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Computation of the Bivariate Normal Integral

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Abstract. This paper presents a simple and efficient computation for the bivariate normal integral based on direct computation of the double integral by the Gauss quadrature method.

1. Introduction. The probability distribution of the normalized Normal Distribution is: [1]

$$\Phi(h, k, \rho) = \Pr\{(x_1 < h) \cap (x_2 < k)\},$$

(1)

$$\Phi(h, k, \rho) = (2\pi\sqrt{1-\rho^2})^{-1} \int_{-\infty}^{h} \int_{-\infty}^{k} \exp\left[-\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2(1-\rho^2)}\right] dx_1\, dx_2.$$

Substitute:

$$u_1 = \frac{h - x_1}{[2(1-\rho^2)]^{1/2}}; \qquad u_2 = \frac{k - x_2}{[2(1-\rho^2)]^{1/2}}. \tag{2}$$

Define:

$$h_1 = \frac{h}{[2(1-\rho^2)]^{1/2}}; \qquad k_1 = \frac{k}{[2(1-\rho^2)]^{1/2}}. \tag{3}$$

Then:

$$\Phi(h, k, \rho) = \frac{(1-\rho^2)^{1/2}}{\pi} \int_0^\infty \int_0^\infty \exp[-u_1^2]\exp[-u_2^2]$$

(4)

$$\exp[h_1(2u_1 - h_1) + k_1(2u_2 - k_1) + 2\rho(u_1 - h_1)(u_2 - k_1)]\, du_1\, du_2.$$

By Gauss quadrature [2]:

$$\Phi(h, k, \rho) \doteq \frac{(1-\rho^2)^{1/2}}{\pi} \sum_{i,j=1}^{k} A_i A_j f(x_i, x_j), \tag{5}$$

where

$$f(x, y) = \exp[h_1(2x - h_1) + k_1(2y - k_1) + 2\rho(x - h_1)(y - k_1)]. \tag{6}$$

The values of A_i, x_i for $k = 2, \ldots, 15$, can be found in [3]. If $h, k, \rho \leqslant 0$, then $0 < f(x, y) \leqslant 1$, and the error in (5) is relatively small. We will make use of the following formulae in order to calculate the double integral for $h, k, \rho \leqslant 0$.

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2. The Method. The following formulae can be found in [1]:

$$\Phi(h, k, \rho) = \phi(h) + \phi(k) - 1 + \Phi(-h, -k, \rho), \tag{7}$$

$$\Phi(h, k, \rho) = \phi(k) - \Phi(-h, k, -\rho), \tag{8}$$

$$\Phi(h, k, \rho) = \phi(h) - \Phi(h, -k, -\rho), \tag{9}$$

where

$$\phi(h) = \frac{1}{2\pi} \int_{-\infty}^{h} \exp\left[-\frac{x^2}{2}\right] dx. \tag{10}$$

For $h, k \neq 0$

$$\Phi(h, k, \rho) = \Phi(h, 0, \rho(h, k)) + \Phi(k, 0, \rho(k, h)) - \delta_{hk}, \tag{11}$$

where

$$\rho(h, k) = \frac{(\rho h - k)\mathrm{Sgn}(h)}{\sqrt{h^2 - 2\rho hk + k^2}}, \quad \delta_{hk} = \frac{1 + \mathrm{Sgn}(h) \cdot \mathrm{Sgn}(k)}{4} \tag{12}$$

and

$$\mathrm{Sgn}(x) = \left\{ \begin{array}{rl} 1, & x \geqslant 0 \\ -1, & x < 0 \end{array} \right\}.$$

Algorithm. If $h \cdot k \cdot \rho \leqslant 0$, then do one of the following:

(a) If $h \leqslant 0, k \leqslant 0, \rho \leqslant 0$ compute directly.

(b) If $h \leqslant 0, k \geqslant 0, \rho \geqslant 0$ use (9).

(c) If $h \geqslant 0, k \leqslant 0, \rho \geqslant 0$ use (8).

(d) If $h \geqslant 0, k \geqslant 0, \rho \leqslant 0$ use (7).

If $h \cdot k \cdot \rho > 0$, use (11). Note that every computation of Φ will now satisfy $h \cdot k \cdot \rho = 0$, (since the new k equals 0).

3. Results. An advantage of this method is that for every ρ (even close to 1) there is no convergence problem. In Table 1 we present results of the average run time and maximum-error for various values of k in (5).

Low values of the exponent in (6) cause $f(x, y)$ to vanish. To save computational effort, if the exponent is lower than the values in Column 4, Table 1, we assume that $f(x, y)$ is zero. The values in the table have been set such that maximum-error remains the same to two significant digits. In Column 5 we present the reduced run time. In order to compare with existing results [4], [5] we take $k = 5$. Note here that in $k = 3, 4, 5$ a further reduction of 0.7 m.s. can be achieved by using the approximation in [6] for the error function instead of using the function erf.

By a regression on Column 3 we can deduce that computations outside the double integration are the same for every k and require approximately 1.6 m.s., so for $k = 5$ and the approximated error function the average computation time is 2.2 m.s. The double integration requires an average 1.3 m.s.

TABLE 1. *Results*

k	maximum error	run time (10^{-3}sec)	limit of exponent	reduced run time
3	1.1×10^{-4}	3.0	−8	1.8
4	8.1×10^{-6}	4.1	−10	2.2
5	5.5×10^{-7}	5.5	−12	2.9
6	3.8×10^{-8}	7.2	−15	3.6
7	3.0×10^{-9}	9.2	−17	4.6
8	2.2×10^{-10}	11.5	−20	6.0
9	1.5×10^{-11}	14.2	−22	7.5
10	1.1×10^{-12}	17.0	−25	9.4

All time data are based on a CDC/6400.

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