Asian Research Journal of Mathematics

16(3): 8-19, 2020; Article no.ARJOM.54909

ISSN: 2456-477X

Computational Analysis of the Stability of 2D Heat Equation on Elliptical Domain Using Finite Difference Method

Mehwish Naz Rajput[1], Asif Ali Shaikh[2] and Shakeel Ahmed Kamboh[3*]

[1]Govt Khan Bahadur Girls Degree College, Hyderabad, Sindh, Pakistan.
[2]Department of Basic Sciences and Related Studies, MUET, Jamshoro, Sindh, Pakistan.
[3]Department of Mathematics and Statistics, QUEST, Nawabshah, Sindh, Pakistan.

Authors' contributions

This work was carried out in collaboration among all authors. Author MNR is a postgraduate student at MUET, Jamshoro who proposed the idea of this study, performed the derivations of the stability condition and wrote the first draft of the manuscript. Authors AAS and SAK are her supervisor and co-supervisor respectively. Author AAS managed the literature search and discussion of the results. Author SAK contributed for the implementation of methodology on MATLAB and obtained the simulation profiles. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/ARJOM/2020/v16i330177
Editor(s):
(1) Danilo Costarelli, University of Perugia, Italy.
Reviewers:
(1) Noreliza Abu Mansor, Universiti Tenaga Nasional, Malaysia.
(2) Sie Long Kek, Universiti Tun Hussein Onn Malaysia, Malaysia.
(3) Hamid El Ouardi, University of Hassan II Casablanca, Morocco.
Complete Peer review History: http://www.sdiarticle4.com/review-history/54909

Received: 02 January 2020
Accepted: 25 February 2020
Published: 10 March 2020

Original Research Article

Abstract

Aims: The aim and objective of the study to derive and analyze the stability of the finite difference schemes in relation to the irregularity of domain.
Study Design: First of all, an elliptical domain has been constructed with the governing two dimensional (2D) heat equation that is discretized using the Finite Difference Method (FDM). Then the stability condition has been defined and the numerical solution by writing MATLAB codes has been obtained with the stable values of time domain.
Place and Duration of Study: The work has been jointly conducted at the MUET, Jamshoro and QUEST, Nawabshah Pakistan from January 2019 to December 2019.
Methodology: The stability condition over an elliptical domain with the non-uniform step size depending upon the boundary tracing function is derived by using Von Neumann method.

**Corresponding author: E-mail: shakeel.maths@yahoo.com;*

Results: From the results it was revealed that stability region for the small number of mesh points remains larger and gets smaller as the number of mesh nodes is increased. Moreover, the ranges for the time steps are defined for varied spatial step sizes that help to find the stable solution.

Conclusion: The corresponding stability range for number of nodes N=10, 20, 30, 40, 50, and 60 was found respectively. Within this range the solution remains smooth as time increases. The results of this study attempt to provide the stable solution of partial differential equations on irregular domains.

Keywords: Modeling and simulation; stability analysis; computational analysis; finite difference method; elliptical domain; heat equation.

1 Introduction

The Partial Differential Equations (PDEs) are widely used in many fields of science and engineering and considered as the principal sources of providing the mathematical models to govern the physical situations [1]. The 2D heat equation is a parabolic partial differential equation which is widely used in many scientific and engineering problems for the purpose of simulating the time dependent diffusion of heat or energy in the physical domains. For a simple one-dimensional case it is represented mathematically as follows:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

where u is the dependent variable and c is the thermal diffusivity constant. Solution of heat equation is computed by variety methods including analytical and numerical methods [2]. But when the heat equation is considered for 2-dimensional and 3-dimensional problems then the analytical solution becomes difficult or impossible in some cases. Then the numerical methods are best choice to solve the problem in 2D and 3D. The errors in the numerical methods are akin to the convergence behavior of the solution algorithm that may accumulate abruptly if the proper values of the time steps or mesh spacing are not selected. The consequences of such errors lead to the instability in the numerical solution and the situation becomes worst when the problem is defined over irregular domain. Without having any prior knowledge, it becomes difficult to give the guarantee of convergence. For the convergence of solution by a numerical finite difference scheme the consistency and stability are the necessary and sufficient requirements respectively. The stability of finite difference numerical schemes can be investigated by a procedure known as Von Neumann stability analysis or Fourier method [3]. In the following section some related works have been reviewed that discuss the solution of heat equation by different methods and the stability of numerical schemes used.

Since this study is concerned with the solution of heat equation over irregular boundaries therefore the attempt is made to highlight previous works that have been used for irregular boundaries. In this regard [4] and [5] have investigated a finite difference scheme for solving the variable coefficient Poisson and heat equations on irregular domains with Dirichlet boundary conditions. They considered non-graded Cartesian grids (grids for which the difference in size between two adjacent cells is not fixed) and employed a second order implicit discretization in time. A parallel solution approach for 2D heat equation was presented by Verena and Peter [6] and they showed that the good numerical approximations can be obtained using finite difference method. A systematic and practical overview of the numerical solution of 1D heat equation using finite difference method was given by Gerald [7]. The author has used the MATLAB codes to find the differences between explicit finite time, centered space (FTCS) and implicit backward time, centered space (BTCS) and implicit Crank-Nicolson methods. The semi discretized heat equations over irregular domains were solved by Kazufumi et al. [8]. They used second and fourth order grid based finite difference methods derived from multivariable Taylor series expansion and included the idea of eigenvalues. Their methods offer systematic treatment of the general boundary conditions in two and three dimensions. A new method to solve the steady state heat equation in 2D on irregular domains has been proposed by [9]. They applied the method on two different types of meshes viz. irregular and semi irregular and concluded that their method can be efficiently used for solving PDEs over irregular domains. A mesh free method was used by [10,11] for solving 3D heat equation by explicit scheme and the stability of the scheme was addresses by taking irregularity of the points in account. Their results showed the improvement in the accuracy of the solution.

The first relationship between stability and convergence was hinted at by Courant, Friedrichs and Lewy (hence known as CFL condition) in the 1920's [12-14]. Then it was clearly identified Von Neumann in the 1940 [15]. Later it was brought into organized form by Lax and Richtmyer in the 1950s by stating a fundamental theorem Lax Equivalence Theorem [16]. However, in most of the cases Von Neumann stability analysis which is based on the Fourier series briefly described by [17] is applied.

In literature, a number of studies can be found where the Von Neumann stability analysis is applied to devise or analyze the well posedness of the problems of interest. In this regard [18- 31] have done extensive work to either utilize, modify or establish stability conditions akin to Von Neumann stability analysis. A comprehensive review of the recent and past technique can be found in [28]. However, the stability analysis of heat equation becomes difficult when the domain under consideration has nonlinear boundaries leading to irregular mesh spacing.

A significant work has been done to investigate the stability of 1D, 2D and 3D heat equation for different finite difference schemes ranging from explicit to implicit methods. However, the stability of heat equation can be difficult when it is applied on nonlinear boundaries or domains. This issue provides motivation for research in the present state of the art by applying 2D heat equation over nonlinear domain specifically over elliptical domain.

2 Methodology

In this study a 2D elliptic domain Ω with boundary $\partial\Omega$ is considered by using general equation of ellipse. Suppose that the domain is made of some thermally conductive material with diffusion coefficient c^2 and heated in some way by applying the initial and boundary conditions. Mathematically, this problem is governed by 2D heat equation as given by Eq. (2) below:

$$\frac{\partial u(x,y,t)}{\partial t} = c^2\left(\frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2}\right), \tag{2}$$

where $u(x,y,t)$ represents the temperature at any point $P(x,y)$ at specific time t. As particular case the temperature at the boundary of the ellipse is set as $u = 100$ initially at time t=0 the temperature is $u = 0$ on other than the boundary nodes (as shown in Fig. 1). The governing 2D heat equation (2) is discretized by using the explicit forward Euler and centered finite difference schemes for time and space parameters respectively [32].

$$u_{j,k,n+1} = u_{j,k,n} + \Delta t.c^2\left(\begin{array}{l}\frac{1}{(\Delta x)^2}\left(u_{j+1,k,n} - 2u_{j,k,n} + u_{j-1,k,n}\right)+ \\ \frac{1}{\Delta y_1 \Delta y_2\left(\Delta y_1 + \Delta y_2\right)}\left(2\Delta y_1 u_{j,k+1,n} - 2\left(\Delta y_1 + \Delta y_2\right)u_{j,k,n} + 2\Delta y_2 u_{j,k-1,n}\right)\end{array}\right), \tag{3}$$

where $\Delta y_1 = y_i - y_{i-1}$ and $\Delta y_2 = y_{i+1} - y_i$. Equation (3) finds the numerical solution on each interior node *(i, j)* at the time *(n+1)* th time step based on the solution of previous n^{th} time step. Fig. 2 shows the finite difference mesh of the discretized domain with N=50 cells along x-axis and N=50 cells along y-axis. In order to reduce the computational cost the exterior cells are removed and solution be computed only on the interior nodes (see Fig. 3). The computational analysis of the mesh elements and mesh nodes for different choices of N is given in Table 1.

Table 1. Analysis of the finite difference mesh parameters

S. no	N	Δx	Total Nodes, TN	Number of exterior nodes, EN	Number of boundary nodes, BN	Number of interior nodes, IN	Number of cell, CN	Number of exterior cells, EC	Number of interior cells, IC	Number of common cells, CC
1	10	0.6000	121	60	22	39	100	40	60	22
2	20	0.3000	441	220	42	179	400	180	220	42
3	30	0.2000	961	480	62	419	900	420	480	62
4	40	0.1500	1681	840	82	759	1600	760	840	82
5	50	0.1200	2601	1300	102	1199	2500	1200	1300	102
6	60	0.1000	3721	1860	122	1739	3600	1740	1860	122

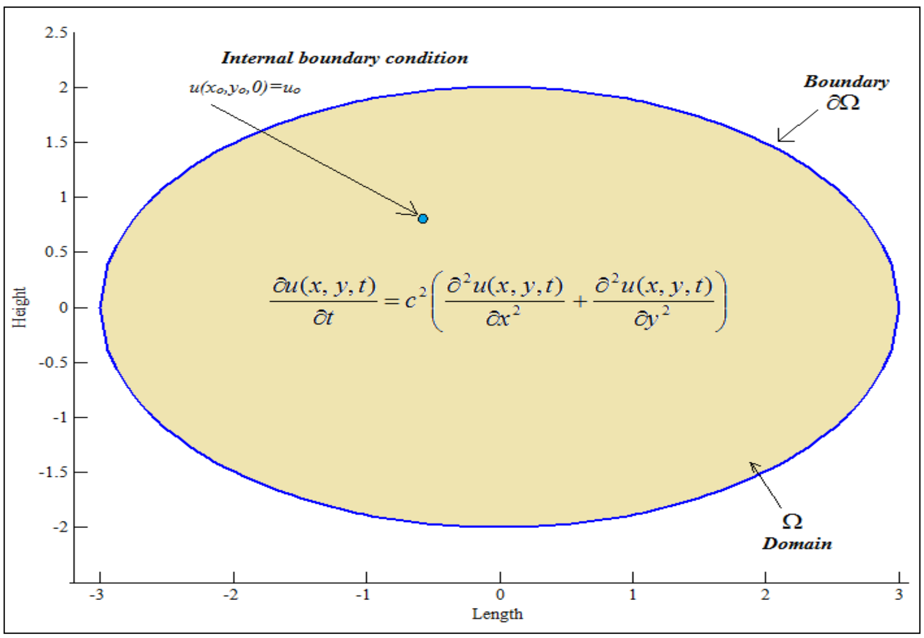

Fig. 1. Schematic of elliptic domain with applied heat equation and boundary conditions

With the aim of finding the stability condition for the heat equation with unequal mesh spacing the Von-Neumann stability method [28] is redefined by taking the average of Δy_1 and Δy_2. Thus, by substituting $u_{j,k,n} = e^{il\Delta x(j)}.e^{im\left(\frac{\Delta y_1+\Delta y_2}{2}\right)(k)}$ in Eq. (3) yields the new equation as

$$u_{j,k,n+1} = e^{il\Delta x(j)}.e^{im\left(\frac{\Delta y_1+\Delta y_2}{2}\right)(k)} + \frac{\Delta t}{(\Delta x)^2}c^2\left[e^{il\Delta x(j+1)}.e^{im\left(\frac{\Delta y_1+\Delta y_2}{2}\right)(k)} - 2e^{il\Delta x(j)}.e^{im\left(\frac{\Delta y_1+\Delta y_2}{2}\right)(k)} + e^{il\Delta x(j-1)}.e^{im\left(\frac{\Delta y_1+\Delta y_2}{2}\right)(k)}\right] +$$
$$\frac{\Delta t}{\Delta y_1 \Delta y_2 (\Delta y_1 + \Delta y_2)}c^2\left[2\Delta y_1 e^{il\Delta x(j)}.e^{im\left(\frac{\Delta y_1+\Delta y_2}{2}\right)(k+1)} - 2(\Delta y_1 + \Delta y_2)e^{il\Delta x(j)}.e^{im\left(\frac{\Delta y_1+\Delta y_2}{2}\right)(k)} + 2\Delta y_2 e^{il\Delta x(j)}.e^{im\left(\frac{\Delta y_1+\Delta y_2}{2}\right)(k-1)}\right], \quad (4)$$

by taking $e^{im\left(\frac{\Delta y_1+\Delta y_2}{2}\right)(k)}$ common and re-arranging the terms the following Eq. (5) is obtained,

$$u_{j,k,n+1} = e^{il\Delta x(j)}.e^{im\left(\frac{\Delta y_1+\Delta y_2}{2}\right)(k)}\left[\begin{array}{l} 1+\frac{\Delta t}{(\Delta x)^2}c^2\left(e^{il\Delta x}-2+e^{-il\Delta x}\right)+ \\ \frac{\Delta t}{\Delta y_1\Delta y_2\left(\Delta y_1+\Delta y_2\right)}c^2\left(2\Delta y_1 e^{im\left(\frac{\Delta y_1+\Delta y_2}{2}\right)}-2\left(\Delta y_1+\Delta y_2\right)+2\Delta y_2 e^{-im\left(\frac{\Delta y_1+\Delta y_2}{2}\right)(k)}\right)\end{array}\right], \tag{5}$$

Now assume that the expression in the square brackets is a function of k^{th} time step and is denoted by *G(k)* as given in Eq. (6)

$$G(k) = 1-2\frac{\Delta t}{(\Delta x)^2}c^2\left[1-\cos(l\Delta x)\right]+\frac{\Delta t}{\Delta y_1\Delta y_2\left(\Delta y_1+\Delta y_2\right)}c^2\left(2\Delta y_1 e^{im\left(\frac{\Delta y_1+\Delta y_2}{2}\right)}+2\Delta y_2 e^{-im\left(\frac{\Delta y_1+\Delta y_2}{2}\right)(k)}-2\left(\Delta y_1+\Delta y_2\right)\right) \tag{6}$$

Then expanding exponential Euler formula the Eq. (6) takes the following form,

$$G(k) = 1-2\frac{\Delta t}{(\Delta x)^2}c^2\left[1-\cos(l\Delta x)\right]+\frac{\Delta t}{\Delta y_1\Delta y_2\left(\Delta y_1+\Delta y_2\right)}c^2\left[\begin{array}{l} 2\left(\Delta y_1\cos\left(m\frac{\Delta y_1+\Delta y_2}{2}\right)+\iota\sin\left(m\frac{\Delta y_1+\Delta y_2}{2}\right)\right) \\ +\Delta y_2\left(\cos\left(m\frac{\Delta y_1+\Delta y_2}{2}\right)+\iota\sin\left(m\frac{\Delta y_1+\Delta y_2}{2}\right)\right) \\ -2\left(\Delta y_1+\Delta y_2\right)\end{array}\right], \tag{7}$$

or can be expressesd as follows,

$$G(k) = 1-2\frac{\Delta t}{(\Delta x)^2}c^2\left[1-\cos(l\Delta x)\right]+\frac{\Delta t}{\Delta y_1\Delta y_2\left(\Delta y_1+\Delta y_2\right)}c^2\left[2\left(\begin{array}{l}\cos\left(m\frac{\Delta y_1+\Delta y_2}{2}\right)\left(\Delta y_1+\Delta y_2\right) \\ +\iota\sin\left(m\frac{\Delta y_1+\Delta y_2}{2}\right)\left(\Delta y_1-\Delta y_2\right) \\ -2\left(\Delta y_1+\Delta y_2\right)\end{array}\right)\right], \tag{8}$$

On further simplification Eq. (8) takes the following form,

$$G(k) = 1-2\frac{\Delta t}{(\Delta x)^2}c^2\left[1-\cos(l\Delta x)\right]+\frac{\Delta t}{\Delta y_1\Delta y_2\left(\Delta y_1+\Delta y_2\right)}c^2\left[\begin{array}{l}2\left(\Delta y_1+\Delta y_2\right)\left(\cos\left(m\frac{\Delta y_1+\Delta y_2}{2}\right)-1\right)+ \\ 2\iota\sin\left(m\frac{\Delta y_1+\Delta y_2}{2}\right)\left(\Delta y_1-\Delta y_2\right)\end{array}\right], \tag{9}$$

or it can be written as

$$G(k) = 1-2\frac{\Delta t}{(\Delta x)^2}c^2\left[1-\cos(l\Delta x)\right]-2\frac{\Delta t}{\Delta y_1\Delta y_2}c^2\left[1-\cos\left(m\frac{\Delta y_1+\Delta y_2}{2}\right)\right]+ \tag{10}$$
$$\frac{\Delta t}{\Delta y_1\Delta y_2\left(\Delta y_1+\Delta y_2\right)}c^2\left[2\iota\sin\left(m\frac{\Delta y_1+\Delta y_2}{2}\right)\left(\Delta y_1-\Delta y_2\right)\right].$$

Consider if worst case $l\Delta x = \pi = m\left(\frac{\Delta y_1 + \Delta y_2}{2}\right)$, then $G(k) = 1 - 4\frac{\Delta t}{(\Delta x)^2}c^2 - 4\frac{\Delta t}{\Delta y_1 \Delta y_2}c^2$, or

$$G(k) = 1 - 4\left[\frac{\Delta t}{(\Delta x)^2}c^2 + \frac{\Delta t}{\Delta y_1 \Delta y_2}c^2\right].$$

Then by definition of stability $|G| \le 1,\ \forall l, m$ which yields the stability condition according to the Von-Neumann stability condition, finally the result is obtained as follows:

$$\frac{\Delta t}{(\Delta x)^2}c^2 + \frac{\Delta t}{\Delta y_1 \Delta y_2}c^2 \le \frac{1}{2} \Leftrightarrow \Delta t \le \frac{1}{2c^2}\left[\frac{1}{(\Delta x)^2} + \frac{1}{\Delta y_1 \Delta y_2}\right]^{-1}, \tag{11}$$

which is the required stability condition for the numerical solution of Eq. (2).

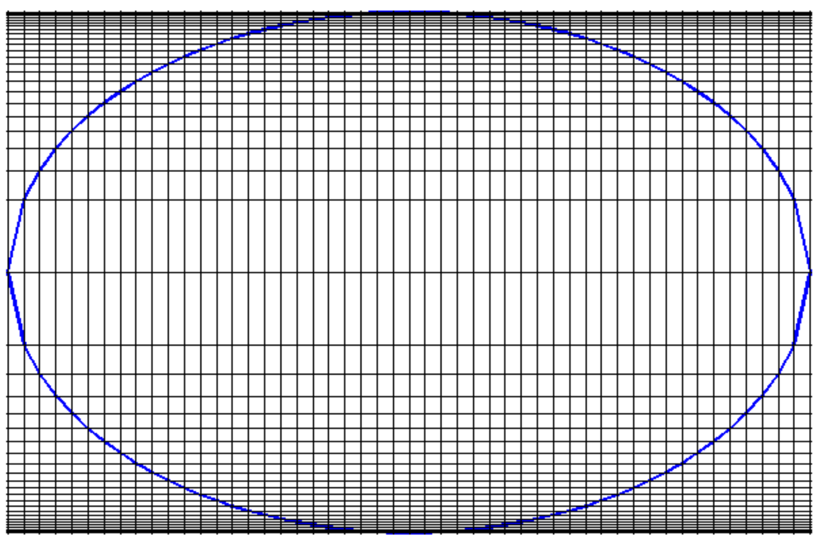

Fig. 2. Schematic of discretized domain for 50 x 50 mesh

3 Results and Discussion

The numerical solution for the defined problem is computed with the explicit finite difference scheme with the stability condition (12) by writing a user defined code on MATLAB. First of all for the six different choices of *N* the different meshes have been generated and the stable ranges of the time increment Δt have been found.

The following Table 2 shows the variation in the stability range in relation to different step sizes and functional increments. The interval of stability range for time step gets much smaller as the number of mesh nodes increases. The same behavior of stabile time step in relation to the varied increment Δy_1 along y-axis has shown in Fig. 4. A more clear 3D representation of the Δt depending upon Δy_1 and Δy_2 has exhibited by Fig. 3 which reveals that the stability region for variable spatial increments scales down as the number of mesh points are increased. In order to validate the smoothness of temperature diffusion the simulation profiles from the numerical solution have been obtained and shown in the Fig. 6 (a) through Fig. 6 (j). The simulation profiles are taken for *t=0* to *t=1;* and then the solution is interpolated for *t=0.1, t=0.2, ...,t=1.0.* From the figures it can be bee seen that the effect of the heat diffuses from boundary to interior region as the time increases. If the time is let to further increase the time dependent diffusion will lead to the stationary behavior.

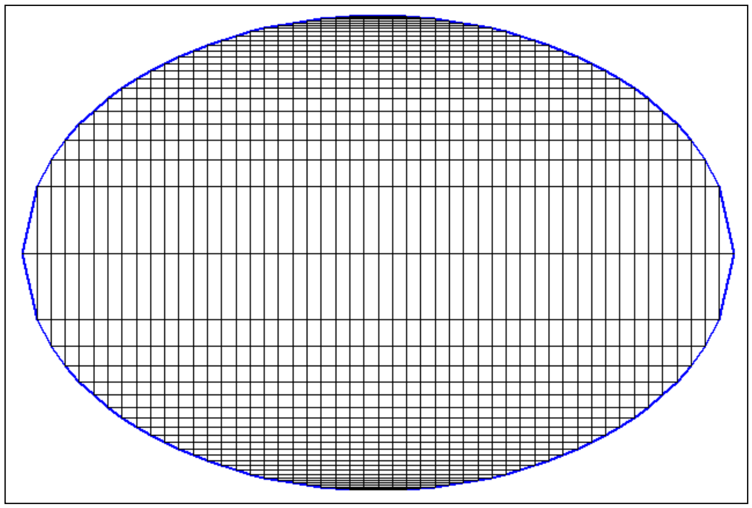

Fig. 3. Simplified mesh of the domain, where extra cells are removed

Table 2. Analysis of the Stable range of the time step

S. No	N	Δx	min(Δy)	max(Δy)	mean(Δy)	min(Δt)	max(Δt)	Stable range of Δt
1	10	0.6000	0.04040800	1.20000000	0.62020410	0.00252124	0.14400000	[0.00252124, 0.144]
2	20	0.3000	0.01002500	0.87177900	0.44090000	0.00015000	0.04023500	[0.00015, 0.040235]
3	30	0.2000	0.00444930	0.71802100	0.36123500	0.00002978	0.01856000	[2.98E-5, 0.01856]
4	40	0.1500	0.00250150	0.62449900	0.31350000	0.00000940	0.01063630	[9.40E-6, 0.0106363]
5	50	0.1200	0.00160000	0.56000000	0.28080000	0.00000385	0.00688300	[3.85E-6, 0.006883]
6	60	0.1000	0.00111142	0.51207638	0.25659390	0.00000185	0.00481633	[1.8542E-06, 0.0048]

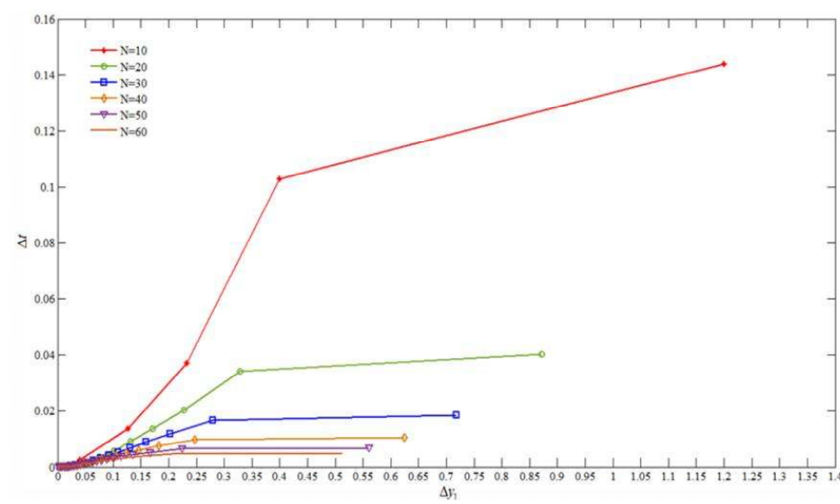

Fig. 4. Relation between functional increment and the time step at different mesh size

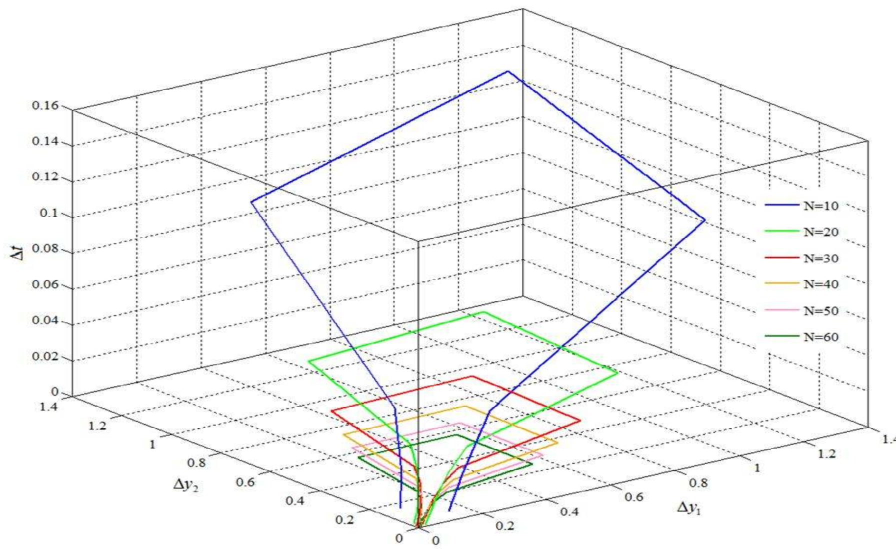

Fig. 5. The stability region for time step at different mesh size with respect to functional increments

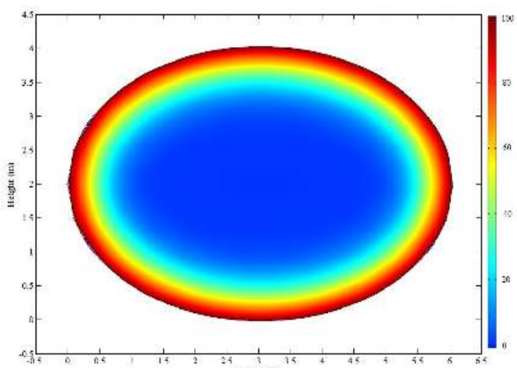

Fig. 6 (a). Simulation of temperature diffusion in the domain at *t=0.1*

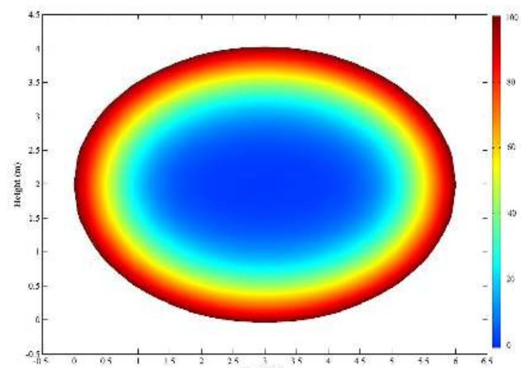

Fig. 6 (b). Simulation of temperature diffusion in the domain at *t=0.2*

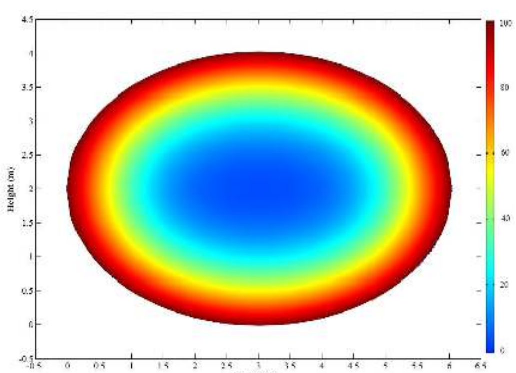

Fig. 6 (c). Simulation of temperature diffusion in the domain at *t=0.3*

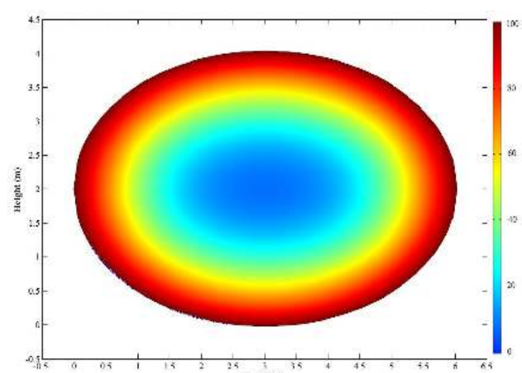

Fig. 6 (d). Simulation of temperature diffusion in the domain at *t=0.4*

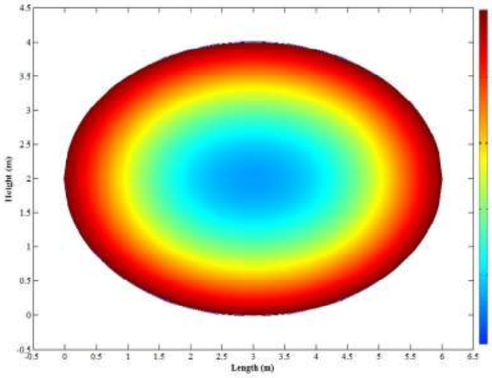

Fig. 6 (e). Simulation of temperature diffusion in the domain at *t=0.5*

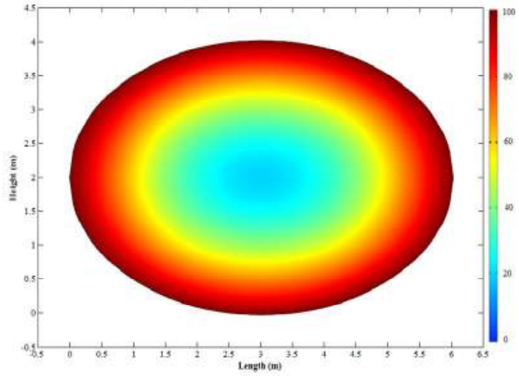

Fig. 6 (f). Simulation of temperature diffusion in the domain at *t=0.6*

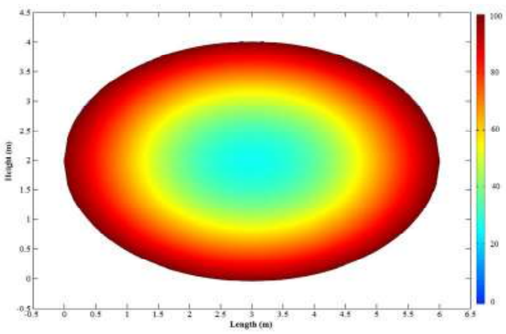

Fig. 6 (g). Simulation of temperature diffusion in the domain at *t=0.7*

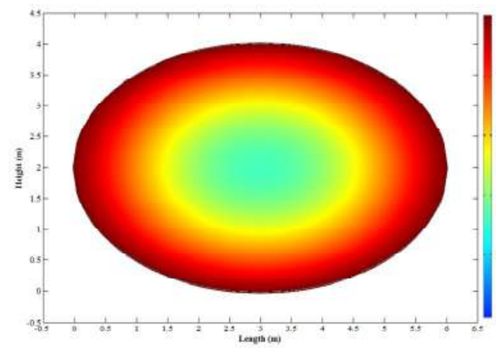

Fig. 6 (h). Simulation of temperature diffusion in the domain at *t=0.8*

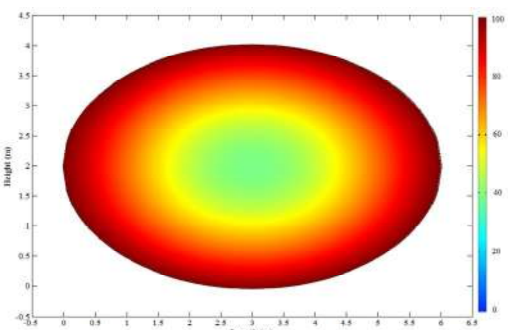

Fig. 6 (i). Simulation of temperature diffusion in the domain at *t=0.9*

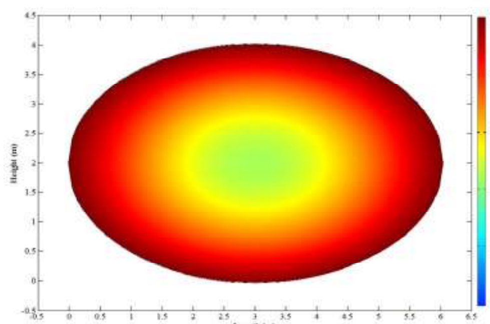

Fig. 6 (j). Simulation of temperature diffusion in the domain at *t=1.0*

4 Conclusion

In this study the stability analysis of the finite difference solution of 2D heat equation was investigated. The main purpose was to find out the stability criteria for the explicit finite difference scheme on irregular domain. Where the domain boundary was constructed by using the equation of ellipse. The problem of stability occurs when the mesh size is unequal along the y-axis due to functional increments. Therefore, the finite difference scheme was redefined for unequal step size along y-axis, the analogous Von-Neumann stability analysis was worked out and the general formula for such problem was obtained. From the results it was revealed that stability region for the small number of mesh points remains larger and then stability region gets smaller as the number of nodes are increased. The corresponding stability range for *N=10, 20, 30, 40, 50,* and *60* was found respectively. Within that range the solution remains smooth as time increases. The results of this study attempt to provide the stable and accurate solution of partial differential equations on irregular domains. The similar work can be done for other types of PDEs such as hyperbolic, elliptical, etc; and the methodology can be extended to 3D.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Available:https://en.wikipedia.org/wiki/partial_differential_equation
[Retrieved in May 2016]

[2] Olsen-Kettle L. Numerical solution of partial differential equations; 2011.

[3] Isaacson E, Keller HB. Analysis of Numerical Methods. 1994;541.

[4] Han Chen, Cho Hong Min, Frederick Gibou. A Supra-convergent finite difference scheme for the poisson and heat equations on irregular domains and Non-Graded Adaptive Grids; 2006.

[5] Peter McCorquodale, Phillip Colella, Hans Johansen, A cartesian grid embedded boundary method for the heat equation on irregular domains.
DOI: 10.1006/jcph.2001.6900
Available:http://www.idealibrary.com

[6] Verena Hora, Peter Gruber. Parallel Numerical Solution of 2-D Heat Equation, Parallel Numerics. 2005;47-56.

[7] Gerald W. Recktenwald. Finite-difference approximations to the heat equation; 2011.

[8] Kazufumi Ito, Zhilin Li, Yaw Kyei. Higher-order, Cartesian grid based finite difference schemes for elliptic equations on irregular domains. SIAM Journal on Scientific Computing. 2005;27(1): 346-367.

[9] Izadian J, Karamooz N. New method for solving poisson equation on irregular domains. Applied Mathematical Sciences. 2012;6(8):369–380.

[10] Gavete L, Benito JJ, Urena F. Generalized finite differences for solving 3D elliptic and parabolic equations. Applied Mathematical Modelling. 2016;40(2):955-965.

[11] Gilberto E. Urroz. Convergence, stability, and consistency of finite difference schemes in the solution of partial differential equations; 2004.

[12] Courant R, Friedrichs K, Lewy H. Über die partiellen Differenzengleichungen der mathematischen Physik", Mathematische Annalen (in German). 1928;100(1):32-74.

[13] Courant R, Friedrichs K, Lewy H. On the partial difference equations of mathematical physics. AEC Research and Development Report, NYO-7689, New York: AEC Computing and Applied Mathematics; 1956.

[14] Courant R, Friedrichs K, Lewy H. On the partial difference equations of mathematical physics. IBM Journal of Research and Development. 1928;11(2):215-34.

[15] Charney JG, Fjörtoft R, Von Neumann J. Numerical integration of the barotropic vorticity equation. In The Atmosphere—A Challenge. American Meteorological Society, Boston, MA. 1990;267-284.

[16] Lax PD, Richtmyer RD. Survey of the stability of linear finite difference equations. Comm. Pure Appl. Math. 1956;9:267-293 MR 79204.
DOI: 10.1002/cpa.3160090206

[17] Crank J, Nicolson P. A practical method for numerical evaluation of solutions of partial differential equations of heat conduction type. Proc. Camb. Phil. Soc. 1947;43.

[18] Yuste SB. Weighted average finite difference methods for fractional diffusion equations. Journal of Computational Physics. 2006;216:264–274.

[19] Yuste SB, Acedo L. An explicit finite difference method and a new von neumann-type stability analysis for fractional diffusion equations. SIAM Journal on Numerical Analysis. 2005;42(5):1862-1874.

[20] Bilbao S. Modeling of complex geometries and boundary conditions in finite difference/finite volume time domain room acoustics simulation. In IEEE Transactions on Audio, Speech, and Language Processing. 2013;21(7):1524-153.

[21] Jincheng Ren, Zhi-zhong Sun, Xuan Zhao. Compact difference scheme for the fractional sub-diffusion equation with Neumann boundary conditions. Journal of Computational Physics. 2013; 232(1):456-467.
[ISSN 0021-9991]
Available:http://dx.doi.org/10.1016/j.jcp.2012.08.026

[22] Ercília Sousa, Can Li. A weighted finite difference method for the fractional diffusion equation based on the Riemann–Liouville derivative. Applied Numerical Mathematics. 2015;90:22-37.
[ISSN 0168-9274]
Avilable:http://dx.doi.org/10.1016/j.apnum.2014.11.007

[23] Ercília Sousa. A second order explicit finite difference method for the fractional advection diffusion equation, Computers & Mathematics with Applications. 2012;64(10):3141-3152.
[ISSN 0898-1221]
Available:http://dx.doi.org/10.1016/j.camwa.2012.03.002

[24] Sweilam NH, Khader MM, Mahdy AMS. Crank-nicolson finite difference method for solving time-fractional diffusion equation. Journal of Fractional Calculus and Applications. 2012;2(2):1-9.

[25] Quintana-Murillo J, Yuste SB. A finite difference method with non-uniform time steps for fractional diffusion and diffusion-wave equations. The European Physical Journal Special Topics. 2013;222(8): 1987–1998.

[26] Norihiro Watanab, Olaf Kolditz. Numerical stability analysis of two-dimensional solute transport along a discrete fracture in a porous rock matrix, Water Resources Research. 2015;51(7):5855–5868.

[27] Jiequan Li, Zhicheng Yang. The von Neumann analysis and modified equation approach for finite difference schemes. Applied Mathematics and Computation. 2013;225:610-621.
[ISSN 0096-3003]
Available:http://dx.doi.org/10.1016/j.amc.2013.09.046

[28] Ehlers W, Zinatbakhsh S, Markert B. Stability analysis of finite difference schemes revisited: A study of decoupled solution strategies for coupled multifield problems. Int. J. Numer. Meth. Engng. 2013; 94:758–78.

[29] Baudouin L, Seuret A, Gouaisbaut F, Dattas M. Lyapunov stability analysis of a linear system coupled to a heat equation. IFAC-PapersOnLine. 2017;50(1):11978-11983.

[30] Konangi S, Palakurthi NK, Ghia U. Von Neumann stability analysis of first-order accurate discretization schemes for one-dimensional (1D) and two-dimensional (2D) fluid flow equations. Computers & Mathematics with Applications. 2018;75(2):643-665.

[31] Jahangir K, Rehman SU, Ahmad F, Pervaiz A. Sixth-Order Stable Implicit Finite Difference Scheme for 2-D Heat Conduction Equation on Uniform Cartesian Grids with Dirichlet Boundaries. Punjab Univ. J. Math. 2019;51(5):27-42.

[32] Singh AK, Bhadauria BS. Finite difference formulae for unequal sub-intervals using Lagrange's interpolation formula. Int. J. Math. Anal. 2009;3(17):815.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://www.sdiarticle4.com/review-history/54909