Monograph:
El-Ghezawi, O.M.E., Zinober, A.S.I., Owens, D.H. et al. (1 more author) (1982) Computation of the Zeros and Zero-Directions of Linear Multivariable Systems. Research Report. ACSE Report 174 . Department of Control Engineering, University of Sheffield, Mappin Street, Sheffield

COMPUTATION OF THE ZEROS AND ZERO-DIRECTIONS OF LINEAR MULTIVARIABLE SYSTEMS

BY

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Research Report No. 174

February 1982

SUMMARY

A new geometric method of calculating multivariable system zeros and zero-directions is presented by considering a particular choice of state feedback control law. This control law has been motivated by the study of variable structure systems in the sliding mode. The cases of singular and non-singular CB have been treated separately.

1. Introduction

A new algorithm for the calculation of the multivariable zeros and zero-directions of the square system S(A, B, C) is derived for the case of equal number of inputs and outputs. It is shown that the zeros are a subset of the eigenvalues of the matrix $A_k = [A - B(CA^{k-1}B)^{-1}CA^k]$; $1 \leq k \leq n$, where k is defined in (12) and (13) and is equal to 1 if $|CB| \neq 0$. The matrix A_k with k=1 is formed by a particular state feedback control law associated with variable structure systems in the sliding mode (El-Ghezawi et al, 1981 , Utkin 1977). Once the zeros have been calculated the state and input zero-directions can be determined independently and without resorting to the null space of the (n+m)th order system matrix defined in Kouvaritakis and MacFarlane (1976).

The paper begins with a brief overview of variable structure systems in the sliding mode. The main results are then presented in the form of three theorems. The relationship between the algorithm given for the case where $|CB| \neq 0$ and the NAM algorithm due to Kouvaritakis and MacFarlane (1976) is then discussed. Finally worked examples are included to illustrate the validity of the methods.

2. Calculation of the system zeros

Consider the square linear time-invariant multivariable system S(A,B,C)

$$\dot{x} = Ax + Bu$$
$$y = Cx \tag{1}$$

where $x \in R^n$, $u \in R^m$, $y \in R^m$ and B, C are full rank matrices. The algorithms presented below for the calculation of the zeros and zero-directions of S(A,B,C) are motivated by the observation that a particular feedback control law

$$u = -(CA^{k-1}B)^{-1}CA^k x \tag{2}$$

which yields the closed loop system

$$\dot{x} = \left[A - B(CA^{k-1}B)^{-1}CA^{k}\right]x = A_k x \tag{3}$$

suggests a particularly simple method of computing the zeros. The control equation (2) (with k = 1) arises in the analysis of variable structure systems in the sliding mode and is known as the equivalent control. Further details of the relationship between variable structure systems and the calculation of multivariable system zeros are given in El-Ghezawi et al. (1981). A brief overview of variable structure systems in the sliding mode is presented in the next section.

3. Variable Structure Control Systems in the Sliding Mode

Variable structure systems are characterized by a discontinuous control action which changes structure upon reaching a set of switching surfaces. The control has the form

$$u_i = \begin{cases} u_i^+(x) & s_i(x) > 0 \\ u_i^-(x) & s_i(x) < 0 \end{cases}$$

where u_i is the ith component of u and $s_i(x)$ is the ith of the m switching hyperplanes which satisfy $s(x) = 0, s \in R^m$. Sliding motion occurs when the control constrains the state to remain on the switching hyperplanes $s(x) = 0$ (Utkin 1977) and then

$$s(x) = Cx = 0 \text{ and } \dot{s}(x) = C\dot{x} = 0$$

The discontinuous control u can then be substituted by a continuous function termed the 'equivalent control' u_{eq}. The equivalent control is obtained by setting $\dot{s}(x) = 0$ and then solving for u to yield

$$u_{eq} = -(CB)^{-1}\, CAx.$$

Substituting back in equation (1) yields a reduced system described by

$$\dot{x} = \left[I - B(CB)^{-1}C\right]Ax = A_{eq}x. \tag{4}$$

Observe that variable structure systems in the sliding mode are closely related to the output zeroing problem as defined by MacFarlane and Karcanias (1976). This follows by interpreting the switching functions s as the system outputs y.

Calculation of the system zero's using variable structure systems is based on the observation that if $y(t) = 0$ for $t > o$ then $\dot{y}(t) = 0$ for $t \geq 0$. This yields an algorithm that consists of determining the eigenvalues of A_{eq} which arises when the feedback control which ensures that $\dot{y}(t) = 0$ is applied to (1).

4. Calculation of Zeros and Zero-directions

The problem of calculating the zeros and zero-directions has been tackled using three different approaches depending upon the form of CB. The three cases considered are:

Case 1: $|CB| \neq 0$; This is dealt with in Theorem 1.

Case 2: $CA^{i-1}B = 0$ for all $i = 1,2, \ldots, k-1$ and $|CA^{k-1}B| \neq 0$; This is known as the uniform rank case (Owens 1979) and is dealt with in Theorem 2.

Case 3: A non-uniform rank case with $|CB| = 0$ is discussed in Theorem 3.

4.1 Case 1: $|CB| \neq 0$

Theorem 1: For the system $S(A,B,C)$ with $|CB| \neq 0$

(i) the n-m zeros z_i of the system are given by

(a) $\{z_i\} = sp(A_{eq}) - \{0\}^m$

or (b) are the eigenvalues of the (n-m)th order matrix $M^g A_{eq} M$ where M is a basis matrix for $N\{C\}$ and M^g is a generalized inverse of M.

(ii) the state zero directions ω_i associated with the zeros z_i are

(a) the corresponding eigenvectors of the matrix A_{eq}

and (b) satisfy $\omega_i = M\alpha_i$ where the α_i are the eigenvectors of $M^g A_{eq} M$.

(iii) the input zero-directions are $g_i = -(CB)^{-1} CA\omega_i = -(CB)^{-1} CAM\alpha_i$.

Proof:

(i) let ν^* be the maximal (A,B)-invariant subspace in the kernel or null space of C. Then $\nu^* = N(C)$ and $R^n = \nu^* \oplus R(B)$. Since

$$CA_{eq} = CA - (CB)(CB)^{-1}CA = 0 \tag{5}$$

it therefore follows that $A_{eq} R^n \subset \nu^*$ and in particular

$$A_{eq} \nu^* \subset \nu^*. \tag{6}$$

Since zeros are invariant under state feedback, the n-m zeros of $S(A,B,C)$ and the closed-loop system $S(A_{eq}, B,C)$ are equal and therefore from (6) the zeros are a subset of the eigenvalues of A_{eq}. The relation $A_{eq}R^n \subset \nu^*$ implies that all other eigenvalues of A_{eq} are zero proving (i)(a).

Furthermore, if z_i is a zero, there exists a non-zero eigenvector $\omega_i \in N\{C\}$ such that

$$A_{eq} w_i = z_i \omega_i \qquad i = 1,2, \ldots, n-m \tag{7}$$

Let M be a basis matrix for N(C) and write $\omega_i = M\alpha_i$ then

$$A_{eq} M\alpha_i = M\alpha_i z_i \tag{8}$$

and

$$M^g A_{eq} M \alpha_i = z_i M^g M\alpha_i \tag{9}$$

$$= z_i \alpha_i \tag{10}$$

where M^g is any generalized inverse of M satisfying $M^g M = I_{n-m}$. It follows from (9) that the zeros of S(A,B,C) are given by the eigenvalues of $M^g A_{eq} M$ which has order n-m. This proves (i)(b).

(ii) This follows from equations (7) and (10).

(iii) Substituting $x = \omega_i = M\alpha_i$ in (2) with $k = 1$ gives

$$g_i = -(CB)^{-1} CA \, \omega_i = -(CB)^{-1} CA \, M \, \alpha_i \tag{11}$$

which completes the proof of the theorem.

Using section (i)(a) in the theorem yields a technique for determining the (n-m) zeros by finding the eigenvalues of the specified matrix A_{eq}, without the need for calculating the annihilator matrices M and N , c.f. the NAM algorithm of Kouvaritakis and MacFarlane (1976). The method of section (i)(b) has links with the NAM algorithm (see section 5). For both cases (a) and (b) the zero-directions are calculated without resorting to the determination of the null space of the (n+m) th order system matrix as defined in Kouvaritakis and Macfarlane (1976).

For the case where n-m is large, method (a) is computationally simpler than (b). For n-m small, i.e. $n \approx m$, method (b) may be a reasonable alternative.

4.2 Case 2: Uniform Rank $CA^{i-1}B = 0$ $i = 1,\ldots,k-1$ and $|CA^{k-1}B| \neq 0$

A system is said to have uniform rank k (Owens 1979) if

$$CA^{i-1}B = 0 \qquad 1 \leq i < k \tag{12}$$

and

$$|CA^{k-1}B| \neq 0 \; . \tag{13}$$

R^n is then decomposed as the direct sum

$$R^n = B \oplus AB \oplus \ldots \oplus A^{k-1}B \oplus \nu^* \tag{14}$$

and

$$\nu^* = \bigcap_{i=1}^{k} N(CA^{i-1}) \tag{15}$$

Theorem 2: Given the feedback control

$$u = -(CA^{k-1}B)^{-1} CA^k x \tag{16}$$

and

$$A_k = A - B(CA^{k-1}B)^{-1}CA^k \tag{17}$$

then

(i) the (n-km) zeros z_i of S(A,B,C) are given by

(a) $\{z_i\} = sp(A_k) - \{0\}^{km}$

or (b) the eigenvalues of the matrix $M_k^g A_k M_k$ where M_k is a basis matrix for ν^*

(ii) the state zero-directions ω_i associated with the zeros z_i are

(a) The corresponding eigenvectors of the matrix A_k

and (b) satisfy $\omega_i = M_k \alpha_i$ where the α_i are eigenvectors of $M_k^g A_k M_k$

(iii) the input zero-directions are given by $g_i = -(CA^{k-1}B)^{-1}CA^k M_k \alpha_i$

Proof: (i) The zeros of S(A,B,C) are invariant under state feedback and are therefore equal to the zeros of $S(A_k,B,C)$. Noting that

$$CA^{i-1}A_k = CA^i - CA^{i-1}B(CA^{k-1}B)^{-1}CA^k = \begin{cases} CA^i & 1 \leq i < k \\ 0 & i = k \end{cases}$$

and therefore

$$CA^{i-1}A_k \nu^* = 0 \qquad 1 \leq i \leq k \tag{18}$$

we have

$$A_k \nu^* \subset \nu^* . \tag{19}$$

and hence from (15) n-km eigenvalues of A_k are the zeros of S(A,B,C). If z_i is a zero of S(A,B,C) then

$$A_k w_i = z_i w_i. \tag{20}$$

Writing

$$\begin{pmatrix} C \\ CA \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ CA^{k-1} \end{pmatrix} A_k = \begin{pmatrix} O & I & \cdots & & O \\ & O & & I \cdots & O \\ & & & & I \\ & & & & \\ O & \cdots & & O & O \end{pmatrix} \begin{pmatrix} C \\ CA \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ CA^{k-1} \end{pmatrix}$$

and noting that $S(A^T,C^T,B^T)$ has uniform rank k and hence, in a similar manner to (14),

$$\text{rank} \begin{pmatrix} C \\ CA \\ \cdot \\ \cdot \\ \cdot \\ CA^{k-1} \end{pmatrix} = km,$$

we conclude that A_k has km zero eigenvalues. If S(A,B,C) has no zeros at the origin then (i)(a) follows trivially since the (n-km) zeros must be the non-zero eigenvalues of A_k. If S(A,B,C) has a zero at the origin, note that (i)(a) is true for $S(A - \alpha I,B,C)$ for $0 < \alpha < \delta$ and some suitably small δ. So (i) (a) follows by continuity letting $\alpha \to 0$.

Now let M_k be a basis for ν^*. Then

$$\omega_i = M_k \alpha_i \quad , \qquad \alpha_i \neq 0 \tag{21}$$

$$A_k M_k \alpha_i = M_k \alpha_i z_i$$

and

$$M_k^g A_k M_k \alpha_i = z_i M_k^g M_k \alpha_i$$

$$= z_i \alpha_i \tag{22}$$

where M_k^g is any matrix satisfying $M_k^g M_k = I_{n-km}$. Therefore, the system zeros are given by the eigenvalues of

$$M_k^g A_k M_k \tag{23}$$

(ii) From (21) $\omega_i = M_k \alpha_i$

(iii) Substituting in (16) with x replaced by $\omega_i = M_k \alpha_i$ yields

$$g_i = (CA^{k-1}B)^{-1} CA^k M_k \alpha_i \tag{24}$$

4.3: Case 3: The non-uniform rank case

The uniform rank case is completely resolved in the preceding sections. The case of a non-uniform rank system is more complex. The following result does however, identify a condition when Theorems 1 and 2 have a natural generalization.

Theorem 3

Let $k \geq 2$, $|CA^{i-1}B| = 0$ for $i=1,2,\ldots k-1$, $|CA^{k-1}B| \neq 0$ and $\nu^* \subset \bigcap_{i=1}^{k} N(CA^{i-1})$. Then, $A_k \nu^* \subset \nu^*$ and the zeros of S(A,B,C) are a subset of the eigenvalues of A_k. The zero-directions are then calculated as in Theorem 2.

Proof: Let ν be a basis matrix for ν^* then

$$A\nu = \nu J + \hat{B}. \tag{25}$$

By the definition of ν^* there exists an m x n matrix F such that

$$\hat{B} = BF\nu. \tag{26}$$

From (25)

$$CA^i \nu = CA^{i-1} \nu J + CA^{i-1}\hat{B} \qquad 1 \leq i \leq k$$

If $i < k$ then $CA^{i-1}\hat{B} = 0$.

If $i = k$ then $CA^k \nu = CA^{k-1}\hat{B}$

$$= CA^{k-1}BF\nu$$

i.e. a valid solution for F is

$$F = (CA^{k-1}B)^{-1} CA^k.$$

Also,

$$[A - B(CA^{k-1}B)^{-1}CA^k]\nu = A\nu - B(CA^{k-1}B)^{-1}CA^k\nu$$

$$= A\nu - B(CA^{k-1}B)^{-1}CA^{k-1}(\nu J + \hat{B})$$

$$= A\nu - B(CA^{k-1}B)^{-1}CA^{k-1}BF\nu \ ; \quad \text{since } CA^{k-1}\nu = 0$$

$= A\nu - BF\nu$

$= (A - BF)\nu$

which has range in ν^*. This proves that $A_k \nu^* \subset \nu^*$ and the remainder of the theorem follows in a similar manner to theorem 2.

This result has a similar interpretation to Theorems 1 and 2 but may be difficult to apply in practice as k is not necessarily known and, although the zeros are a subset of the eigenvalues of A_k, we have not identified which subset. In this sense the result is primarily of theoretical interest.

5. The Relationship with the NAM algorithm

The algorithm (b) presented for the case $|CB| \neq 0$ resembles the NAM method of Kouvaritakis and MacFarlane (1976) in the sense that both methods determine the system zeros as the eigenvalues of an (n-m)-dimensional matrix. Furthermore, it can be easily shown that the matrix $N_e = M^g[I-B(CB)^{-1}C]$ qualifies for the matrix N in the NAM algorithm. This is because

$$N_e B = 0$$

and

$$N_e M = I_{n-m}$$

where the matrix M is the same in both methods (El-Ghezawi et al, 1981). Similar comments apply in the case of $CA^{i-1}B = 0$, $i < k$ and $|CA^{k-1}B| \neq 0$ when compared with Owens (1979).

6. Examples

We shall next give some worked examples to illustrate the algorithms proved in the previous sections.

Example 1: Consider the following example which was originally introduced by Patel (1977) where $|CB| \neq 0$.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 1 & -3 \end{pmatrix}; \; B = \begin{pmatrix} 0 & 0 \\ 1 & 2 \\ 0 & 1 \end{pmatrix}; C = \begin{pmatrix} 1 & 1 & -2 \\ 1 & 0 & 1 \end{pmatrix}$$

$$CB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2.$$

We now use the results of Theorem 1.

$$A_{eq} = [I - B(CB)^{-1}C]A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

Using (a) the zeros are given by sp $(A_{eq}) - \{0\}^2 = \{-3,0,0\} - \{0,0\}.$
$= -3$

From (b) $M = N(C) = [1 \quad -3 \quad -1]^T$ and $M^g = [1 \quad 0 \quad 0]$ giving

$$M^g A_{eq} M = [1 \quad 0 \quad 0] \begin{pmatrix} 0 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix} = -3.$$

The system, therefore has a single zero at -3 and this agrees with the value obtained by Patel (1977). The state zero-direction ω_1 is given by

$$\omega_1 = \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix} \alpha_1$$

where α_1 is an arbitrary scalar since n - m = 1. The input zero direction α_1 given by (11) is

$$g_1 = -(CB)^{-1}CA\,\omega_1$$
$$= -\begin{pmatrix} 1 & -2 & 8 \\ 0 & 2 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Example 2: Consider the uniform rank system

$$A = \begin{pmatrix} 0 & 0 & -2 & 0 & 0 \\ 1 & 0 & -5 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 \\ 0 & 0 & 2 & 0 & -4 \\ 0 & 0 & 1 & 1 & -4 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad C = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

with CB = O and

$$CAB = \begin{pmatrix} 1 & 0 & -4 & 1 & -4 \\ 0 & 1 & -3 & 1 & -4 \end{pmatrix} B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

We may use Theorem 2 with k = 2. Using (a) the zeros are given by

$$sp(A_k) - \{0\}^{km} = \{-3,0,0,0,0\} - \{0,0,0,0\}$$
$$= -3.$$

The algorithm (b) yields

$$M_k = \nu^* = N(C) \cap N(CA) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cap \begin{pmatrix} 4 & -1 & 4 \\ 3 & -1 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ -2 \\ -1 \end{pmatrix}$$

From (17)

$$A_k = \begin{pmatrix} 1 & 1 & -9 & 0 & 0 \\ 1 & 0 & -5 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 \\ -1 & 3 & -3 & 4 & -16 \\ 0 & 0 & 1 & 1 & -4 \end{pmatrix}$$

$$M_k^g A_k M_k = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix} A_k \cdot \nu^*$$

$$= \begin{bmatrix} 1 & 0 & -5 & 0 & 0 \end{bmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \\ -2 \\ -1 \end{pmatrix} = -3$$

Therefore the system has a single zero at z = -3. The state zero-direction is equal to $\omega_1 = M_k \alpha_1 = \begin{bmatrix} 2 & 1 & 1 & -2 & -1 \end{bmatrix}^T \alpha_1$ where α_1 is an arbitrary scalar since n-km=1.

The input zero-direction obtained from (24) is

$$g_1 = (CAB)^{-1} CA^2 M_k$$
$$= - (CAB)^{-1} CA.AM_k$$

$$= \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 4 \end{pmatrix} \begin{pmatrix} -2 \\ -3 \\ -3 \\ 6 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \end{pmatrix}$$

Example 3:

For

$$A = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 4 & -5 & 0 & 0 \\ -2 & 3 & -6 & 2 \\ 0 & 0 & 1 & -2 \end{pmatrix}; \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 4 \\ 6 & 0 \\ 0 & 0 \end{pmatrix}; \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

$|CB| = \begin{vmatrix} 0 & 1 \\ 0 & 4 \end{vmatrix} = 0$ but $CB \neq 0$. We now use Theorem 3.

$$CAB = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 4 & -5 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 4 \\ 6 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 6 & -3 \\ 0 & -16 \end{pmatrix}.$$

Clearly $|CB| = 0$, $|CAB| \neq 0$ and $k = 2$ since it is easily verified that

$$\nu^* \subset N(C) \cap N(CA) \subset \text{span}\left(\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right) \cap \text{span}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \\ -1 \\ 0 \end{pmatrix}\right) = \text{span}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right)$$

Let $M_k = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

and then

$$M_k^g = [0 \quad 0 \quad 0 \quad 1].$$

Note that $M_k^g A_k M_k$ can be easily calculated since $M_k^g A_k M_k = M_k^g \left[I - B(CAB)^{-1} CA\right] AM_k$ giving

$$M_k^g A_k M_k = -2.$$

The system therefore has a single zero at $z_1 = -2$ and this agrees with the results of Patel (1977). The state zero-direction is given by (21)

$$\omega_1 = M_k \alpha_1 = [0 \quad 0 \quad 0 \quad 1]^T$$

where α_1 is taken to be unity.

The input zero direction is obtained from (24)

$$g_1 = -(CAB_1)^{-1} CA^2 = \frac{1}{96} \begin{pmatrix} 16 & -3 \\ 0 & -6 \end{pmatrix} \begin{pmatrix} -5 & 7 & -5 & 2 \\ -16 & 21 & 4 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1/3 \\ 0 \end{pmatrix} .$$

8. Conclusions

A new geometric method for the calculation of system zeros and zero-directions has been presented for the system S(A,B,C) with an equal number of inputs and outputs for both the cases of CB singular and non-singular. The method presented is based upon a particular choice of state feedback control law motivated by variable structure systems in the sliding mode. The determination of the zeros is achieved by finding a subset of the eigenvalues of an nth order matrix A_k or by calculating the eigenvalues of an (n-m)th order matrix $M^g A_k M$. The algorithms provided offer the advantage over known techniques of the ability to calculate the state and the input zero-directions independently of each other without resorting to the determination of the null space of the (n+m)th order system matrix.

9. References

El-Ghezawi, O.M.E., Billings, S.A., and Zinober, A.S.I., 1981, Research Report No. 157, Department of Control Engineering, Sheffield University.

Kouvaritakis, B., and MacFarlane, A.G.J., 1976, Int.J. Control, 23, 149.

Macfarlane A.G.J., Karcancas, 1976, Int. J. Control, 24, 33.

Owens, D.H., 1977, Int.J. Control, 26, 537.

Owens, D.H., 1979, Proc. IEE, 126, No. 12, 1335.

Patel, R.V., 1977, in Control systems design by pole zero assignment, edited by F. Fallside, Academic Press, 145-165.

Utkin, V.I., 1977, IEEE. Trans. Automat. Control, 22, 212.