JOURNAL OF RESEARCH of the National Bureau of Standards — B. Mathematical Sciences
Vol. 78B, No. 3, July–September 1974

Computation of the Field of Values of a 2 X 2 Matrix *

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(May 17, 1974)

It is known that the field of values, $F(A) \equiv \{x^*Ax : x^*x = 1,\ x \epsilon C^2\}$, of a 2×2 matrix A is a convex set whose boundary is an ellipse. This is used to compute $F(A)$ explicitly from the entries of A when A is 2×2 and real. Employing known properties of the field of values, this may then be used to estimate $F(A)$ when A is $n \times n$ and real.

Key words: Eigenvalues; ellipse; field of values; square matrix.

For an arbitrary $n \times n$ complex matrix A, the *field of values*, $F(A)$ is defined by

$$F(A) = \{x^*Ax : x^*x = 1, \qquad x \epsilon C^n\}.$$

It is well known that $F(A)$ is a closed, convex subset of the complex plane. It arises in several practical and theoretical settings, and, in particular, $F(A)$ contains each eigenvalue of A. Furthermore, in case $n = 2$, the boundary of $F(A)$ is an ellipse (or, in degenerate cases, a line segment or circle). In [1] [1] a method is given for determining this ellipse by first solving for the eigenvectors of A. In this note we determine the ellipse directly as a function of the entries for any real (2×2) matrix A.

THEOREM: *Suppose* $\mathrm{A} = \begin{bmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{bmatrix}$, a, b, c, d *real, and let* x, y, z, w *be the four points in the complex plane:*

$$\mathrm{x} = \frac{1}{2}\left(\mathrm{a} + \mathrm{d} - ((\mathrm{a} - \mathrm{d})^2 + (\mathrm{b} + \mathrm{c})^2)^{1/2}\right)$$

$$\mathrm{y} = \frac{1}{2}\left(\mathrm{a} + \mathrm{d} + ((\mathrm{a} - \mathrm{d})^2 + (\mathrm{b} + \mathrm{c})^2)^{1/2}\right)$$

$$\mathrm{w} = \frac{\mathrm{a} + \mathrm{d}}{2} + \mathrm{i}\left|\frac{\mathrm{b} - \mathrm{c}}{2}\right|$$

$$\mathrm{z} = \frac{\mathrm{a} + \mathrm{d}}{2} - \mathrm{i}\left|\frac{\mathrm{b} - \mathrm{c}}{2}\right|.$$

Then the boundary of the convex set F(A) *is the (possibly degenerate) ellipse whose axes are the line segments joining* x *to* y *and* w *to* z *respectively.*

AMS Subject Classification: 15–A63, 65–F99.

*This work was done while the author was a National Academy of Sciences—National Research Council Postdoctoral Research Associate at the National Bureau of Standards, Washington, D.C. 20234.

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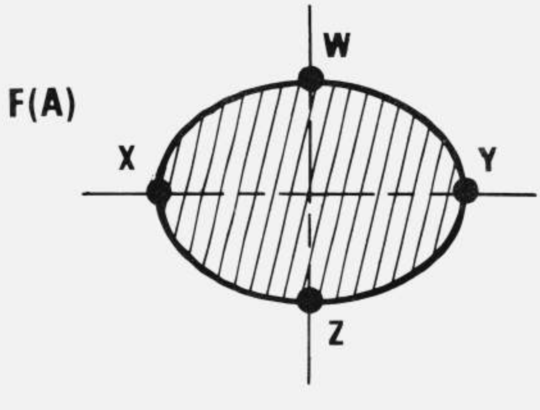
[1] Figures in brackets indicate the literature references at the end of this paper.



PROOF: We know that the boundary of $F(A)$ is an ellipse. Since A is real, the ellipse is symmetric with respect to the real axis of the complex plane. Therefore, one of the axes of the ellipse lies on the real line. It is the line segment whose endpoints are the eigenvalues of

$$\operatorname{Re}(A)=\frac{A+A^T}{2}=\begin{bmatrix} a & \frac{b+c}{2} \\ \frac{b+c}{2} & d \end{bmatrix}.$$

A computation yields that these two values are the numbers x and y. The remaining axis of the ellipse must be parallel to the imaginary axis of the complex plane. The length of this axis is just the distance between the eigenvalues of

$$\operatorname{Im}(A)=\frac{A-A^T}{2}=\begin{bmatrix} 0 & \frac{b-c}{2} \\ \frac{c-b}{2} & 0 \end{bmatrix}.$$

This implies that the second axis has endpoints w and z as asserted. This completely determines the boundary ellipse of $F(A)$ and thus $F(A)$ itself.

In case (1) $b=c$ (A symmetric) or (2) $a=d=0$ and $c=-b$ (A skew-symmetric), the given ellipse degenerates into a line segment. In case $(a-d)^2+4bc=0$, the ellipse degenerates into a circle and A has only a single repeated eigenvalue $\lambda=\frac{a+d}{2}\cdot$ In all other cases the boundary of $F(A)$ is a legitimate ellipse.

For an arbitrary $n\times n$ matrix A, the normalization

$$B=A-\frac{\operatorname{Tr}(A)}{n}I$$

produces a matrix B of trace 0 whose eigenvalues and field of values are merely a translation of those of A by the scalar $\frac{\operatorname{Tr}(A)}{n}\cdot$ For a real 2×2 matrix A, this normalization allows us to give the equation of the boundary ellipse of $F(A)$ in standard form, since the translated ellipse will be centered at the origin.

COROLLARY 1: *Suppose* A *is* 2×2, *real*, $\mathrm{Tr}(A)=0$, *and* $A=\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$.

Then the boundary of F(A) *has one of the following forms:*

(*i*) *The line segment joining*

$$\pm [a^2+b^2]^{1/2} \textit{ if } c=b;$$

(*ii*) *The line segment joining*

$$\pm i \left|\frac{b-c}{2}\right| \textit{ if } a=0 \textit{ and } c=-b; \textit{ and}$$

(*iii*) $\left\{u+iv : \dfrac{u^2}{u_0^2}+\dfrac{v^2}{v_0^2}=1\right\}$ *where*

$$u_0=\left[a^2+\left(\frac{b+c}{2}\right)^2\right]^{1/2},\ v_0=\left|\frac{b-c}{2}\right|;\ u, v \textit{ real},$$

otherwise.

A computation then shows that in case (i) and (ii), the eigenvalues of A occur at the endpoints of the given line segments. In case (iii) the eigenvalues of A occur at the foci (center) of the given ellipse (circle). In each of the cases, the eigenvalues of A are

$$\pm (a^2+bc)^{1/2}.$$

For arbitrary n, the theorem above and the convexity of $F(A)$ may be used to obtain an "inner" approximation of $F(A)$ for an arbitrary real $n\times n$ matrix $A=(a_{ij})$. Let

$$A_{i,j}=\begin{bmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{bmatrix}$$

denote the 2×2 principal submatrix of A determined by i and j. Then $F(A_{i,j})$ is determined explicitly by the theorem when A is real. Let also Co(S) denote the closed convex hull of a subset S of the complex plane. We then have:

COROLLARY 2: *Suppose* $A=(a_{ij})$ *is* $n\times n$, *then*

$$\mathrm{Co}\left(\bigcup_{1\le i<j\le n} F(A_{i,j})\right) \subseteq F(A).$$

PROOF: The assertion is valid since the field of values of any principal submatrix of A is contained in $F(A)$ and $F(A)$ is convex. Its efficacy lies in the fact that the left hand side is easily computed from the entries of A when A is real.

An outer approximation for $F(A)$ was developed in [2, 3] using Gersgorin estimates. Thus by combining our corollary 2 and the main theorem of [3], convex sets $s(A)$ and $S(A)$ may be computed directly from the entries of A so that

$$s(A)\subseteq F(A)\subseteq S(A).$$

References

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(Paper 78B3–406)