*

where "*" denotes an unknown value. The calculation may still proceed, keeping track of unknown values:

$n = 6$	0	272	0	1232	0	*	
$n = 7$	272	0	3968	0	*		
$n = 8$	0	7936	0	*			
$n = 9$	7936	0	*				etc.

In this way we may compute the values of about twice as many tangent numbers as were produced before overflow occurred, avoiding much of the calculation of the $T_{n,k}$.

Since the numbers T_n become very large (T_{835} has 1866 digits, and T_n is asymptotically $2^{n+2}n!/\pi^{n+1}$ when n is odd), care needs to be taken for storage allocation of the numbers $T_{n,k}$ if we are to make efficient use of memory space. The program we prepared makes use of two rather small areas of memory (say A and B) each of which is capable of holding any one of the numbers $T_{n,k}$, plus a large number of consecutive locations used for all the remaining values. By sweeping cyclically through this large memory area, it is possible to store and retrieve the values in a simple manner.

For the sake of illustration let us suppose the word size of our computer is very small, so that only one decimal digit may be stored per word; and suppose there are just 14 words of memory used for the table of $T_{n,k}$. After the calculation of the values for $n = 4$, the memory might have the following configuration:

(13)

6	.	1	6	,	4	0	,	2	4	.	,	8	,
		↑									↑		
		P									Q		

Here P and Q represent variables in the program that point to the current places of interest in the memory; P points to the number that will be accessed next, and Q points to the place where the next value is to be written. Only locations from P to Q contain information that will be used subsequently by the program. The symbols "." and "," represent special negative codes in the table which delimit the numbers in an obvious fashion. As we begin the calculation for $n = 5$, we set area A to zero and a variable k to 1. The basic cycle is then:

(a) Set area B to k times the next value indicated by P, and move P to the right.

(b) Store the value of $A + B$ into the locations indicated by Q, and move Q to the right.

(c) Transfer the contents of B to area A.

(d) Increase k by 2.

In the case of (13) we would change the memory configuration to

(14)

6	.	1	6	,	4	0	,	2	4	.	1	6	,
↑ Q					↑ P								

$k = 3 \quad A = 16 \quad B = 16$

Notice that the value 16 has been stored, the pointer Q has moved to the right and (treating the memory as a circular store) then to the far left. The next two iterations of steps (a)–(d) give

(15)

1	3	6	,	2	4	0	,	2	4	.	1	6	,
								↑ Q			↑ P		

$k = 7 \quad A = 120 \quad B = 120$

Now since the terminating "." was sensed, the program attempts to store the value from area A; but since this would make pointer Q pass P, the "memory overflow" condition is sensed, and the memory configuration becomes

(16)

1	3	6	,	2	4	0	,	*	2	0	1	6	,
									↑ Q		↑ P		

where "*" is another internal code symbol. The computation for $n = 6$ is similar but it uses a different initialization since n is even; after $n = 6$ has been processed we would have

(17)

2	3	2	,	*	4	0	,	*	2	7	2	,	1
					↑ Q				↑ P				

and so on.

The above discussion has been slightly simplified for purposes of exposition. In the actual program, it is preferable to keep the numbers stored with least significant digit first, so that for example (16) would really be

(18)

6	3	1	,	0	4	2	,	*	2	1	6	1	,
									↑ Q		↑ P		

in order to simplify the multiple-precision operations. A few other changes in the sequence of operations were made in order to use memory a little more efficiently (for example the value T_{n0} need never be retained).

A similar method may be used for E_n. This arrangement of the computation gives a substantial advantage over Joffe's method [3] because of the "*", and it also has advantages over (10) for the same reason.

It remains to consider the calculation of the Bernoulli number B_{2n} from T_{2n-1}. Consider formula (12); if p is an odd prime, $2^{p-1} \equiv 1$ (modulo p), hence if $(p - 1)\backslash 2n$, then $2^{2n} - 1$ is divisible by p. So we first compute the integer

$$(19) \qquad N = (-1)^{n-1} 2n T_{2n-1} + \sum_{p \text{ prime}; (p-1) \backslash 2n} \frac{(2n)(2^{2n})(2^{2n} - 1)}{p}$$

by referring to an auxiliary table of primes that may be calculated at the beginning of the program. Then it is merely a question of computing

$$(20) \qquad C_{2n} = N/2^{2n}(2^{2n} - 1) = N/2^{4n} + N/2^{6n} + N/2^{8n} + \cdots .$$

The calculation of $N/2^k$ is of course merely a "shift right" operation in a binary computer, so all the terms of the infinite series on the right side of (20) are readily computed. This series converges very rapidly, and we know C_{2n} is an integer, so we need only carry out the calculation indicated in (20) until it converges one word-size (35 bits) to the right of the decimal point. It is simple to check at the same time that C_{2n} is indeed very close to an integer, in order to verify the computations.

4. Periodicity of the Sequences. Examination of the tables produced by the computer program shows that the unit's digits of the nonzero tangent numbers repeat endlessly in the pattern 2, 6, 2, 6, 2, 6, starting with T_3; furthermore the two least significant digits ultimately form a repeating period of length 10: 16, 72, 36, 92, 56, 12, 76, 32, 96, 52, 16, 72, ... The three least significant digits have a period of length 50, and for four digits the period-length is 250. These empirical observations suggest that theoretical investigation of period-length might prove fruitful.

THEOREM 1. *Let p be an odd prime, and let λ be the period-length of the sequence $\langle T_n \bmod p \rangle$. Then*

$$(21) \qquad \lambda = \begin{cases} p - 1, & p \equiv 1 \pmod 4 \\ 2(p - 1), & p \equiv 3 \pmod 4 \end{cases}$$

and

$$(22) \qquad T_{n+\lambda} \equiv T_n \pmod p \quad \textit{for all} \quad n \geqq 0 .$$

Proof. It is clear from the recurrence relation (7) that the sequence $\langle T_n \bmod p \rangle$ is determined by the recurrence equation

$$(23) \qquad y_{n+1} = A y_n$$

where the vector y_n and the matrix A are defined by

$$(24) \qquad A = \begin{bmatrix} 0 & 2 & & & & & \\ 1 & 0 & 3 & & & & \\ & 2 & 0 & 4 & & & \\ & & 3 & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & 0 \\ & & & & & 0 & p-1 \\ & & & & & p-2 & 0 \end{bmatrix}, \qquad y_n = \begin{bmatrix} T_{n,1} \\ T_{n,2} \\ \cdot \\ \cdot \\ \cdot \\ T_{n,p-1} \end{bmatrix}.$$

For $T_{n,k}$ can contribute nothing to any subsequent value of T_n when $k \geqq p$.

We will show below that the minimum polynomial equation satisfied by A is

$$A^{p-1} - (-1)^{(p-1)/2} I \equiv 0 \text{ (modulo } p) \; ; \tag{25}$$

hence (22) is valid for the value of λ given by (21). It remains to show that λ is the true period-length of the sequence, not merely a multiple of the period.

Accordingly, suppose $T_{n+\lambda'} \equiv T_n (\text{mod } p)$ for some positive $\lambda' \leqq \lambda$ and all large n. In view of (22) this congruence must hold for all $n \geqq 0$. Let $y = y_{\lambda'} - y_0$; then $p(A^n y) \equiv 0$ for all $n \geqq 0$ where p denotes the projection onto the first component of the vector $A^n y$. But this implies $n! \alpha_n \equiv 0$ (mod p) for all components α_n of y, hence $y \equiv 0$, i.e., $y_0 \equiv y_{\lambda'} = A^{\lambda'} y_0$. It follows that $y_n \equiv A^{\lambda'} y_n$ for all $n \geqq 0$, and since the vectors $y_0, \cdots, y_{p-2}$ are obviously linearly independent we must have $A^{\lambda'} \equiv I$ (modulo p). Therefore, λ' is $\geqq \lambda$, and the proof is complete.

It remains to verify (25), which seems to be a nontrivial identity. Clearly, the minimum polynomial of A must be of degree $p - 1$, since $y_0, \cdots, y_{p-2}$ are linearly independent; therefore, it suffices to calculate the characteristic polynomial of A. Let

$$D_n = \det \begin{bmatrix} x & -(n-1) & & & & \\ -n & x & -(n-2) & & & \\ & -(n-1) & \cdot & & & \\ & & & \cdot & & \\ & & & & x & -1 \\ & & & & -2 & x \end{bmatrix} ; \tag{26}$$

then $D_n = xD_{n-1} - (n-1)nD_{n-2}$ so we have

$$\begin{aligned} D_1 &= x \, , \\ D_2 &= x^2 - 1\cdot 2 \, , \\ D_3 &= x^3 - (1\cdot 2 + 2\cdot 3)x \, , \\ D_4 &= x^4 - (1\cdot 2 + 2\cdot 3 + 3\cdot 4)x^2 + 1\cdot 2\cdot 3\cdot 4 \, , \\ D_5 &= x^5 - (1\cdot 2 + 2\cdot 3 + 3\cdot 4 + 4\cdot 5)x^3 + (1\cdot 2\cdot 3\cdot 4 + 1\cdot 2\cdot 4\cdot 5 + 2\cdot 3\cdot 4\cdot 5)x \, , \end{aligned}$$

and in general

$$D_n = x^n - s_{n1}x^{n-2} + s_{n2}x^{n-4} - s_{n3}x^{n-6} + \cdots \, , \tag{27}$$

where

$$s_{nk} = \sum a_1(a_1+1)a_2(a_2+1)\cdots a_k(a_k+1) \tag{28}$$

is summed over all values $1 \leqq a_1 \ll a_2 \ll \cdots \ll a_k < n$. (Here $u \ll v$, for integers u, v, denotes $v \geqq u + 2$.) Thus, s_{nk} is the sum of all products of k of the pairs $1\cdot 2, 2\cdot 3, \cdots, (n-1)\cdot n$ with no "overlapping" pairs allowed in the same term.

To evaluate $s_{(p-1)k}$ mod p, it is convenient to allow also the pairs $(p-1)\cdot p$ and $p \cdot 1$, since these contribute nothing to the sum. Thus for example,

$$s_{62} \equiv 1\cdot2\cdot3\cdot4 + 1\cdot2\cdot4\cdot5 + 1\cdot2\cdot5\cdot6 + 1\cdot2\cdot6\cdot7 + 2\cdot3\cdot4\cdot5 + 2\cdot3\cdot5\cdot6$$
$$+ 2\cdot3\cdot6\cdot7 + 2\cdot3\cdot7\cdot1 + 3\cdot4\cdot5\cdot6 + 3\cdot4\cdot6\cdot7 + 3\cdot4\cdot7\cdot1$$
$$+ 4\cdot5\cdot6\cdot7 + 4\cdot5\cdot7\cdot1 + 5\cdot6\cdot7\cdot1$$

(modulo 7). Let us say two terms $a_1(a_1 + 1) \cdots a_k(a_k + 1)$ and $a_1'(a_1' + 1) \cdots a_k'(a_k' + 1)$ are "equivalent" if, for some r and t and for all j, $a_j \equiv a'_{(j+r) \bmod p} + t$; thus, in the above example the terms $1\cdot2\cdot4\cdot5$, $2\cdot3\cdot5\cdot6$, $3\cdot4\cdot6\cdot7$, $4\cdot5\cdot7\cdot1$, $5\cdot6\cdot1\cdot2$, $6\cdot7\cdot2\cdot3$, $7\cdot1\cdot3\cdot4$ are mutually equivalent. It is impossible for a term to be equivalent to itself when $0 < t < p$, since this would imply $a_1 + \cdots + a_k \equiv a_1 + \cdots + a_k + kt$, and $t \equiv 0$. Therefore, each equivalence class has precisely p terms in it. When $k < (p-1)/2$ the sum over an equivalence class has the form

$$\sum_{0 \leqq t < p} (a_1 + t)(a_1 + t + 1) \cdots (a_k + t)(a_k + t + 1)$$

where the summand is a polynomial of degree $\leqq p - 2$ in t. Any such summation may be expressed modulo p as a sum of terms of the form

$$c \sum_{0 \leqq t < p} \binom{t}{j} = c\binom{p}{j+1} \equiv 0\,, \quad \text{since} \quad 0 \leqq j < p - 1\,,$$

so $s_{kp} \equiv 0$. It follows that

$$D_{p-1} \equiv x^{p-1} + (-1)^{(p-1)/2}(p-1)!\ (\text{modulo } p) \tag{29}$$

and an application of Wilson's theorem completes the proof of (25).

THEOREM 2. *Let p be an odd prime, and let λ be the period-length of the sequence $\langle E_n \bmod p\rangle$. Then*

$$\lambda = \begin{cases} p - 1\,, & p \equiv 1 \pmod 4 \\ 2(p-1)\,, & p \equiv 3 \pmod 4 \end{cases} \tag{30}$$

and

$$E_{n+\lambda} \equiv E_n \pmod p \quad \textit{for all} \quad n \geqq 1\,. \tag{31}$$

Proof. Make the following changes in the proof of Theorem 1:

$$A = \begin{bmatrix} 0 & 1 & & & & & \\ 1 & 0 & 2 & & & & \\ & 2 & 0 & 3 & & & \\ & & 3 & \cdot & \cdot & & \\ & & & \cdot & & \cdot & \\ & & & & \cdot & & p-1 \\ & & & & & p-1 & 0 \end{bmatrix}, \quad y_n = \begin{bmatrix} E_{n,0} \\ E_{n,1} \\ \vdots \\ E_{n,p-1} \end{bmatrix}. \tag{32}$$

Then the minimum polynomial equation satisfied by A is

$$A^p - (-1)^{(p-1)/2}A \equiv 0\ (\text{modulo } p)\,. \tag{33}$$

The proof is a straightforward modification of the proof of Theorem 1.

The congruences (22) and (31) were obtained long ago by Kummer (see for example [5, p. 270]), but it was not shown that the true period-length could not be a proper divisor of the number λ given by (21), (30). More general congruences given by Kummer make it possible to establish further results about the period-length:

THEOREM 3. *Let p be an odd prime, and let λ be given by* (30). *Then*

$$(34) \qquad T_{n+\lambda p^{k-1}} \equiv T_n \ (modulo\ p^k)\ , \qquad n \geqq k\ ,$$

$$(35) \qquad E_{n+\lambda p^{k-1}} \equiv E_n \ (modulo\ p^k)\ , \qquad n \geqq k\ .$$

Proof. Assume $n \geqq k$ and define the sequence $\langle u_m \rangle$ by the rule

$$(36) \qquad u_m = (-1)^{(p-1)m/2} T_{n+(p-1)m}\ , \qquad m \geqq 0\ .$$

Kummer's congruence for the tangent numbers may be written

$$(37) \qquad \Delta^k u_m \equiv 0 \ (\text{modulo } p^k)\ , \qquad m \geqq 0\ , \qquad k \geqq 1\ ,$$

where $\Delta^k u_m$ denotes

$$u_{m+k} - \binom{k}{1} u_{m+k-1} + \binom{k}{2} u_{m+k-2} - \cdots + (-1)^k u_m\ .$$

We will prove that (37) implies

$$(38) \qquad u_{m+p^{r-1}} \equiv u_m \ (\text{modulo } p^r)\ , \qquad m \geqq 0\ , \qquad r \geqq 1\ ,$$

and this will establish (34). Eq. (35) follows in the same way if we let

$$u_m = (-1)^{(p-1)m/2} E_{n+(p-1)m}\ .$$

Assume Eq. (37) is valid for some sequence of real numbers (not necessarily integers) $u_0, u_1, \cdots$; thus, $\Delta^k u_m$ is an integer multiple of p^k when $k \geqq 1$, but not necessarily when $k = 0$. We will prove that the sequence $u_m/p, u_{m+p}/p, u_{m+2p}/p, \cdots$, for fixed m also satisfies Eq. (37), and this suffices to prove (38) by induction on r.

Let E be the operator $Eu_m = u_{m+1}$. Eq. (37) may be written $(E - 1)^k u_m \equiv 0$ (modulo p^k), and our goal as stated in the preceding paragraph is to show that $(E^p - 1)^k (u_m/p) \equiv 0$ (modulo p^k), i.e. $(E^p - 1)^k u_m \equiv 0$ (modulo p^{k+1}). Let $f(E) = E^{p-2} + 2E^{p-3} + \cdots + (p - 2)E + (p - 1)$; then $E^p - 1 = (E - 1)(p + f(E)(E - 1))$, hence

$$(E^p - 1)^k u_m = \sum_{0 \leqq j \leqq k} \binom{k}{j} p^j (E - 1)^{2k-j} f(E)^{k-j} u_m$$

and each term in the sum on the right is an integer multiple of p^{2k}. Hence, we have proved in fact that $(E^p - 1)^k u_m \equiv 0$ (modulo p^{2k}), which is more than enough to complete the proof of the theorem.

Note that Eqs. (34), (35) do not necessarily give the true period-length of the sequence mod p^k when $k > 1$; although (34) is "best possible" when $p = 5$ and $k = 2, 3, 4$, the tangent numbers have the same period-length modulo 9 as they do modulo 3.

The tangent number T_{2n+1} is divisible by 2^n, so the period length of T_n mod 2^r is 1 for all r. Eq. (35) is valid for $\lambda = 2$ when $p = 2$, since Kummer's congruence (37) holds for $u_m = E_{n+2m}$. In particular, we may combine the results proved above to show that for any modulus m the sequences T_n mod m, E_n mod m are periodic, and the period-length divides $2\phi(m)$.

TABLE 1. *The first 60 nonzero tangent numbers*

n	T_n					
1	1.					
3	2.					
5	16.					
7	272.					
9	7936.					
11	353792.					
13	22368256.					
15	1903757312.					
17	20	9865342976.				
19	2908	8885112832.				
21	495149	8053124096.				
23	101542388	6506852352.				
25	2	4692148019	0207983616.			
27	702	5160160394	3959887872.			
29	231191	8418780959	7841473536.			
31	87139627	5712516929	6170811392.			
33	3	7294077037	2052957109	7509625856.		
35	1798	6516934508	8878007175	0349094912.		
37	970982	8107850591	1237939970	7952152576.		
39	583203324	9173100439	4319164162	5494290432.		
41	38	7635983772	0830318280	1462400217	5135645696.	
43	28372	7921907431	9093041833	1629578783	7183229952.	
45	22768137	9129930886	4886002843	3631616460	3920777216.	
47	1	9950025215	7859031027	1604996431	9565816634	0757225472.
49	1901	6956465792	8428175235	4450739249	2859204777	5873499136.
51	1965356	9491567180	8914892880	7269899849	6749880539	8829268992.
53	2195234391	0676159128	0258358007	9642459877	5270244950	5540243456.
55	264	2394112879	0088327017	8745605712	6484887310	5817055122
		3831232512.				
57	341838	3013357185	8035017444	9297951396	8470814438	2678544895
		2307122176.				
59	474090194	3513421559	7452258201	0891145370	3030136589	7345704226
		3121068032.				
61	70	3237958001	3937369998	9682771463	4659411015	0902726842
		2783100116	1763127296.			
63	111325	5345330866	7003397462	1804708828	0783690575	3945388146
		9938250594	2970007552.			
65	187693125	2103779423	7441079285	2094015924	1846377457	0015344479

		4255826189	9287330816.			
67	33	6377818583	5623734685	6877233329	3113418998	6914776308
		2252421375	2870352046	8549763072.		
69	63965	4705763027	8735849943	4326866459	2938843795	4320994101
		2547258153	1136084889	9380084736.		
71	128843416	9576789446	5395544961	6369630081	3886273130	1380715918
		9122218698	6494595788	9839726592.		
73	27	4459227344	4371901652	3516486805	2457457867	9114870464
		3680696607	0734990348	6020216502	7058221056.	
75	61734	9449829416	7370744681	8719138436	0603858737	8520405728
		4992863438	2622751313	8292113188	8174170112.	
77	146418390	5893124310	4329214126	9909391400	0717441472	0630707472
		2368442192	9633491577	1113545177	2434251776.	
79	36	5660095844	0682610727	0144430968	0705805634	2702370300
		3980054983	7863741179	3701774928	6422729861	4949445632.
81	96031	3027666248	1075266957	0567541956	3027855649	3741030484
		6277121485	9942977938	4149713570	7706863741	6399568896.
83	264889663	2807323067	7773107678	1904830963	2044764392	8688298159
		2211638598	1067039286	1466558923	6754520862	2811185152.
85	76	6519961272	4801366153	1794130757	5349971865	4634906097
		3607290907	9076131721	2953998553	6700552799	6370878975
		4131644416.				
87	232434	9433745362	2193600440	8780902217	0812714593	4478585859
		6661674060	9119191978	8758468215	4130828009	6173838087
		0441500672.				
89	737792682	4740564012	1148258616	2341769921	8108362187	7149167320
		4777725652	1520277361	6583155193	4594712417	0347433028
		3945230336.				
91	244	8941953051	5642227874	9861738770	6905643965	0950687716
		2622951953	3266404250	3431622461	3187841278	4592740313
		9690334355	4866184192.			
93	849199	0762262559	4428836230	2662250632	3194267305	8660849712
		7041570684	4923128691	6739105340	0799032391	5652777784
		9540333030	1807558656.			
95	3073415080	2897237829	5231282906	4508016361	8836921783	9513190225
		7879025458	2349092340	9861969626	7158091384	4033104110
		5856582593	0001907712.			
97	1159	9103698421	3924402863	9929718418	3467010271	7097769722
		6102768897	5098301378	8854712967	5792955877	0355004525
		4382396370	6687569562	6637541376.		
99	4560851	6616801111	8210438295	3145169718	5581949239	8924144787

TABLE 1—*Continued*

n	T_n					
		7042772066	0171869441	0047936547	8229870027	6817088804
		9937408986	6899187030	6963423232.		
101	1	8669279906	6534977615	9928982810	6174743255	0509816834
		2879473499	3871837828	1122719555	1754542669	2529915556
		3954320693	5127146296	4772550115	8308970496.	
103	7949	2326836383	7296825215	8440590799	5150283969	6539875280
		5263054499	4140134793	4632528787	7257788852	3638482311
		6116040368	1135561434	7362070730	0450762752.	
105	35180993	0277448013	2955727650	0727464271	4639405654	6029941974
		2060080598	1901789276	3499985629	4332559603	0384517339
		8006679743	5816406207	0224471177	9077718016.	
107	16	1717858874	5215971711	0186706465	2513397720	9248162391
		7128707035	1972782143	4957959108	6004226144	6628003769
		3458730392	9487648121	8799405564	9433217448	6336438272.
109	77155	7828380939	9490537680	2460595806	7574980560	2111631319
		7321330938	3312556087	6114515653	9836404772	1015654470
		4162805881	5541548168	6784721593	5856757008	9952935936.
111	381807444	4196147801	8966239285	9619830441	1240712916	4430020735
		8125472693	6727895619	3951931102	1421845843	7007093225
		2703193275	6005581575	1662244551	3279305256	8372641792.
113	195	8398663290	4131567170	1199172580	7974770028	4375913985
		4340241234	2038670128	5665524839	5047696425	9280405845
		2218319170	1679429091	1384499992	9227926705	5414739516
		6332256256.				
115	1040552	6070691740	8391389087	3747623961	0007069533	0048288233
		9319091564	2977785601	0534109858	0945677436	0653272241
		4725860275	1459533577	7733542817	6107197749	0669176471
		6738445312.				
117	5723585022	9555589879	7004078003	1278606958	7871036923	3804707134
		9116874639	2466184489	3499287007	8763836938	6390752650
		8633084261	1483758302	7014497286	3537856412	6193750216
		2020990976.				
119	3257	2969544137	3711110813	9491520587	0894578681	8558730200
		7333881055	9724342116	8172307776	6222847786	5964664757
		3601851664	4828218413	9690510871	7176120451	6527175740
		8580920993	7947000832.			

TABLE 2. *The first* 61 *nonzero Euler numbers*

n	E_n				
0	1.				
2	1.				
4	5.				
6	61.				
8	1385.				
10	50521.				
12	2702765.				
14	199360981.				
16	1	9391512145.			
18	240	4879675441.			
20	37037	1188237525.			
22	6934887	4393137901.			
24	1551453416	3557086905.			
26	40	8707250929	3123892361.		
28	12522	5964140362	9865468285.		
30	4415438	9324902310	4553682821.		
32	1775193915	7953928943	6664789665.		
34	80	7232992358	8789806216	8247453281.		
36	41222	0603395177	0212234707	9671259045.		
38	23489580	5270431082	5201782857	6198947741.		
40	1	4851150718	1149800178	7715678140	5826684425.	
42	1036	4622733519	6121193979	5730474518	5976310201.	
44	794757	9422597592	7036080405	1008807061	9519273805.	
46	666753751	6685544977	4350284747	7374819752	4107684661.	
48	60	9627864556	8542158691	6857428768	4315397653	9044435185.
50	60532	8524818862	1896314383	7851116490	8810349822	5146815121.
52	65061624	8668460884	7715870634	0808229834	8364423676	5385576565.
54	7	5466599390	0873909806	1432565889	7367442122	4002471169
		9858645581.				
56	9420	3218964202	4120420228	6237690583	2272093888	5259964600
		9394905945.				
58	12622019	2518062187	1990340923	7287489255	4823410611	9182559406
		9964920041.				
60	1	8108911496	5792304965	4580774165	2158688733	4873492363
		1410600809	5454231325.			
62	2775	7101702071	5805973669	8090837152	7449233019	5948009175
		7803378276	6889782501.			
64	4535810	3330017889	1747468878	7156776236	6351861519	4703688814

TABLE 2—*Continued*

n	E_n					
		6884383791	9695760705.			
66	7886284206	6617894181	0072074223	9990423947	8162972003	7689327097
		5749485716	7945376961.			
68	1456	1844380139	6315007150	4700949423	2666186081	2858314932
		9864476977	6806459548	8862902085.		
70	2850517	8322369771	8732198729	5567393395	0425524177	8255239879
		3532111069	8042754623	5397447421.		
72	5905747207	7754436545	5135032296	4395713720	3301618182	2954929765
		9721536598	0505026450	1891063465.		
74	1292	9736641878	6417049760	3235938698	7540761705	1912367260
		6411370597	3437870353	3180819573	1850937881.	
76	2986928	1832845769	5093074365	2217140605	6929223693	7068070281
		3812833466	8980381720	1565580896	0288452845.	
78	7270601714	0168641438	0328065169	9281851647	2342880492	0790510830
		9583687335	6880176415	4619109500	9395592341.	
80	1862	2915758412	6970444824	9230304312	6011920010	1945185560
		6357710109	5681956123	5462014428	3229383700	5396878225.
82	5013104	9408109796	6129086936	7888100942	0083336722	2205397659
		7359623656	1571401154	6997615522	5318908480	9951554801.
84	1	4165255759	7856259916	7220694100	2167040547	5845492837
		9123907001	4684537456	7994390844	9771259876	7502043638
		0612547605.				
86	4196	6431640402	4471322573	4140694188	9181896262	8391683907
		0392122285	4903292185	3217838146	6080538087	8636544057
		0254969261.				
88	13021595	9052404639	8125858691	3308186813	5675761398	6610030678
		0957582424	0428663372	9262297123	6771997435	9174800620
		4646868985.				
90	4	2272406861	3990906470	5589929214	5931029338	4538867236
		9082676644	5426502482	2836959052	5634078984	3021532175
		0794578239	6923579721.			
92	14343	2127919765	8340613368	2640578565	8585798821	4884315911
		1106574955	5097901968	1261825484	8857854461	5507146314
		4403492151	7907250365.			
94	50817990	7245804251	6455975764	3090736003	4824356715	1341392681
		3239886828	2108762470	7489775212	2164140484	8819075342
		9706818956	5042330181.			
96	18	7833293645	2930264020	0757918417	9892539001	4449970053
		6163708087	0116823642	6457556016	7857968115	9136078780

		8122338310	3537309752	8077899745.		
98	72365	3438103385	7776571876	6173678229	2986259565	1810672327
		6071243105	5015669043	2246475917	9223614145	2770950810
		8421919498	1419813489	7708964641.		
100	290352834	6661097497	0546038347	6443587507	7553006646	1589450804
		9231914699	7643370625	0238893534	4712996735	4174648294
		7485105535	2869245763	2980625125.		
102	121	2293737892	9218210539	2954978560	9880769588	0456925359
		8783740312	5205142532	4802983023	2591646618	9556246560
		2169682143	3854077446	4648305790	3979627101.	
104	526306	4249616990	7060022407	3584236661	3920044251	8073843505
		4049404498	0000330206	3127338662	3384973914	9760949941
		9661748773	6605063835	0121582193	8795634505.	
106	2374073071	9367663470	3461698760	6526516334	5442841928	7776842645
		2795281647	8787229741	7395385346	7057217045	7147825505
		6084593759	5862589203	7569585468	8654461561.	
108	1111	8900942482	8230249702	3358817578	9328258295	3446870840
		3452929407	1740642849	8024481433	1727650672	0170124648
		2942622666	9534375777	2589331464	9416971071	9722335885.
110	5403078	6597952932	0561911549	4263476990	4748827182	9642019740
		8516010249	8147118459	6327880271	7760330206	6276288863
		4782738106	8652979312	0790759867	3617958139	9343212021.
112	2	7223410855	7222702137	1534144589	0954891150	1924525644
		6700836600	5911126658	4900943112	6511622934	5791312467
		2736615820	6146238929	6672527429	0471204776	0583423048
		1160137265.				
114	14213	0105480096	6981180852	0457223188	2248930706	6995326299
		1709670127	9605797073	3300193420	6948612331	3307158077
		6905678286	5303800832	2871113514	7775762116	5153920474
		4250822481.				
116	76842618	2064690265	3170956283	6664779364	0120107733	7276440867
		0420540847	5538272770	3976472123	6896168602	1466396893
		8327890952	8067067756	1989835334	2711105340	1493015019
		2489246645.				
118	42	9962192543	9749642818	8903364863	2755023029	6183651057
		0835366233	6035433477	1425729606	5830552349	7893533611
		6864465814	8786736548	3786235212	4705254397	3611068831
		8626950069	8123036941.			
120	248839	1574782987	1631690245	5408489408	2372867090	7090814055
		5499968530	1842243985	7255460434	6369071792	7997103011
		5914025391	0784871444	2940830046	2747699810	6540373770
		6481607384	7531472025.			

TABLE 3. *The first* 250 *Bernoulli numbers*

$B_0 = 1, B_1 = -1/2, B_{2n+1} = 0$ for $n \geqq 1$, and the values of B_{2n} for $1 \leqq n \leqq 125$ appear below in the form $C_{2n} - \{p_1, p_2, \cdots, p_k\}$. This notation stands for $C_{2n} - 1/p_1 - \cdots - 1/p_k$; thus $B_4 = 1 - \{2, 3, 5\} = 1 - 1/2 - 1/3 - 1/5 = -1/30$. The Bernoulli numbers have been expressed in this form here, since the numbers C_{2n} have not been tabulated before.

n	B_n					
2	1	−{2,3}				
4	1	−{2,3,5}				
6	1	−{2,3,7}				
8	1	−{2,3,5}				
10	1	−{2,3,11}				
12	1	−{2,3,5,7,13}				
14	2	−{2,3}				
16	−6	−{2,3,5,17}				
18	56	−{2,3,7,19}				
20	−528	−{2,3,5,11}				
22	6193	−{2,3,23}				
24	−86579	−{2,3,5,7,13}				
26	1425518	−{2,3}				
28	−27298230	−{2,3,5,29}				
30	601580875	−{2,3,7,11,31}				
32	−1	5116315766	−{2,3,5,17}			
34	42	9614643062	−{2,3}			
36	−1371	1655205087	−{2,3,5,7,13,19,37}			
38	48833	2318973594	−{2,3}			
40	−1929657	9341940067	−{2,3,5,11,41}			
42	84169304	7573682616	−{2,3,7,43}			
44	−4033807185	4059455412	−{2,3,5,23}			
46	21	1507486380	8199160561	−{2,3,47}		
48	−1208	6626522296	5259346026	−{2,3,5,7,13,17}		
50	75008	6674607696	4366855721	−{2,3,11}		
52	−5038778	1014810689	1413789302	−{2,3,5,53}		
54	365287764	8481812333	5110430844	−{2,3,7,19}		
56	−2	8498769302	4508822262	6914643290	−{2,3,5,29}	
58	238	6542749968	3627644645	9819192193	−{2,3,59}	
60	2 −21399	9492572253	3366581074	4765191096	−{2,3,5,7,11,13,31,61}	
62	050097	5723478097	5699217330	9567231026	−{2,3}	
64	−209380059	1134637840	9095185290	0279701846	−{2,3,5,17}	
66	2	2752696488	4635155596	4926035276	9264581471	−{2,3,7,23,67}
68	−262	5771028623	9576047303	0497361582	0208144899	−{2,3,5}

70	32125	0821027180	3251820479	2304264985	2435219412	−{2,3,11,71}	
72	−4159827	8166794710	9139170744	9526235893	6689603010	−{2,3,5,7,13,19,37,73}	
74	569206954	8203528002	3883456219	1210586444	8051297182	−{2,3}	
76	−8	2183629419	7845756922	9065346861	7333014550	8927628859	−{2,3,5}
78	1250	2904327166	9930167323	3982970289	5524177196	3644484776	−{2,3,7,79}
80	−200155	8323324837	0274925329	1988132987	6872422013	2825915914	
		−{2,3,5,11,17,41}					
82	33674982	9153643742	3339667690	3338753016	2195989471	9384367233	−{2,3,83}
84	−5947097050	3135447718	6604968440	5154084057	9071565106	9049904703	
		−{2,3,5,7,13,29,43}					
86	110	1191032362	7977559564	1307904376	9160463051	1444223148	
		8626999498	−{2,3}				
88	−21355	2595452535	0118865838	5019041065	6789732987	3916346921	
		1804590303	−{2,3,5,23,89}				
90	4332889	6986641192	4196166130	5937920621	8451368511	8091091449	
		8655788034	−{2,3,7,11,19,31}				
92	−918855282	4166932822	6200555215	5018971389	6038891627	1995959100	
		4487113436	−{2,3,5,47}				
94	20	3468967763	2907449345	5027990220	0200659751	4025337827	
		7023936918	4214108242	−{2,3}			
96	−4700	3833958035	7310785752	5553500606	0654596737	3697590579	
		1513976356	4120483353	−{2,3,5,7,13,17,97}			
98	1131804	3445484249	2706751862	5773393426	7890365954	7507479181	
		7899354166	5491176374	−{2,3}			
100	−283822495	7069370695	9264156336	4817647382	8468092801	2882128228	
		5317144648	6511107027	−{2,3,5,11,101}			
102	7	4064248979	6788506297	5082714092	0984176879	7317880887	
		0667311610	0348748532	8441210856	−{2,3,7,103}		
104	−2009	6454802756	6044834656	1967271536	3186867270	8225328766	
		2434613019	8921356500	9779698882	−{2,3,5,53}		
106	566571	7005080594	1445719346	0305193569	6141946828	7510420621	
		3875644521	5246086197	2277798401	−{2,3,107}		
108	−165845111	5413621691	5823713374	3199123014	9496261472	5464727402	
		4668155898	7813771265	0743149938	−{2,3,5,7,13,19,37,109}		
110	5	0368859950	4923774192	8942191518	0154812442	3742649032	
		1414152565	1322528310	9767429893	2791785388	−{2,3,11,23}	
112	−1586	1468237658	1863693634	0157296643	8782740978	4127789638	
		8047286451	4297311365	0988500683	1200945120	−{2,3,5,17,29,113}	
114	517567	4361754562	6984073240	6825071225	6124084923	5930550859	
		0621669403	1810829579	6651549771	8776632445	−{2,3,7}	
116	−174889218	4021711733	9690025877	6181591451	4147616182	6544872627	

TABLE 3—*Continued*

n	B_n						
		3472158762	1228952384	0015332666	6438279520	−{2,3,5,59}	
118	6	1160519994	9521852558	2452526426	4167780767	7268467832	
		0071684324	0112735747	5076344103	1489529605	9086182634	−{2,3}
120	−2212	2776912707	8349422883	2345671293	2445573185	0549877801	
		5056655269	3027736635	0025726591	0252803139	1154956835	
		−{2,3,5,7,11,13,31,41,61}					
122	827227	7679877096	9854221062	4599845957	3120465051	8433566283	
		8488529885	8447202350	0718881721	8561301633	9661427406	−{2,3}
124	−319589251	1141570958	3591634369	1808148735	2627667109	9112273184	
		5042431195	3111814531	4804543981	2034228242	2969820299	
		−{2,3,5}					
126	12	7500822233	8779298231	0024302926	6798669571	9179638977	
		3295160585	7353822073	1833362242	1938478819	1283226347	
		5958141510	−{2,3,7,19,43,127}				
128	−5250	0923086774	1338994028	2462456517	5446919894	0377552432	
		6078013452	2227018183	3065745383	0640452814	1149421273	
		7075399446	−{2,3,5,17}				
130	2230181	7894241625	2098692981	9883872814	3738272150	8758785424	
		9055078103	8036345171	2245962893	1773876814	5763813725	
		8286208932	−{2,3,11,131}				
132	−976845219	3095520443	8633513398	9802393011	6690267498	5678971000	
		1706618959	8371132984	4759158434	4882999447	8018574251	
		2315481909	−{2,3,5,7,13,23,67}				
134	44	0983619784	5295427227	2622874813	1691918757	5426552811	
		4735319759	1401112942	6528175678	7997886065	2087390581	
		1078243989	1580698362	−{2,3}			
136	−20508	5708864640	8883972933	7727583015	4864565966	9040083595	
		3087398275	4818594264	3022208918	6918602388	7468948154	
		5442476427	3682977286	−{2,3,5,137}			
138	9821443	3279791277	1075729696	0209752104	1491857990	7241070558	
		3196274811	6831819391	1585658026	7855114057	4147212665	
		3082120499	9429964679	−{2,3,7,47,139}			
140	−4841260079	8208880508	7891967099	6341276113	0549942324	6203851158	
		5625800263	1506521525	5521783095	3721687111	4312353272	
		7957252622	4667309228	−{2,3,5,11,29,71}			
142	245	5308880148	0982609783	4674040886	9039967369	5039984404	
		4603247983	2901257676	9302738510	9499436486	8624711701	
		5805824257	3375943467	8897524866	−{2,3}		

144	−128069	2680408474	7548782513	2786017857	2181183417	1196320118
		0952142990	8427882645	3276869447	0578038003	7383050883
		0586284403	5889432674	5245777737	−{2,3,5,7,13,17,19,37,73}	
146	68676167	1046685811	9210188859	8464400436	0924268134	7568589956
		3628000579	8377113920	7141426350	0143698420	6381706690
		5802338371	6450621194	7331637478	−{2,3}	
148	−3	7846468581	9691046949	7899541637	9556814489	5492650402
		9979455214	0400826798	0129451551	0704298643	4146783802
		5357177777	9275574483	0826629638	2227717750	−{2,3,5,149}
150	2142	6101250665	2915508713	2313514827	2096660152	6029650951
		4155963489	3447829324	8460575061	2130066048	0116095571
		7270014726	4310210906	0678384924	1293313386	−{2,3,7,11,31,151}
152	−1245672	7137183695	0070196429	6163760721	9458298438	2964431725
		6169042394	1365771094	9069481888	4546604176	7134041184
		5820630576	9850941641	9528102954	3793193519	−{2,3,5}
154	743457875	5100015254	3679668394	0520613117	8071487290	2675608643
		3078967455	1693845553	4843831051	1182799109	2931489774
		0934831271	8600733639	7682180940	4713103509	−{2,3,23}
156	−45	5357953046	4170489406	3333223321	2748767721	1453427716
		0901794185	5635546610	9267970425	3013898315	1091718700
		8442342331	9549792635	2989124863	3112539372	6615424110
		−{2,3,5,7,13,53,79,157}				
158	28612	1128168588	6834536384	7251017232	5229189870	4567159402
		5083355877	5513580328	9309199986	7644645062	3355482880
		4349737827	0596646795	4014203994	9587511379	5593003154
		−{2,3}				
160	−18437723	5520338697	2768820265	3628785487	5414029263	3526027003
		4458408149	3932458494	8472610290	3484283419	3543196674
		1361091019	1555877583	4147615579	4428844016	5302368314
		−{2,3,5,11,17,41}				
162	1	2181154536	2210466995	0131650659	9521355817	4306631670
		1506035197	1806696491	0818057404	2748253800	1277493077
		7128266667	7752505278	9561241031	3002485844	6445814484
		0696728275	−{2,3,7,19,163}			
164	−824	8218718531	4121548481	8457296893	4473014189	1659231506
		2977243870	8405761023	4203215225	7185798736	5839684671
		9571577485	1579320680	5258967279	0604924064	3114348648
		0062730952	−{2,3,5,83}			
166	572258	7793783294	3329651649	8142978615	9186848661	2327430125
		4799292545	2097589473	1766114802	4526218482	4544007231
		8449214212	7912059306	4575317447	7528081717	8119178510

TABLE 3—*Continued*

n	B_n					
		1612590641	−{2,3,167}			
168	−406685305	2505910472	6767969383	1158655602	1957212176	4308330500
		0247754105	0243613769	3861568178	3983391160	3693482276
		7391874851	0229357659	3840334537	7310111326	6018436817
		0811876203	−{2,3,5,7,13,29,43}			
170	29	5960920646	4205006287	5269581585	1870426379	2990166491
		3214415150	8647495422	9161923004	0558813869	1443409958
		3868641966	9420754538	1714359738	7801102773	0113620408
		9633208761	3030846881	−{2,3,11}		
172	−22049	5225651894	5750903117	5227344598	4836378545	3956150622
		6888744024	4032520852	8888444081	1880467501	5447047651
		0302666979	1530497497	1252796339	0550202209	5516797031
		9607122917	2550624182	−{2,3,5,173}		
174	16812597	0728895998	0583115251	5136066575	4463854648	2066752238
		9752558827	3115549539	5993071507	1102880678	3998483734
		6914908538	3009714494	9671480205	1322184598	7480256819
		5443304812	2073989964	−{2,3,7,59}		
176	−1	3116736213	5569576486	4528063558	1715300443	1236000708
		2912686129	5969870858	7851050763	6768715702	2774063895
		1709335322	8985838740	8194471986	6442176508	5970669286
		1071847423	8734706022	2109905110	−{2,3,5,17,23,89}	
178	1046	7894009478	0380821832	8539298230	8964382872	1388535128
		0532629144	8964895965	6275209074	4556955976	1295923123
		0755541348	7361911412	4938813919	3241409316	9410477684
		4453714786	4958750394	0825192833	−{2,3,179}	
180	−854328	9357883370	7718598254	6299082774	5932701078	9872701906
		0442429685	2469746205	6996355108	3486627920	7004565489
		4333747091	9889765654	5462169412	4070630607	2177774046
		0524064563	4756508130	7079104923	−{2,3,5,7,11,13,19,31,37,61,181}	
182	712878213	2248654235	2288406677	1438224721	2446893047	3371527694
		7169770408	6099982659	1845215263	5072706453	5232578272
		5977587673	0721498128	8699480152	4058827108	8555947991
		9354243243	3839298821	7775167786	−{2,3}	
184	−60	8029314555	3589930008	4711868647	7458461988	1600926988
		3618118131	1550128675	4024922061	6030431157	7645148677
		0513367440	1539809892	2709241642	5453965536	1359613968
		5245930722	4294536494	4930250501	8633342016	−{2,3,5,47}
186	52996	7764248499	2393009429	1004324726	6228489712	3456032897

		6063501191	9038386327	2020855955	4228186636	5036916065
		5050538246	7055952587	5890896211	9945430322	1971803519
		8560643844	5876162116	7362303222	7002399941	−{2,3,7}
188	−47194259	1687458626	4436462290	1337991110	3760787757	2492683808
		8406765706	6708691597	3596491237	1873883437	0055520607
		5218109736	8245350938	7323230602	0438746878	2859946730
		2385753064	6862291852	9811936260	3979610699	−{2,3,5}
190	4	2928413791	4029810894	1682965410	7466904552	0981012117
		7363278709	8118229333	4484492274	2133451584	5766966132
		8438797939	2909861710	7109337670	4029317883	7678744934
		3197582408	6108199054	4401946615	9172774476	9921781652
		−{2,3,11,191}				
192	−3987	6744968232	2074434477	6555429387	9510665147	8560005423
		1823516318	1283658237	0982439948	9570515870	3033629100
		4292500115	6449718495	4863866512	4615902544	7020534114
		2323956805	8179865950	9608665263	6481422248	1410299256
		−{2,3,5,7,13,17,97,193}				
194	3781978	0419358882	7138944181	1613933278	9822023821	6264722872
		1644587349	6090237057	9449477199	9599969353	0294345595
		8971887316	7427492089	9363952932	0718684812	1525149224
		9505176738	8606080695	7543999535	6099438963	4889003542
		−{2,3}				
196	−3661423368	3681191243	6858082151	1973487551	9606834302	9904344422
		8899411740	7456818588	2959826276	6472873338	1245017672
		5368256809	2199927707	7382315070	1229725802	1548317388
		2180261011	2733870153	8667195714	9158887055	8284840257
		−{2,3,5,29,197}				
198	361	7609027237	2862348855	4609298914	0894775414	7596881957
		2031634249	1298519646	2855137114	4863312914	3587611834
		7972287556	0266199064	2027674174	3185245140	3213909430
		1486010629	5965773896	7139333352	5454387835	3071239515
		0222961685	−{2,3,7,19,23,67,199}			
200	−364707	7264519135	4362138308	8655499449	0486823468	6191058737
		6827309036	2185020610	7709861390	9538979906	0387496808
		7981794273	0412840299	5376210844	3710719800	5516180654
		9309586210	1213464712	6652378010	6263015084	0451297095
		4087034966	−{2,3,5,11,41,101}			
202	375087554	3645440909	8345241010	4814189306	8417407585	4118603710
		5235280258	2967550389	1934499292	2399095820	4298817680
		3007125338	6372603520	6143515790	5731464299	0768640406
		4497333142	1585898926	2493045836	9525193533	4283250606
		9589513246	−{2,3}			

TABLE 3—*Continued*

n	B_n					
204	−39	3458672964	3902826948	9128853371	3429355657	1403660905
		2496665007	4757739143	3417411584	1178927312	4036225303
		9254127818	0009611241	4727926748	9281199287	3335083984
		8070313868	4333801645	3321797763	8224372602	4150845715
		5749309182	9949979112	−{2,3,5,7,13,103}		
206	42088	2111481900	8200465711	7111149489	8242731374	8983148899
		0926738611	5262230714	1130489366	8334448739	3615611074
		1203897906	3986637022	4655275319	4679418664	6682265708
		2173032350	7933278680	1228521411	1840875841	2535715340
		6602483011	7664139458	−{2,3}		
208	−45902296	2206179186	5598029405	7332559105	9370917366	3618746795
		3585172559	3799606865	5970041175	9308402125	9096461499
		3328384490	8802899119	2186757099	4384998570	4423384432
		3462124353	7170675162	8536213237	9360250776	4120246691
		1454621569	2193925929	−{2,3,5,17,53}		
210	5	1031725772	6295759279	1981851064	9676853975	9962892161
		9631485229	8959297983	4925158772	7204489612	7090496935
		0689608783	2201695440	4553092124	5210024031	0159699351
		1088954708	5827189886	2459182485	1689483118	7470399162
		0426012819	7938441124	0725752244	−{2,3,7,11,31,43,71,211}	
212	−5782	2762303656	9554015377	2712429171	4251219952	0385256050
		8576732638	1720376565	1689837517	8826924331	4587153964
		0779505551	0123632318	8057368512	1550902842	4409830023
		7546775110	4213714925	3445893517	1858879998	4318266834
		8799571247	2760829621	9105036976	−{2,3,5,107}	
214	6676248	2167835881	0322637794	4128093634	5107953790	8103711340
		3421242076	3304042329	5091056631	6499243600	5693167818
		4911170586	1809297698	3100932819	7867121979	7119475720
		7678480685	3084385157	3109320947	9856004904	0741960957
		8186348324	1367729728	8448950602	−{2,3}	
216	−7853530764	4450416322	5916259639	3124444282	2957000240	1395817760
		8314734816	2128994899	8561746180	2936510368	0530514353
		9547117531	4002045806	3281621851	7344439893	1506681060
		9995523753	2584232544	7916932745	4283324504	8399788068
		5495854177	2690855641	2817155661	−{2,3,5,7,13,19,37,73,109}	
218	941	0689406705	8725524544	3288258762	4852939477	9681195006
		2604571110	5610846395	2039233284	5981956117	8018896163
		7477680645	9040581129	6983885871	3098394359	2743108529

		1769116375	3413686776	8545665041	7772745635	4356010999
		1753289288	4875336862	6769177869	5021253934	−{2,3}
220	−1148493	3873465183	9938498599	2068055925	4835379696	4714139788
		4210113949	5708268220	8399738934	9974452549	7367883712
		9461290535	0793890370	2921608803	1307012711	0695460686
		2596755565	2249577996	7089908426	3152138601	3855375811
		7119153292	3110885047	6401867092	5355305441	−{2,3,5,11,23}
222	1427295874	2848785677	1416320087	1224998971	7991300396	6452873285
		1045408568	2263510522	2850318091	5783619081	8912755517
		5693032632	9197981112	0382549614	1975027633	6559804985
		0819895653	4963952936	8205790276	5386744234	4443377036
		7136781907	0037356974	4842136656	7512564136	−{2,3,7,223}
224	−180	5955958690	9309014228	5728117654	5609267193	9480952331
		2742566785	2475999079	8082620122	6361991190	4617400054
		6901247289	6416991911	8036907453	4931915475	9313949607
		0714040934	0937943854	7362681132	6992433312	7147855342
		7099813261	3598855839	6032846537	0376562367	7153769200
		−{2,3,5,17,29,113}				
226	232615	3530766080	5216129798	5184708876	1617363833	1726936408
		1987699998	8452631715	3581861945	5112354774	6947154327
		9661317679	2671578428	2504601643	8929726215	3089565855
		8219608074	4723944327	1504659390	3980456248	6544681615
		6333669851	7437321421	9464916240	9434172746	5296001281
		−{2,3,227}				
228	−304957517	1549959476	8194281926	1542593785	3270373100	7776186897
		5138058297	3758251258	3074615208	2253504769	0479552881
		4604199461	4109365080	6392836448	3066072567	6098663456
		1807840321	0677907520	3402243744	2309724077	0474029979
		5494865873	5436614108	7565109793	1427313172	9435753210
		−{2,3,5,7,13,229}				
230	40	6858060764	3397344240	1212412493	7318633684	3107545216
		5580263344	8454630775	2980369933	1020735696	8180151498
		7467110796	4869898086	3240064474	1961740985	9552136116
		2223064607	2185060378	6948409495	2820977951	0490781228
		2479879036	7669303934	9965846795	6084050515	4210865512
		0657424844	−{2,3,11,47}			
232	−55231	0313219743	6162523200	4409318639	2324279514	4462697421
		4855771006	3188681185	3706834819	3594302895	0685889421
		1330610119	6313289429	8186765908	9391468737	5002965153
		5401993003	3316776886	5616716658	6380513825	0188758085
		7958437505	8248612456	3974606583	1655771141	7225503375

TABLE 3—*Continued*

n	B_n					
		4381668883	−{2,3,5,59,233}			
234	76277279	3964343924	8699496902	0496121553	3858633733	3094314992
		2602156915	5404657656	0059673430	5378477493	7166852319
		3172824104	4824672538	7679230961	3234297171	0355300226
		2769721082	5308894750	2186829704	9269056086	3590200953
		7940104635	7094152574	4783953140	6553823465	9776824872
		8101363903	−{2,3,7,19,79}			
236	−10	7155711196	9788631327	9352400106	5396932666	8761158145
		5455800640	1453678013	1050047902	8996165766	0468153629
		4242852577	4436006216	1743438848	0821803894	5482278121
		0656408493	4415996760	2233854058	9979906147	8065707036
		1932180305	9982485664	9902290017	1134761196	7831390856
		4885838903	5244104859	−{2,3,5}		
238	15310	2008959691	8844534409	1615335533	4355847353	7267467117
		9474080648	1780121607	8482715373	2479581379	0999964338
		6407684728	6660910064	2596622727	3199004907	7787555531
		2112977159	6097118162	7760380806	3882142768	4946257333
		4099183377	0247322956	9895401431	9651749746	3517067124
		8782241352	5979034353	−{2,3,239}		
240	−22244891	6821798346	6766023488	6504851082	4835127801	6901932057
		8856338459	1178982564	7646719073	1955204204	3262555126
		9575436649	7661464998	0072673324	0943796791	6800349276
		6675814212	9545464419	1440291140	1882638087	6235632262
		6097467497	0219781136	8568253844	9175520779	5120926143
		9553024713	8574463721	−{2,3,5,7,11,13,17,31,41,61,241}		
242	3	2862679190	6901391668	1897364368	9527536518	2790039834
		4087566626	0584304963	6976103299	6408617489	5285569627
		3396921301	6945311830	2594958319	8052492243	4780786847
		1048283783	0512022452	6520446365	6142590630	1564731368
		1813645163	6729184891	6461470053	5164667652	7378396322
		7192329317	6646330108	5549078253	−{2,3,23}	
244	−4935	5928955960	3449020711	9381915759	6349699881	9967717969
		8708705972	5604467196	9633417732	5441813083	8585332169
		9668527213	2945193753	2837808099	2715624519	5426701817
		6662639276	6618976366	7390596305	0805318122	4601261827
		2025136515	1101918685	1568094572	7733353488	0829389397
		8043072083	2964232337	2756268499	−{2,3,5}	
246	7534957	1200832506	7212266049	7792839567	2782441082	4047848115

		8629135810	0586514612	5521159057	3572502170	1660668994
		6714386726	2918577221	9905853925	0141553115	5047192006
		2584159153	7974881907	3193814078	3061938087	9790766520
		0890893653	9580078697	1227136624	8264605129	3314709961
		3048103966	1982928154	5191164668	−{2,3,7,83}	
248	−1	1691485154	5841777278	0889247316	5504178389	9537111655
		7393365156	2527151686	4037772408	1413341139	8220377591
		3792860024	5560819655	4041431243	6529831158	4013811373
		2371174050	5486569829	5510092809	7355456887	6734383380
		4166388754	5474825140	9659524107	5021479246	8850325243
		8697409702	5058507517	1554253442	1158331599	−{2,3,5}
250	1843	5261467838	9394126646	2015977022	3239649247	7000034429
		4569758971	5176258684	2335348377	7655844047	4285697821
		8329220039	6978835905	6020603056	3580448568	1973565915
		1510513686	8316837867	5226653094	2856333382	8622890759
		5799693397	1198209110	9285643939	6181295360	9407215690
		8622535217	4286407738	3938476752	5254881572	−{2,3,11,251}

Mathematics Department
California Institute of Technology
Pasadena, California 91109

1. THOMAS CLAUSEN, "Theorem," *Astr. Nachrichten*, v. 17, 1840, cols. 351–352.

2. S. A. JOFFE, "Calculation of the first thirty-two Eulerian numbers from central differences of zero," *Quart. J. Math.*, v. 47, 1916, pp. 103–126.

3. S. A. JOFFE, "Calculation of eighteen more, fifty in all, Eulerian numbers from central differences of zero," *Quart. J. Math.*, v. 48, 1917–1920, pp. 193–271.

4. D. H. LEHMER, "An extension of the table of Bernoulli numbers," *Duke Math. J.*, v. 2, 1936, pp. 460–464.

5. NIELS NIELSEN, *Traité Élémentaire des Nombres de Bernoulli*, Paris, 1923.

6. J. PETERS & J. STEIN, *Zehnstellige Logarithmentafel*, Berlin, 1922.

7. S. Z. SEREBRENNIKOFF, "Tables des premiers quatre vingt dix nombres de Bernoulli," *Mém. Acad. St. Petersbourg* 8, v. 16, 1905, no. 10, pp. 1–8.

8. K. G. C. VON STAUDT, "Beweis eines Lehrsatzes die Bernoullischen Zahlen betreffend," *J. für Math.*, v. 21, 1840, pp. 372–374.