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# Computational Complexity and Tort Deterrence

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**Abstract**

Standard formulations of the economic model of tort deterrence constitute the injurer as the unboundedly rational bad man. Unbounded rationality implies that the injurer can always compute the solution to his care-taking problem. This in turn implies that optimal liability rules can provide robust deterrence, for they can always induce the injurer to take socially optimal care. In this paper I examine the computational complexity of the injurer's care-taking problem. I show that the injurer's problem is computationally tractable when the precaution set is unidimensional or convex, but that it is computationally intractable when the precaution set is multidimensional and discrete. One implication is that the standard assumptions of unidimensional and convex care, though seemingly innocuous, are pivotal to ensuring that tort law can provide robust deterrence. It is therefore important to recognize situations with multidimensional discrete care, where robust tort deterrence may not be possible.

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# 1 Introduction

In many ways the economic approach to law embraces Oliver Wendell Holmes' "bad man" theory of the law (Holmes 1897). Holmes argeed that if "you want to know the law and nothing else, you must look at it as a bad man," to whom the notion of legal duty means "a prophecy that if he does certain things he will be subjected to disagreeable consequences by way of imprisonment or compulsory payment of money" (Holmes 1897, pp. 459 & 461). In other words, if you want to understand the law, you must focus on the deterrence effects of legal rules.

Nowhere is the focus on the deterrence effects of legal rules more prominent than in the economic analysis of tort law. The economic model of tort deterrence posits that the tortfeasor, known as the injurer, chooses safety precautions, or care, when engaging in a risky activity to minimize his cost of care plus his expected liability to a potential victim (Shavell 1987, ch. 2). The social goal, by comparison, is to minimize the injurer's cost of care plus the victim's expected loss. Under the economic model, a liability rule is deemed optimal if the solution to the injurer's care-taking problem under that rule coincides with the solution to the social problem.

Standard formulations of the economic model of tort deterrence constitute the injurer as the unboundedly rational bad man. Unbounded rationality implies that the injurer can always compute the solution to his care-taking problem. This in turn implies than an optimal liability rule can provide robust deterrence, for it can always induce the injurer to take socially optimal care (Cooter and Ulen 2012, ch. 7).

Beginning with Simon (1955, 1957), however, economists have questioned the assumption of unbounded rationality and explored the implications of bounded rationality for standard economic analysis. Legal economists have followed suit (see, for example, Jolls, Sunstein, and Thaler 1998; Korobkin and Ulen 2000). An important aspect of bounded rationality is limited computational capacity. This aspect of bounded rationality refers not only to humans' limited cognitive ability or skill with respect to computation, but also to the theoretical and practical limits of computability even when aided by machines (Simon 1976, 1990). While the former limit is the subject of behavioral economics and psychology, the latter limits are the subjects of computability theory and computational complexity theory, respectively.

In this paper I examine the computational complexity of the injurer's care-taking problem under strict liability and negligence, the two basic liability rules of Anglo-

American tort law. I start by defining several concepts from computational complexity theory, including what it means for a problem to be computationally tractable. In short, a problem is computationally tractable if it can be solved in polynomial time, i.e., if the number of elementary steps required to compute the solution grows at a polynomial rate (or slower) with the size of the problem.[1]

In the heart of the paper I analyze tort deterrence using the unilateral care model with fixed activity level (the UCFA model).[2] The standard UCFA model includes four assumptions that impact the computational complexity of the injurer's problem: (i) there is a single dimension of care; (ii) care is a continuous variable; (iii) the marginal cost of care is increasing; and (iv) the marginal benefit of care (i.e., the marginal reduction in expected loss) is decreasing (see, for example, Shavell 1987, ch. 2; Cooter and Ulen 2012, ch. 6).[3] For obvious reasons I refer to the first assumption as the "unidimensional care" assumption. I refer to the second as the "convex care" assumption because it implies that the injurer's precaution set is convex. The third and fourth assumptions are motivated and implied by the law of diminishing returns; I therefore refer to them collectively as the "diminishing returns" assumption.

Throughout the analysis I maintain the diminishing returns assumption, which is based on a "fundamental law of economics" (Samuelson 1980, p. 25), and probe the unidimensional and convex care assumptions, which are assumptions of convenience that lack comparable economic foundations.[4] I show that (i) either assumption is sufficient to ensure that the injurer's problem is computationally tractable, but that (ii) when we relax both assumptions—and assume that care is multidimensional and discrete—the injurer's problem is computationally intractable.[5]

[1] The definition of computational tractability is based on the limit behavior of the solution (i.e., on a worst-case analysis) to make it robust to variation in problem specifics. Time is measured in elementary steps to make it robust to variation in computing power.

[2] The UCFA model is the foundational model upon which other economic models of tort deterrence are built. In cases of unilateral care, the injurer, but not the victim, can take care to reduce the victim's expected loss. In cases of unilateral care with fixed activity level, the injurer can reduce the victim's expected loss only by taking care and not by modulating his activity level.

[3] For the avoidance of doubt, unless they are qualified by "strictly," the terms "increasing" and "decreasing" (and similar terms like "positive" and "negative") are used in the weak sense.

[4] More precisely, I maintain the implications of the diminishing returns assumption. When care is unidimensional the diminishing returns assumptions implies that the social cost function is convex and supermodular. In order to preserve these implications when care is multidimensional I assume that there are diminishing returns to care within and across each safety dimension.

[5] The first result holds without qualification for the model analyzed in Section 3, which assumes that the social cost function has a quadratic form. For the general model analyzed in the Appendix, which does not specify a functional form for the social cost function, the first result holds for social

The intuition for this result is the following. With convex care the injurer's problem is a convex optimization problem—the minimization of a convex function on a convex choice set. Due to the magic of calculus, this problem can generally be solved in polynomial time whether the choice set is unidimensional or multidimensional. With discrete care, by contrast, the injurer's problem entails the minimization of a supermodular function on a discrete choice set. In general the only known solution algorithm for this problem is the brute-force method—evaluate the objective function at every point in the choice set. When the choice set is unidimensional, the size of the problem is governed by its cardinality, and thus the problem is computationally tractable (because brute-force scales linearly in the cardinality of the choice set). When the choice set is multidimensional, however, the size of the problem is governed by its dimension, and thus the problem is computationally intractable (because brute-force scales exponentially in the dimension of the choice set).

The paper is structured as follows. The brief primer on computational complexity theory appears in Section 2. At the end of the section I provide several examples of real-world problems in which the choice set is multidimensional and discrete. The examples are my attempt to convince the skeptical reader that multidimensional discrete choice sets are an important phenomenon in the real world.

Section 3 contains the tort deterrence analysis. In the section I analyze a "toy" version of the UCFA model—driver-pedestrian accident with quadratic social cost. My analysis of this simple model illustrates the main takeaways of the paper. For the interested reader I also analyze a general model in the Appendix. The general model maintains the shape restrictions on the social cost function that are implied by the diminishing returns assumption, but does not assume a specific functional form.

In Section 4 I tie up three loose ends. First, I address the (dim) prospect of approximating the solution to the injurer's care-taking problem when care is multidimensional and discrete. Second, I critique the common justifications for the unidimensional and convex care assumptions. Third, I revisit the diminishing returns assumption and consider the (unlikely) possibility of increasing returns to care.

I conclude the paper in Section 5 with a discussion of several implications of my analysis for the theory and practice of tort law.

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cost functions that satisfy a minimal computability assumption (in addition to the diminishing returns assumption). The second results hold generically without qualification for both models.

The paper contributes to two strands of the law and economics literature. The first is the strand that analyzes the deterrence properties of tort liability rules, including strict liability and negligence. The pioneers of this strand include Calabresi (1961, 1970), Posner (1972a,b), Brown (1973), Diamond (1974a,b), Green (1976), Landes and Posner (1980, 1987), and Shavell (1980, 1987). Surveys of this strand are provided by Shavell (2007), Schäfer and Müller-Langer (2009), and Arlen (2017).

The second is the behavioral strand that explores the implications of bounded rationality for the standard economic analysis of tort deterrence. Jolls, Sunstein, and Thaler (1998) and Korobkin and Ulen (2000) were among the early calls for the modification of standard law and economics models to reflect bounded rationality. Zamir and Teichman (2018) provide a comprehensive textbook treatment of the emergent field of behavioral law and economics, including a chapter on the behavioral analysis of tort law. Faure (2010), Halbersberg and Guttel (2014), and Luppi and Parisi (2018) provide surveys of behavioral models of tort law.

To my knowledge this is the first paper in the law and economics literature to explore how computational complexity impacts tort deterrence. Legal economists have long recognized that "people's decision-making capabilities are relevant to the design of tort law" (Zamir and Teichman 2018, p. 330). In their early paper on strict liability, for instance, Guido Calabresi and Jon Hirschoff argue that the choice among tort liability rules should depend not on the theoretical ability of injurers and victims to optimize, but rather on their actual abilities taking into account the "psychological or other impediments" to optimizing (Calabresi and Hirschoff 1972, p. 1059; see also Faure 2008).[6] Subsequently, legal economists have studied the implications for tort law of various aspects of bounded rationality, including ambiguity (Teitelbaum 2007; Chakravarty and Kelsey 2017; Franzoni 2017) and unawareness (Chakravarty, Kelsey, and Teitelbaum 2019). As far as I am aware, however, no other paper in the literature has studied the implications of computational complexity for tort law.

The paper also contributes to the literature on legal complexity (see, for example, Ehrlich and Posner 1974; Schuck 1992; Kaplow 1995; Ruhl and Katz 2015). Papers in this literature study various kinds of legal complexity, including the intricacy of legal

[6]Legal economists have also long recognized that people's limited computational capacity is relevant to contracts. In his oft-cited paper on the transaction cost approach to the study of economic organization, for example, Oliver Williamson argues that "incomplete contracting is the best that can be achieved" because "organizational man," unlike "economic man," is "boundedly rational" and subject to "limits in formulating and solving complex problems" (Williamson 1981, pp. 553-554).

rules and the legal system. To my knowledge the only other paper in this literature that studies computational complexity is Kades (1997). In contrast to this paper, Kades does not focus on the tractability of compliance problems arising under tort law. Rather, he focuses on the tractability of adjudication problems in selected cases arising under bankruptcy law, commercial law, contract law, corporate law, criminal law, property law, and tax law. He also invokes computational complexity to explain judges' aversion to multiparty disputes and the existence of private property.[7]

# 2 Computational Complexity

Computational complexity theory is a subfield of computer science that studies the tractability of computational problems, including decision problems (i.e., yes-no problems) and optimization problems. In this section I introduce several concepts from computational complexity theory that are relevant for my analysis.[8]

## 2.1 Algorithms and Efficiency

An *algorithm* is a step-by-step procedure for solving a computational problem. The *time complexity* of an algorithm, denoted by $\tau(n)$, gives the maximum number of elementary steps that the algorithm requires to produce its output, expressed as a function of the size of its input, denoted by $n$. Algorithms are classified according to the rate at which $\tau(n)$ grows with $n$. Algorithms for which $\tau(n)$ grows with $n$ at a polynomial rate (or slower) are said to run in *polynomial time*. Polynomial-time algorithms are considered to be *fast* or *efficient*. Algorithms for which $\tau(n)$ grows with $n$ at a faster rate (e.g., exponential) are considered to be *slow* or *inefficient*.

The efficiency of polynomial-time algorithms is manifested by a comparison with exponential-time algorithms. Table 1 displays the running times for selected input sizes $n \leq 100$ of a polynomial-time algorithm that requires $n^2$ steps and an exponential-time algorithm that requires $2^n$ steps, assuming one calculation per step and 200,000 trillion calculations per second (the peak speed of the world's fastest

[7] There are papers in the economics and computer science literatures that study the computational complexity of economic models, including papers that study the tractability of computing Nash equilibria in games (Daskalakis 2009; Roughgarden 2010) and of the consumer's utility-maximization problem (Echenique, Golovin, and Wierman 2011; Gilboa, Postlewaite, and Schmeidler 2021).

[8] For a more rigorous introduction to these concepts, see, for example, Garey and Johnson (1979), Schrijver (2003), or Kleinberg and Tardos (2006).

Table 1: Polynomial Time versus Exponential Time

| Input size ($n$) | Running time: Polynomial-time algorithm ($n^2$ steps) | Running time: Exponential-time algorithm ($2^n$ steps) |
|---|---|---|
| 50 | Less than a second | Less than a second |
| 75 | Less than a second | More than two days |
| 80 | Less than a second | More than two months |
| 85 | Less than a second | More than six years |
| 90 | Less than a second | Almost 200 years |
| 95 | Less than a second | More than 6,000 years |
| 100 | Less than a second | More than 200,000 years |

Note: Assumes one calculation per step and 200,000 trillion calculations per second.

supercomputer). The polynomial-time algorithm runs in less than a second for all input sizes $n \leq 100$ (and, indeed, for all $n \leq 447,213,595$). By contrast, the running time of the exponential-time algorithm increases from less than a second for $n = 50$ to more than 200,000 years for $n = 100$.

## 2.2 Problems and Tractability

In light of the efficiency of polynomial-time algorithms, computational problems that can be solved in polynomial time are considered to be *easy* or *tractable*. In what follows I describe the concept of $\mathcal{NP}$-hardness, which is the defining property of computational problems that are considered to be hard or intractable.

I begin with two classes of decision problems known as $\mathcal{P}$ and $\mathcal{NP}$. $\mathcal{P}$ is the class of decision problems that can be solved in polynomial time (i.e., *efficiently solved*). $\mathcal{NP}$ is the class of decision problems for which it can be verified in polynomial time whether a proposed solution is correct (i.e., *efficiently verified*). Every problem in $\mathcal{P}$ is necessarily also in $\mathcal{NP}$ (i.e., $\mathcal{P} \subseteq \mathcal{NP}$), for if a problem can be efficiently solved that efficiently verifies whether a proposed solution correct. Whether $\mathcal{P} = \mathcal{NP}$ (i.e., whether every decision problem that can be efficiently verified can also be efficiently solved) is an open question—indeed, it is a Millennium Prize Problem (Jaffe 2000). It is conjectured and presumed that $\mathcal{P} \neq \mathcal{NP}$.

I now come to the concept of $\mathcal{NP}$-hardness. Any computational problem—including, in particular, an optimization problem—is *$\mathcal{NP}$-hard* if every problem in $\mathcal{NP}$ is reducible to it. One problem is *reducible* to a second problem if the existence of an efficient solution to the second problem would imply the existence of an efficient solution to the first problem. In this sense, an $\mathcal{NP}$-hard problem is at least as hard as every problem in $\mathcal{NP}$. It follows that an $\mathcal{NP}$-hard problem can be efficiently solved only if $\mathcal{P} = \mathcal{NP}$, or, equivalently, that no $\mathcal{NP}$-hard problem can be efficiently solved unless $\mathcal{P} = \mathcal{NP}$.[9] Because it is conjectured and presumed that $\mathcal{P} \neq \mathcal{NP}$, all $\mathcal{NP}$-hard problems are considered to be *hard* or *intractable*.[10]

The following are four well-known examples of $\mathcal{NP}$-hard problems.

**Problem 1 (knapsack)** Given a set of items with specified values and weights and a knapsack with a specified capacity, find a subset of the items to pack in the knapsack such that the sum of their values is maximized while the sum of their weights does not exceed the knapsack's capacity.

**Problem 2 (traveling salesman)** Given a set of cities and distances between each pair of cities, find a roundtrip route that visits each city exactly once such that the total distance traveled is minimized.

**Problem 3 (set cover)** Given a set of elements and a collection of subsets that covers the set (i.e., whose union equals the set), find the smallest subcollection of subsets that covers the set.

**Problem 4 (max cut)** Given a graph defined by a set of nodes and edges with specified capacities connecting each pair of nodes, find a cut of the graph (i.e., a subset of nodes) that maximizes the total capacity of the edges severed by the cut.

Each of these problems is an optimization problem where the choice set is multi-dimensional and discrete. For instance, the knapsack problem with $n$ items is a binary linear optimization problem: Choose $\mathbf{x} = (x_1, \ldots, x_n) \in \{0,1\}^n$ to maximize $\sum_{i=1}^{n} v_i x_i$ subject to $\sum_{i=1}^{n} w_i x_i \leq c$, where $v_i$ and $w_i$ are the value and weight of item $i$, respectively, $c$ is the capacity of the knapsack, and $x_i$ indicates whether item $i$ is packed in the knapsack.[11] Meanwhile, the max cut problem for a graph with $n$ nodes

[9] Even if $\mathcal{P} = \mathcal{NP}$, this would imply only that some, but not necessarily all, $\mathcal{NP}$-hard problems can be efficiently solved.

[10] An important subclass of $\mathcal{NP}$-hard problems are $\mathcal{NP}$-complete problems. $\mathcal{NP}$-complete problems are $\mathcal{NP}$-hard problems that are in $\mathcal{NP}$. By definition, all $\mathcal{NP}$-complete problems are decision problems (because they are in $\mathcal{NP}$) and, morevoer, all $\mathcal{NP}$-hard optimization problems are at least as hard as all $\mathcal{NP}$-complete problems.

[11] The knapsack problem has many variants, including variants with a nonlinear value function (e.g., a quadratic function) or multiple capacity constraints (Kellerer, Pferschy, and Pisinger 2004).

is a binary quadratic optimization problem: Choose $\mathbf{x} = (x_1, \ldots, x_n) \in \{0, 1\}^n$ to maximize $\sum_{i,j=1}^{n} \frac{1}{2} w_{ij}(x_i - x_j)^2$, where $w_{ij}$ is the weight of the edge connecting nodes $i$ and $j$ and $x_i$ indicates whether node $i$ is contained in the cut.

## 2.3 Real-world Problems

Many real-world problems are optimization problems with multidimensional discrete choice sets. Here are five examples inspired by Problems 1-4.

**Example 1 (vehicle design)** Tort litigation is replete with suits for defective design of vehicles such as airplanes (e.g., the Boeing 737 MAX) and automobiles (e.g., the Ford Pinto). See, for example, *In re: Lion Air Flight JT 610 Crash* (1:18-cv-07686 N.D. Ill. [2018]); *Grimshaw v. Ford Motor Co.* (119 Cal. App. 3d 757 [1981]). A vehicle can be seen as a collection of systems that differentially impact safety, performance, and cost. The vehicle design problem can thus be seen as a knapsack problem in which there is a set of available systems and the objective is to find a subset that maximizes safety subject to performance and cost constraints. The set of available systems is often large. For instance, there are dozens of available systems for automobiles, including braking systems (e.g., anti-lock, disc, and drum), drive systems (e.g., all wheel, four wheel, front wheel, and rear wheel), frame systems (e.g., backbone tube, ladder, space frame, x-frame, and unibody), lighting systems (e.g., head lights, brake lights, hazard lights, tail lights, and turn signals), restraint systems (e.g., airbags, LATCH, and seat belts), steering systems (e.g., rack-and-pinion, power, and four wheel), suspension systems (e.g., beam axle, dual beam axle, double wishbone, and MacPherson strut), transmission systems (e.g., automatic, automated manual, continuously variable, and manual), and accident mitigation systems (e.g., adaptive cruise control, adaptive headlights, adaptive park assist, automatic emergency braking, automatic high beams, backup camera, bicycle detection, blind sport warning, brake assist, curve speed warning, driver attention monitoring, forward collision warning, high speed alert, hill descent assist, hill start assist, lane centering assist, lane departure warning, lane keeping assist, left turn crash avoidance, night vision, obstacle detection, parking sensors, pedestrian detection, rear cross traffic warning, semi-autonomous driving, shatter resistant glass, temperature warning, tire pressure monitoring, traction control, and traffic sign recognition).

**Example 2 (school safety)** School shootings have become an all-too-common phenomenon. A tragic example is the shooting at Marjory Stoneman Douglas High School in Parkland, Florida on February 14, 2018, which left 17 people dead and another 17 injured (Bakeman 2019). Families of the victims filed numerous tort suits against the Broward County School Board for failing to safeguard its students and employees. See, for example, *Alhadeff v. The School Board of Broward County* (19-008077 Fla. Cir. Ct. [2019]). In the wake of the shooting, the Broward County League of Cities' Community Public Safety Task Force (2018) issued a report documenting the numerous security measures in place at the time of the shooting (e.g., placement of police officers in schools, single point of entry measures, camera surveillance programs, and active shooter training) and making 100 recommendations for additional measures to "maximize safety" at Broward County schools. From this example the school safety problem can be seen as a knapsack problem in which there is a large set of available security measures that vary in terms of their efficacy and cost, and the objective is to find the subset that maximizes school safety subject to a cost constraint.

**Example 3 (carriage of goods)** At common law carriers have a ceteris paribus duty to send goods over the shortest route. In *Miller v. Davis* (213 Iowa 1091 [1932]), for example, a rail carrier who failed to ship unrouted freight (three carloads of grain) over the shortest route between terminal points in Iowa and Tennessee was held liable to the shipper for the difference in the tariff rates on the route taken and the shortest route ($118.64). The carriage of goods problem can thus be seen as a traveling salesman problem.

**Example 4 (medical diagnosis)** Misdiagnosis of illness or injury can subject a physician to liability for medical malpractice. See, for example, *Pike v. Honsinger* (155 N.Y. 201 [1898]). Insofar as physicians follow the rule of diagnostic parsimony (Hilliard et al. 2004), the medical diagnosis problem can be seen as a set covering problem (Reggia, Nau, and Wang 1983). The basic idea is this. A patient presents a set of symptoms. All the disorders that produce one or more of the patient's symptoms form a collection, where each disorder is defined by the subset of symptoms that it produces. The best diagnosis is the smallest subcollection of disorders that covers (explains) the patient's symptoms.

**Example 5 (D-Day)** In the weeks preceding the Allied invasion of Normandy on June 6, 1944, Allied air forces conducted an operation known as the Transportation

Plan in which strategic bombers destroyed key rail lines in France to isolate Normandy from German reinforcements (Ellis 1962). The operation was very effective. According to a contemporaneous German Transport Ministry report, "the raids . . . caused the breakdown of all main lines; the costal defenses have been cut off from the supply bases in the interior . . ." (Ellis 1962, p. 111). The D-Day problem can be seen as a max cut problem where the French rail network is a graph with nodes (stations) and connecting edges (lines) with specified capacities (for carrying German reinforcements), and the Allied objective was to find a cut of the graph containing Normandy that maximized the total capacity of the severed edges.

I now turn to tort deterrence and the computational complexity of the injurer's care-taking problem under the UCFA model. As it turns out, the max cut problem plays an important role in the analysis. In short, I show that the max cut problem is reducible to the social problem with multidimensional discrete care, which establishes that the injurer's problem with multidimensional discrete care is $\mathcal{NP}$-hard.

# 3 Tort Deterrence

I analyze tort deterrence using the unilateral care model with fixed activity level (the UCFA model). In this section I present a "toy" version of the model—driver-pedestrian accident with quadratic social cost—that illustrates the main takeaways of the paper. A general model and analysis are set forth in the Appendix.

In the model there are two agents: a driver and a pedestrian. Both are risk neutral expected utility maximizers. The agents are strangers and not in any contractual relationship. Transaction costs are sufficiently high to preclude Coasian bargaining.

The driver engages in a risky activity—driving. In the event of an accident the pedestrian incurs a loss. The driver, but not the pedestrian, can take precautions against an accident. The set of feasible precautions forms the driver's choice set—the *precaution set*. More specifically, the driver's precaution set is the Cartesian product of $n \geq 1$ sets, where each factor set represents a different type of precaution and the elements of each factor set represent the feasible levels of care within each type.

The governing liability rule determines whether the driver is liable to the pedestrian for her loss in the event of an accident. I consider the two basic liability rules of Anglo-American tort law: strict liability and negligence. Under negligence the driver is liable to the pedestrian if the driver failed to exercise due care (a legal standard set by the court). Under strict liability the driver is liable to the pedestrian whether or not the driver exercised due care.

## 3.1 The Model

There are $n \geq 1$ safety precautions that a driver can take to reduce the expected loss from an accident with a pedestrian.[12] Let $\mathcal{X} \subseteq \mathbb{R}^n_+$ denote the driver's $n$-dimensional precaution set, and let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathcal{X}$ denote his care levels across the $n$ safety dimensions. The driver's cost of care is $c(\mathbf{x}) = \sum_{i=1}^n \alpha_i x_i^2$ where $\alpha_i \geq 0$ for all $i = 1, \ldots, n$, and the pedestrian's expected loss is $\ell(\mathbf{x}) = \left(m - \sum_{i=1}^n \beta_i x_i\right)^2$ where $\beta_i > 0$ for all $i = 1, \ldots, n$ and $m \geq \sum_{i=1}^n \beta_i x_i$ for all $\mathbf{x} \in \mathcal{X}$.[13]

The social cost function is $s(\mathbf{x}) \equiv c(\mathbf{x}) + \ell(\mathbf{x}) = \sum_{i=1}^n \alpha_i x_i^2 + \left(m - \sum_{i=1}^n \beta_i x_i\right)^2$. It can be written in matrix notation as $s(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\intercal H \mathbf{x} - \mathbf{b}^\intercal \mathbf{x} + m^2$ where

$$H = \begin{bmatrix} 2\alpha_1 + 2\beta_1^2 & 2\beta_1\beta_2 & 2\beta_1\beta_3 & \cdots & 2\beta_1\beta_n \\ 2\beta_2\beta_1 & 2\alpha_2 + 2\beta_2^2 & 2\beta_2\beta_3 & \cdots & 2\beta_2\beta_n \\ 2\beta_3\beta_1 & 2\beta_3\beta_2 & 2\alpha_3 + 2\beta_3^2 & \cdots & 2\beta_3\beta_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2\beta_n\beta_1 & 2\beta_n\beta_2 & 2\beta_n\beta_3 & \cdots & 2\alpha_n + 2\beta_n^2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2m\beta_1 \\ 2m\beta_2 \\ 2m\beta_3 \\ \vdots \\ 2m\beta_n \end{bmatrix}.$$

Note that $s(\mathbf{x})$ is quadratic and $H$ is symmetric and positive definite.[14]

This simple model has two key virtues. First, with different specifications of $n$ and $\mathcal{X}$, I can consider the standard case where care is unidimensional and convex ($n = 1$ and $\mathcal{X}$ is an interval in $\mathbb{R}_+$), as well as the generalized cases where care is multidimensional and convex ($n > 1$ and $\mathcal{X}$ is a convex subset of $\mathbb{R}^n_+$), unidimensional and discrete ($n = 1$ and $\mathcal{X}$ is a discrete subset of $\mathbb{R}_+$), or multidimensional and discrete ($n > 1$ and $\mathcal{X}$ is a discrete subset of $\mathbb{R}^n_+$).

[12] For the avoidance of doubt, $n$ is an integer.

[13] For instance, suppose the probability of an accident is $p(\mathbf{x}) = m^{-1}(m - \sum_{i=1}^n \beta_i x_i)$ and the severity of harm in the event of an accident is $h(\mathbf{x}) = m(m - \sum_{i=1}^n \beta_i x_i)$. Then the expected loss is $\ell(\mathbf{x}) = p(\mathbf{x})h(\mathbf{x}) = \left(m - \sum_{i=1}^n \beta_i x_i\right)^2$.

[14] Definitions of selected mathematical terms used in the paper are set forth in the Appendix.

Second, the specification of $s(\mathbf{x})$ ensures that it is (I) convex in all cases where the driver's precaution set is convex and (II) supermodular in all cases.[15] In the standard UCFA model, where care is unidimensional and convex, the diminishing returns assumption implies that the social cost function satisfies restrictions I and II. When care is multidimensional, however, the diminishing returns assumption is no longer sufficient; more is required to preserve these shape restrictions. In my model the specification of $s(\mathbf{x})$ entails that (i) for all $n \geq 1$, the diminishing returns assumption holds within each safety dimension, and that (ii) for all $n > 1$, safety precautions are substitutes across dimensions (i.e., increasing care in one dimension decreases the marginal benefit of care in all other dimensions). When care is multidimensional these assumptions are jointly sufficient to preserve restriction I (convexity) and the second is necessary and sufficient to preserve restriction II (supermodularity).[16]

In the ensuing analysis I say that a liability rule is *optimal* if the solution to the driver's care-taking problem under that rule is *socially optimal* (i.e., coincides with the solution to the social problem). I say that an optimal liability rule can provide *robust deterrence* if the driver's problem under that rule is always easily solved (i.e., computationally tractable). After all, if the driver's problem is computationally intractable, even an optimal liability rule cannot always induce the driver to take socially optimal care.

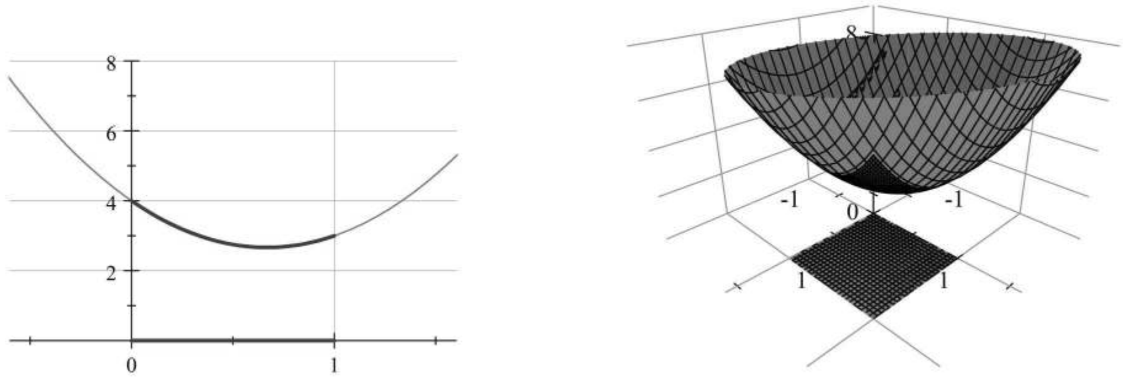
## 3.2 Convex Care

Suppose first that the safety precautions are continuous variables. Without loss of generality let $\mathcal{X} = [0,1]^n$. Note that $\mathcal{X}$ is convex. Each $\mathbf{x} = (x_1, \ldots, x_n) \in \mathcal{X}$ describes a unique array of care. For example, if care is unidimensional ($n = 1$) then perhaps $x = 1 - spd$ where *spd* is the driver's speed (expressed as a fraction of his vehicle's maximum speed), or if care is bidimensional ($n = 2$) then perhaps $x_1 = 1 - spd$ and $x_2 = 1 - bac$ where *bac* is the driver's blood alcohol concentration.

With $\mathcal{X} = [0,1]^n$ as its domain the social cost function $s(\mathbf{x})$ is convex and supermodular. It is also twice continuously differentiable. Figure 1 depicts $s(\mathbf{x})$ for

[15] Intuitively, a function is supermodular if it has increasing differences. For a formal definition, see the Appendix.

[16] The standard bilateral care model with fixed activity levels assumes that care by the two agents are substitutes (Miceli 1997, § 1.2). Accordingly, one could argue that assuming safety precuations are substitutes across dimensions is standard when care is multidimensional (as well as instrumental to preserving supermodularity).




(a) $2x^2 + (2-x)^2$ (b) $3x_1^2 + 3x_2^2 + (3 - x_1 - 2x_2)^2$

Figure 1: Two examples of the social cost function $s(\mathbf{x})$ on $\mathcal{X} = [0,1]^n$.

two cases: (a) the unidimensional case where $s(x) = 2x^2 + (2-x)^2$ and (b) the bidimensional case where $s(\mathbf{x}) = 3x_1^2 + 3x_2^2 + (3 - x_1 - 2x_2)^2$.

### 3.2.1 The Social Problem

The social problem is to find $\mathbf{x} \in \mathcal{X}$ to minimize $s(\mathbf{x})$. Assuming it is interior, the unique solution $\mathbf{x}^*$ is defined implicitly by the first-order condition $\nabla s(\mathbf{x}^*) = \mathbf{0}$, i.e., the gradient of $s(\mathbf{x})$ evaluated at $\mathbf{x}^*$ equals zero.[17] In the standard case of unidimensional care, the first-order condition reduces to $s'(x^*) = 0$, or equivalently $c'(x^*) = -\ell'(x^*)$, i.e., the marginal cost of care equals the marginal reduction in expected loss (the marginal benefit of care). In the case of multidimensional care, the first-order optimality condition can be written in matrix notation as $H\mathbf{x}^* - \mathbf{b} = \mathbf{0}$.

The social problem is easily solved in the unidimensional case. You first compute $s'(x^*) = 2ax^* - 2\beta(m - \beta x^*)$. You then set $2ax^* - 2\beta(m - \beta x^*) = 0$ and solve $x^* = \frac{\beta m}{a+\beta^2}$. This algorithm runs in linear time—the time complexity of each step is a linear function of the number of operations required to compute $s(x)$.

**Example 6** Take the unidimensional case where $s(x) = 2x^2 + (2-x)^2$. Compute $s'(x^*) = 4x^* - 2(2 - x^*)$. Set $4x^* - 2(2-x^*) = 0$ and solve $x^* = \frac{1\times 2}{2+1^2} = \frac{2}{3}$.

[17]The solution $\mathbf{x}^*$ is unique because $s(\mathbf{x})$ is strictly convex on $\mathcal{X} = [0,1]^n$. We know that $\mathbf{x}^* > 0$ because $\nabla s(\mathbf{0}) < \mathbf{0}$ for all admissible parameters. Thus, the solution is interior provided that $\nabla s(\mathbf{1}) > \mathbf{0}$ for the given parameters, because then $\mathbf{x}^* < \mathbf{1}$.

The social problem is also easily solved in the multidimensional case. In this case the size of the social problem is governed by the dimension of the precaution set, $n$. Due to the magic of calculus, there are many known polynomial-time algorithms for solving convex optimization problems with multidimensional choice sets, including several whose time complexities are low-order polynomial functions of $n$.

For instance, we could find $\mathbf{x}^*$ using a gradient descent algorithm, i.e., a "hill climbing" algorithm. The basic idea is that, starting any initial point on the "hill," you climb down in steps, where the size and direction of each step are functions of the gradient, until you reach the "bottom," where the gradient is zero. The conjugate gradient descent (CGD) algorithm is particularly well-suited to this problem (Hestenes and Stiefel 1952). According to CGD, starting from an initial point $\mathbf{x}_0 \in \mathcal{X}$, you first move in the direction of steepest descent, $d_0 = -\nabla s(\mathbf{x}_0)$, until you reach the lowest point $\mathbf{x}_1$ in that direction; you then move in the conjugate direction of steepest descent $d_1$ (i.e., the direction of steepest descent that is orthogonal to $Hd_0$) until you reach the lowest point $\mathbf{x}_2$ in that direction; you then move in the conjugate direction of steepest descent $d_2$ (i.e., the direction of steepest descent that is orthogonal to $Hd_1$) until you reach the lowest point $\mathbf{x}_3$ in that direction; and so forth until you reach the point at which the gradient is zero. In other words, CGD finds $\mathbf{x}^*$ by minimizing $s(\mathbf{x})$ in one conjugate direction (i.e., dimension) at a time. CGD therefore takes $n$ steps to find $\mathbf{x}^*$.[18] The time complexity at each step has order $n^2$, and hence CGD's total time complexity has order $n^3$ (Ryaben'kii and Tsynkov 2007).

**Example 7** Take the bidimensional case where $s(\mathbf{x}) = 3x_1^2 + 3x_2^2 + (3 - x_1 - 2x_2)^2$, in which case $\nabla s(\mathbf{x}) = (8x_1 + 4x_2 - 6, 4x_1 + 14x_2 - 12)$, and start initially at $\mathbf{x}_0 = (0, 1)$. CGD finds $\mathbf{x}^* = (\frac{3}{8}, \frac{3}{4})$ in two steps as follows:

0. Start at $\mathbf{x}_0 = (0, 1)$; $s(\mathbf{x}_0) = 4$, $\nabla s(\mathbf{x}_0) = (-2, 2)$.
1. Move to $\mathbf{x}_1 = (\frac{2}{7}, \frac{5}{7})$; $s(\mathbf{x}_1) = 3\frac{3}{7}$, $\nabla s(\mathbf{x}_1) = (-\frac{6}{7}, -\frac{6}{7})$.
2. Move to $\mathbf{x}_2 = (\frac{3}{8}, \frac{3}{4})$; $s(\mathbf{x}_2) = \frac{4}{3}$, $\nabla s(\mathbf{x}_2) = (0, 0)$.

Figure 2 depicts the two-step convergence path of CGD from $\mathbf{x}_0 = (0, 1)$ to $\mathbf{x}^* = (\frac{3}{8}, \frac{3}{4})$ on a contour plot of $s(\mathbf{x})$.

[18] CGD can take fewer than $n$ steps if the lowest point in one direction is also the lowest point in a conjugate direction. It can also take more than $n$ steps (or fail) due to rounding errors, though with exact arithmetic it is guaranteed to find $\mathbf{x}^*$ within $n$ steps.

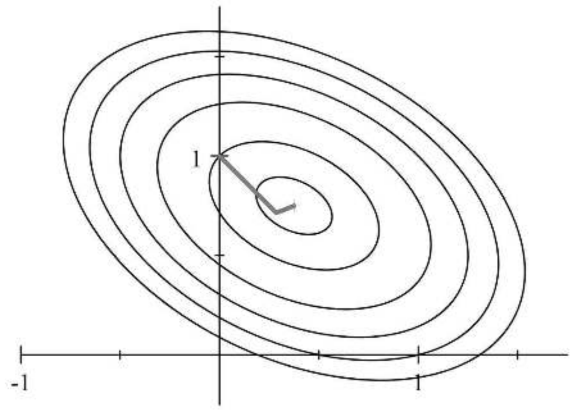



Figure 2: Two-step convergence path of CGD.

We could also find $\mathbf{x}^*$ using a matrix decomposition algorithm. Recall that $\mathbf{x}^*$ is defined implicitly by the first-order condition $H\mathbf{x}^* = \mathbf{b}$. The basic idea of a matrix decomposition algorithm is that you decompose $H$ into factors that can be used to easily solve for $\mathbf{x}^*$ without having to invert $H$. The Cholesky decomposition algorithm is particularly well-suited to this problem. It first decomposes $H$ into the product $LL^\top$ where $L$ is lower triangular. It then uses $L$ and $L^\top$ to solve for $\mathbf{x}^*$ in two steps by forward and backward substitution, respectively. Its total time complexity is $\frac{1}{3}n^3 + 2n^2$ (Boyd and Vandenberghe 2004).

**Example 8** Take the bidimensional case where $s(\mathbf{x}) = 3x_1^2 + 3x_2^2 + (3 - x_1 - 2x_2)^2$, in which case

$$H = \begin{bmatrix} 8 & 4 \\ 4 & 14 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 6 \\ 12 \end{bmatrix}.$$

The Cholesky decomposition algorithm finds $\mathbf{x}^* = (\frac{3}{8}, \frac{3}{4})$ in three steps as follows:

1. Factor $H$ as $H = LL^\top = \begin{bmatrix} 2\sqrt{2} & 0 \\ \sqrt{2} & 2\sqrt{3} \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & \sqrt{2} \\ 0 & 2\sqrt{3} \end{bmatrix}$.
2. Solve $L\mathbf{y} = \mathbf{b}$ by forward substitution:
$$2\sqrt{2}y_1 + 0y_2 = 6 \Rightarrow y_1 = \tfrac{3}{2}\sqrt{2};$$
$$\sqrt{2}\tfrac{3}{2}\sqrt{2} + 2\sqrt{3}y_2 = 12 \Rightarrow y_2 = \tfrac{3}{2}\sqrt{3}.$$
3. Solve $L^\top\mathbf{x} = \mathbf{y}$ by backward substitution:
$$0x_1 + 2\sqrt{3}x_2 = \tfrac{3}{2}\sqrt{3} \Rightarrow x_2 = \tfrac{3}{4};$$
$$2\sqrt{2}x_1 + \sqrt{2}\tfrac{3}{4} = \tfrac{3}{2}\sqrt{2} \Rightarrow x_1 = \tfrac{3}{8}.$$

### 3.2.2 The Driver's Problem

The fact that the social problem is easy solved implies that the driver's problem, whether under strict liability or negligence, is also easy solved.

Under strict liability the driver's problem is identical to the social problem. This is because strict liability forces the driver to internalize the social cost of his activity. Under strict liability, therefore, the driver's problem can be easily solved, and moreover the solution is socially optimal.

Let $\overline{\mathbf{x}} \in \mathcal{X}$ denote the due care standard under negligence. Under negligence the driver faces no liability if he chooses $\mathbf{x} \geq \overline{\mathbf{x}}$, and he effectively faces strict liability if he chooses $\mathbf{x} \ngeq \overline{\mathbf{x}}$.[19] Accordingly, the driver's problem under negligence has two steps. First, the driver finds the solution to the social problem, $\mathbf{x}^*$, which is easy. Second, the driver chooses $\mathbf{x} = \mathbf{x}^*$ if $\mathbf{x}^* \ngeq \overline{\mathbf{x}}$ and $s(\mathbf{x}^*) < c(\overline{\mathbf{x}})$, and he chooses $\mathbf{x} = \overline{\mathbf{x}}$ otherwise. This step is also easy because $s(\mathbf{x}^*)$ and $c(\overline{\mathbf{x}})$ are easily computed. Moreover, if the court sets the due care standard equal to the social optimum, $\overline{\mathbf{x}} = \mathbf{x}^*$, then the driver always chooses $\mathbf{x} = \mathbf{x}^*$. Under negligence, therefore, the driver's problem can be easily solved, and if $\overline{\mathbf{x}} = \mathbf{x}^*$ the solution is always socially optimal.

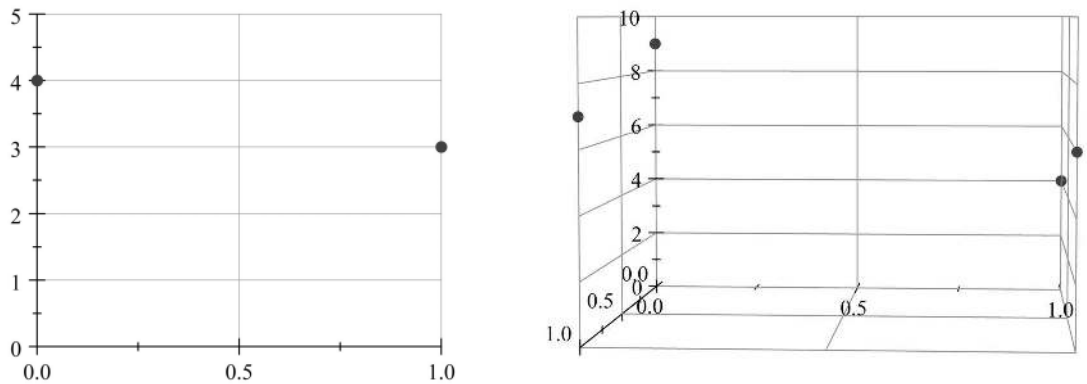
The following proposition recaps the foregoing results.

**Proposition 1** *Suppose that care is convex. The social problem is easily solved when care is unidimensional, and the time complexity of the solution scales polynomially in the dimension of the driver's precaution set. It follows that the driver's problem is computationally tractable. Thus, because strict liability and negligence (with $\overline{x} = x^*$) are both optimal, either rule can provide robust deterrence.*

## 3.3 Discrete Care

Suppose now that the safety precautions are discrete variables. Without loss of generality let $\mathcal{X} = \{0, 1\}^n$. Note that $\mathcal{X}$ is discrete. Each $\mathbf{x} \in \mathcal{X}$ corresponds to a unique combination of safety precautions. For example, if care in unidimensional ($n = 1$) then perhaps $x$ indicates whether the driver's vehicle has anti-lock brakes, or if care is bidimensional ($n = 2$) then perhaps $x_1$ indicates whether the driver's vehicle has anti-lock brakes and $x_2$ indicates whether it has a pedestrian detection system.

[19] For the avoidance of doubt, vector inequalities are componentwise.




(a) $2x^2 + (2-x)^2$

(b) $3x_1^2 + 3x_2^2 + (3 - x_1 - 2x_2)^2$

Figure 3: Two examples of the social cost function $s(\mathbf{x})$ on $\mathcal{X} = \{0,1\}^n$.

With $\mathcal{X} = \{0,1\}^n$ as its domain the social cost function $s(\mathbf{x})$ is supermodular (but not convex). Figure 3 depicts $s(\mathbf{x})$ for (a) the unidimensional case where $s(x) = 2x^2 + (2-x)^2$ and (b) the bidimensional case where $s(\mathbf{x}) = 3x_1^2 + 3x_2^2 + (3 - x_1 - 2x_2)^2$.

As before, the driver's problem under strict liability coincides with the social problem—find $\mathbf{x} \in \mathcal{X}$ to minimize $s(\mathbf{x})$—and his problem under negligence includes the social problem as a first step. And like before, the social problem is easy solved in the unidimensional case. You first compute the two possible values of $s(\mathbf{x})$, $s(0) = m^2$ and $s(1) = \alpha + (m - \beta)^2$, and then determine which is smaller. Indeed, even if there are more than two levels of care—i.e., $\mathcal{X} = \{0, 1, \ldots, k\}$—the social problem can still be easily solved because the number of computations grows linearly in $k$.

Unlike before, however, the social problem cannot be easily solved in the multidimensional case. This is because it is a binary quadratic optimization problem—the minimization of a quadratic function on the vertices of a hypercube—which is known to be $\mathcal{NP}$-hard (see, for example, Li et al. 2010). It also follows from the fact that:[20]

**Claim 1** *The max cut problem is reducible to the social problem with multidimensional discrete care.*

In general, the only known method for solving the social problem in this case is brute-force. That is, you must compute the value of $s(\mathbf{x})$ at every $\mathbf{x} \in \mathcal{X}$ and find the minimizer $\mathbf{x}^*$ which satisfies $s(\mathbf{x}^*) \leq s(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{X}$. In particular, neither a

[20] Claim 1 is proved in the Appendix.

"discretized" version of the gradient descent algorithm, nor a "convexified" version of the social problem, is guaranteed to find $\mathbf{x}^*$.

**Example 9** Consider a descent algorithm where you start at an initial point $\mathbf{x}_0 \in \mathcal{X}$ and then at each step check all points in $\mathcal{X}$ that are one-increment away in any direction/dimension and move to the lowest point thereamong, until you reach a "bottom" point where you cannot move lower. Take the bidimensional case where $s(\mathbf{x}) = 3x_1^2 + 3x_2^2 + (3 - x_1 - \frac{4}{3}x_2)^2$ on $\mathcal{X} = \{\mathbf{x} \in \{0,1,2,3,4\}^2,\ x_1 \neq x_2\}$. The global minimizer, found by brute-force, is $\mathbf{x}^* = (0,1)$ where $s(0,1) = 4$. However, the "one-increment" algorithm may fail to converge to $\mathbf{x}^*$, and instead converge to a local minimizer. For instance, if you start at $\mathbf{x}_0 = (3,0)$ where $s(3,0) = 27$, you first move to $\mathbf{x}_1 = (2,0)$ where $s(2,0) = 13$ (after also checking $s(4,0) = 49$ and $s(3,1) = 34$), and then move to $\mathbf{x}_2 = (1,0)$ where $s(1,0) = 7$ (after also checking $s(2,1) = 16$), at which point you reach bottom. But $\mathbf{x}_2 \neq \mathbf{x}^*$.

Alternatively, suppose you minimize $s(\mathbf{x})$ on the convex hull of $\mathcal{X}$ and then select the point $\mathbf{x} \in \mathcal{X}$ that is closest to the solution $\widehat{\mathbf{x}}$.[21] Take the bidimensional case where $s(\mathbf{x}) = 3x_1^2 + 3x_2^2 + (3 - x_1 - \frac{4}{3}x_2)^2$ on $\mathcal{X} = \{0,1\}^2$. The true minimizer, found by brute-force, is $\mathbf{x}^* = (0,1)$ where $s(0,1) = 5\frac{7}{9}$. However, if you minimize $s(\mathbf{x})$ on $[0,1]^2$, which yields the solution $\widehat{\mathbf{x}} = (\frac{27}{52}, \frac{9}{13})$, and then select the closest point $\mathbf{x} \in \mathcal{X}$, you obtain $\mathbf{x} = (1,1)$ where $s(1,1) = 6\frac{4}{9}$. But $\mathbf{x} \neq \mathbf{x}^*$.

With $n$ safety dimensions, $\mathcal{X} = \{0,1\}^n$ has $2^n$ elements. Hence, the time complexity of the brute-force method grows exponentially with $n$. For instance, with two safety dimensions you must compute four values of $s(\mathbf{x})$,

$$
\begin{aligned}
&s(0,0) = m^2, && s(1,0) = \alpha_1 + (m - \beta_1)^2, \\
&s(0,1) = \alpha_2 + (m - \beta_2)^2, \quad \text{and} && s(1,1) = \alpha_1 + \alpha_2 + (m - \beta_1 - \beta_2)^2,
\end{aligned}
$$

[21] This "convexification" approach assumes that $s(\mathbf{x})$ has an explicit form and that the same functional relation that applies to the points in $\mathcal{X}$ also applies to the points in the convex hull of $\mathcal{X}$. Neither assumption may be true. An alternative approach that does not rely on these assumptions would be to minimize the convex closure of $s(\mathbf{x})$. In general, however, the convex closure of a supermodular function does not have a closed form that is easy to compute (Bach 2013).

Table 2: Discrete Care, Brute-force versus Convex Care, CGD

| Safety dimensions ($n$) | Computations | |
|---|---|---|
| | Discrete care, brute-force ($\tau(n) = 2^n$) | Convex care, CGD ($\tau(n) = n^3$) |
| 2 | 4 | 8 |
| 5 | 32 | 125 |
| 10 | 1,024 | 1,000 |
| 25 | 33,554,432 | 15,625 |
| 50 | 1,125,899,906,842,624 | 125,000 |
| 75 | 37,778,931,862,957,161,709,568 | 421,875 |
| 100 | 1,267,650,600,228,229,401,496,703,205,376 | 1,000,000 |

whereas with 100 safety dimensions you must compute more than one nonillion ($1 \times 10^{30}$) values of $s(\mathbf{x})$.[22] With multidimensional convex care, by comparison, the number of computations grows from eight to one million. See Table 2.

The following proposition recaps the foregoing results.

**Proposition 2** *Suppose that care is discrete. Although the social problem is easily solved when care is unidimensional, the time complexity of the solution scales exponentially in the dimension of the driver's precaution set. It follows that the driver's problem is computationally intractable ($\mathcal{NP}$-hard) under strict liability and negligence. Thus, although strict liability and negligence (with $\overline{x} = x^*$) are both optimal, neither rule can provide robust deterrence.*

In the Appendix I show that the results in Propositions 1 and 2 go through to a general UCFA model which maintains the restrictions on the social cost function that are implied by the diminishing returns assumption, but relaxes the assumption that the social cost function is quadratic. Henceforth, when I refer to the social problem and the injurer's problem, I am referring to the general versions of these problems.

# 4 Loose Ends

In this section I tie up three loose ends by addressing the following questions. (i) Can we approximate the solution to the injurer's problem in polynomial time when care is

[22] From the two-dimensional computations one can already see that $s(\mathbf{x})$ is supermodular (i.e., has increasing differences): $s(1,0)-s(0,0) = \alpha_1 - 2m\beta_1 + \beta_1^2 < \alpha_1 - 2m\beta_1 + \beta_1^2 + 2\beta_1\beta_2 = s(1,1) - s(0,1)$ and $s(0,1) - s(0,0) = \alpha_2 - 2m\beta_2 + \beta_2^2 < \alpha_2 - 2m\beta_2 + \beta_2^2 + 2\beta_1\beta_2 = s(1,1) - s(1,0)$.

multidimensional and discrete? If yes, then tort law can provide approximately robust deterrence in this case. (ii) Can we justify assuming that care is unidimensional and convex? "All models are wrong," after all, "but some are useful" nonetheless (Box 1979, p. 202). (iii) What if there were increasing returns to care?

## 4.1 Approximating the Injurer's Problem

If an optimization problem is $\mathcal{NP}$-hard, implying that it cannot be solved in polynomial time unless $\mathcal{P} = \mathcal{NP}$, there nevertheless may exist a polynomial-time algorithm that approximates the optimal value within some constant factor. Such an algorithm is called a $\rho$-*approximation algorithm*, where $\rho > 1$ denotes the approximation factor. More specifically, if $\mathbf{x}^*$ is a solution to an optimization problem with objective function $f$, a $\rho$-approximation algorithm for the problem yields an output $\widetilde{\mathbf{x}}$ in polynomial time such that (i) $f(\widetilde{\mathbf{x}}) \leq \rho f(\mathbf{x}^*)$, in the case of a minimization problem, or (ii) $f(\widetilde{\mathbf{x}}) \geq \frac{1}{\rho} f(\mathbf{x}^*)$, in the case of a maximization problem. For instance, the traveling salesman problem (with distances that satisfy the triangle inequality) is known to have a $\frac{3}{2}$-approximation algorithm (Christofides 1976), while the max cut problem is known to have a $\frac{1}{.878}$-approximation algorithm (Goemans and Williamson 1995).

As I show in the Appendix, however, when care is multidimensional and discrete it is $\mathcal{NP}$-hard to approximate the optimum of the social problem within any constant factor. Thus, unless $\mathcal{P} = \mathcal{NP}$, there does not exist a $\rho$-approximation algorithm for the injurer's problem with multidimensional discrete care. The proof is by reduction from the max cut problem and relies on a result, due to Håstad (2001), that it is $\mathcal{NP}$-hard to approximate the max cut problem within a factor less than $\frac{17}{16}$.

## 4.2 The Unidimensional and Convex Care Assumptions

The common justification for the unidimensional care assumption is that it is without loss of generality. For example, Shavell (1987, p. 36) remarks: "If care $x$ is a multidimensional variable, the proofs that strict liability and the negligence rule will lead to the socially optimal outcome $x^*$ still apply." This of course is true—given the other assumptions of the standard UCFA model, including in particular the convex care assumption. But, as I have shown, the unidimensional care assumption is not without loss of generality when care is discrete.

There are two common justifications for the convex care assumption. The first is the argument that care is isomorphic to expenditures on care. This argument is implicit in the common choice to model the injurer's precaution set not as the set of feasible safety precautions, but rather as the set of feasible safety expenditures (see, for example, Shavell 1987, ch. 2; Miceli 1997, ch. 1). The purported isomorphism runs into difficulty, however, when we consider the diminishing returns assumption. The law of diminishing returns is traditionally defined in terms of inputs, not expenditures on inputs (see, for example, Samuelson 1980; Brue 1993). Thus, morphing "diminishing returns to care" into "diminishing returns to expenditures on care" requires an auxiliary assumption, namely that "injurers invest first in the most effective precautions and only later turn to less effective measures" (Miceli 2017, p. 42). But this auxiliary assumption begs the very question that I am asking in this paper.

The other common justification for the convex care assumption is the argument that, once you account for time, care is properly modeled as a continuous flow variable, not a discrete stock variable. This is surely correct for some safety precautions—but not all. Take driving. While some discrete precautions, such as wearing a seat belt or using high beams, can plausibly be viewed as a sequence of (arbitrarily) short-term commitments, other discrete precautions, such as equipping your vehicle with anti-lock brakes or power steering, cannot. Even accounting for time, therefore, we cannot always smooth over the knotty fact that some precautions are discrete.

## 4.3 Increasing Returns to Care

The diminishing returns assumption is grounded in the law of diminishing returns, "one of the few generalities of economic theory which might be called a law" (Shephard and Rolf 1974, p. 69). The law of diminishing returns, however, is not an immutable law of nature, and so increasing returns to care are theoretically possible. If there were increasing returns to care this would turn the analysis on its head.[23] In the case of convex care the social problem would entail the minimization of a concave function on a convex choice set, which is $\mathcal{NP}$-hard (see, for example, Benson 1995). In the case of discrete care the social problem would entail the minimization of a submodular function on a discrete choice set, which is easy (Grötschel, Lovász, and

[23] To be clear, if there were increasing returns to care, the marginal cost of care would be decreasing and the marginal benefit of care would be increasing. That is, $c(\mathbf{x})$ and $\ell(\mathbf{x})$ would be concave (in the case of convex care) and submodular.

Schrijver 1981).[24] That being said, it bears repeating that the diminishing returns assumption rests of firm ground. "Economists are not myth-making when we tell our students that, if the law of diminishing returns was not true, the world's food supply could be grown in a flower pot" (Brue 1993, p. 191). It thus seems unlikely that the theoretical possibility of increasing returns to care has much practical importance.

# 5 Discussion

My analysis of the computational complexity of the injurer's care-taking problem has several implications for the theory and practice of tort law.

One implication is that the standard assumptions of unidimensional and convex care, though seemingly innocuous, are pivotal to constituting the injurer as Holmes' unboundedly rational "bad man"—*homo law-and-economicus*—and ensuring that tort law can provide robust deterrence.[25] When care is multidimensional and discrete, however, we cannot necessarily rely on tort law to provide robust deterrence. In these cases the injurer may be less like Holmes' "bad man" and more like H.L.A. Hart's "puzzled man . . . who is willing to do what is required, if only he can be told what it is" (Hart 1961, p. 40).[26]

This leads to a second implication. It is important to recognize situations with multidimensional discrete care, where tort deterrence may not be robust. The examples listed at the end of Section 2 are just the tip of the iceberg. There are many real-world care-taking problems in which the precaution set is multidimensional and discrete. Indeed, some commentators have surmised that this may be the "usual" case (see, for example, Shavell 1987, p. 9; Miceli 2017, p. 49).[27]

In light of the foregoing one could argue that, at least in cases of multidimensional discrete care, we should perhaps elevate other normative approaches to tort law—such

[24]For details, see the remark following the proof of Theorem 2 in the Appendix.

[25]The term *homo law-and-economicus* was coined by Gordon (1997, p. 1014).

[26]Similar implications of other aspects of bounded rationality have been found in related work. For instance, Teitelbaum (2007) finds that neither strict liability nor negligence is generally efficient in the presence of ambiguity, while Chakravarty, Kelsey, and Teitelbaum (2019) find similar inefficiencies in the presence of unawareness. Both papers study tort deterrence using the UCFA model.

[27]Shavell (1987, p. 9): "Suppose, as would be usual, that there is more than one dimension of an injurer's behavior that affects accident risks (not only a driver's speed, but also the frequency with which he looks at the rear-view mirror)." Miceli (2017, p. 49): "One complication in applying marginal analysis to actual accident cases is that care usually does not vary continuously but comes in discrete bundles."

as corrective justice (Coleman 1992; Weinrib 1995) or civil recourse (Goldberg and Zipurski 2020)—over the economic approach and its focus on deterrence and social welfare. Despite the apparent logic of this conclusion, I reject it for two reasons. First, although the injurer's care-taking problem is in general hard to solve when care is multidimensional and discrete, there undoubtedly are particular instances in which the injurer's problem can be easily solved.[28] Second, because I find the Pareto principle to be compelling (if not unassailable), and because any non-welfarist approach to tort law will sometimes require adoption of liability rules that violate the Pareto principle (Kaplow and Shavell 2001), I am loath to elevate any non-welfarist approach over the economic approach.

Finally, complexity analysis may have implications for the "strict liability versus negligence" debate within tort law and economics. My results—including the result that neither strict liability nor negligence can provide robust deterrence when care is multidimensional and discrete—are based on a static analysis. There is an argument to be made, however, that negligence has a dynamic advantage over strict liability when care is multidimensional and discrete. I sketch out the argument in the remainder of this section. I also raise three potential counterarguments. In the end I conclude that the subject warrants future in-depth exploration.

The argument, in brief, is that when care is multidimensional and discrete (i) the injurer's behavior, under strict liability or negligence, moves in the direction of the social optimum over time through a learning-by-experimentation process, but that (ii) negligence accelerates the injurer's learning process "because it generates more public information about the [social optimum]" (Schäfer and Müller-Langer 2009, p. 27; see also Ott and Schäfer 1997; Feess and Wohlschlegel 2006).[29]

The starting point of the argument is the following claim.[30]

**Claim 2** *The decision problem that corresponds to the injurer's problem, whether under strict liability or negligence, is in* $\mathcal{NP}$*.*

[28]Just as there are particular instances—i.e., particular objective functions and/or graph topologies—in which the max cut problem is solvable in polynomial time (see, for example, Ben-Ameur, Mahjoub, and Neto 2014).

[29]Chakravarty, Kelsey, and Teitelbaum (2019) similarly find that negligence, for the same reason, has a dynamic advantage over strict liability in the presence of growing awareness. Although Teitelbaum (2007) does not consider a dynamic environment with learning, he finds that negligence, due to the discontinuity it creates in the injurer's expected liability, has a static advantage over strict liability in the presence of ambiguity.

[30]Claim 2 is proved in the Appendix.

Assume that care is multidimensional and discrete and consider how the injurer's behavior evolves over time as he repeatedly engages in his risky activity, periodically causing an accident. Let $t = 0$ denote the time before the injurer first engages in his activity, let $t = 1$ denote the time after the first accident but before the injurer next engages in his activity, let $t = 2$ denote the time after the second accident but before the injurer next engages in his activity, and so forth. In addition, assume that: (i) each time there is an accident, the circumstances of the accident suggest an array of care $\mathbf{y}_t \in \mathcal{X}$ that would have prevented the accident; (ii) each time there is an accident, the victim brings suit against the injurer before the court; and (iii) the injurer, the victim, and the court all have access to the same computational methods.

Suppose first that the governing liability rule is strict liability. At $t = 0$ the injurer chooses precautions $\mathbf{x}_0 \in \mathcal{X}$. However, because the injurer's problem is hard, it is likely that $\mathbf{x}_0 \neq \mathbf{x}^*$. At $t = 1$ the injurer can efficiently verify whether $s(\mathbf{y}_1) \leq s(\mathbf{x}_0)$, i.e., whether $\mathbf{y}_1$ is superior to $\mathbf{x}_0$. This follows from Claim 2. If $\mathbf{y}_1$ is superior to $\mathbf{x}_0$ then the injurer adopts $\mathbf{x}_1 = \mathbf{y}_1$; otherwise he stands pat at $\mathbf{x}_1 = \mathbf{x}_0$. At $t = 2$ the injurer can efficiently verify whether $\mathbf{y}_2$ is superior to $\mathbf{x}_1$. If it is then the injurer adopts $\mathbf{x}_2 = \mathbf{y}_2$; otherwise he stands pat at $\mathbf{x}_2 = \mathbf{x}_1$. And so forth. In this way, as $t \to \infty$, the injurer's behavior moves in the direction of the social optimum $\mathbf{x}^*$.

Suppose next that the governing liability rule is negligence, with due care standard $\overline{\mathbf{x}}_0$ at $t = 0$. Because the social problem is hard, it is likely that $\overline{\mathbf{x}}_0 \neq \mathbf{x}^*$. Moreover, because the injurer and the court have access to the same computational methods, the injurer always chooses to take due care: $\mathbf{x}_t = \overline{\mathbf{x}}_t$ at all $t$. Thus, we need only consider the evolution of the due care standard. At $t = 1$ the court can efficiently verify whether $\mathbf{y}_1$ is superior to $\overline{\mathbf{x}}_0$. This follows from Claim 2 and the fact that the social problem is identical to the injurer's problem under strict liability. If $\mathbf{y}_1$ is superior to $\overline{\mathbf{x}}_0$ then the court adopts $\overline{\mathbf{x}}_1 = \mathbf{y}_1$; otherwise it stands pat at $\overline{\mathbf{x}}_1 = \overline{\mathbf{x}}_0$. At $t = 2$ the court can efficiently verify whether $\mathbf{y}_2$ is superior to $\overline{\mathbf{x}}_1$. If it is then the court adopts $\overline{\mathbf{x}}_2 = \mathbf{y}_2$; otherwise it stands pat at $\overline{\mathbf{x}}_2 = \overline{\mathbf{x}}_1$. And so forth. In this way, as $t \to \infty$, the injurer's behavior moves in the direction of the social optimum $\mathbf{x}^*$.

So far the model suggests that the evolution of the injurer's behavior over time is the same under strict liability and negligence. Each time there is an accident nature proposes a solution $\mathbf{y}_t$ and the injurer can efficiently verify whether $\mathbf{y}_t$ is superior to the status quo and adapt his behavior accordingly. It is a learning-by-experimentation process, akin to the process of learning the probability distribution of an unfair coin through repeated flips.

The dynamic advantage of negligence emerges when we add another injurer to the model. Suppose that: (i) there is another injurer who engages in the same risky activity as our injurer, periodically causing an accident; (ii) the other injurer's accidents occur at periods $t = \frac{1}{2}$, $t = \frac{3}{2}$, and so forth, but he is otherwise identical to our injurer (same precaution set, same cost of care function, etc.); and (iii) the other injurers' accidents are unobserved by our injurer.

Return first to the case where strict liability is the governing liability rule. Because our injurer does not observe the other injurer's accidents, he does not observe the sequence $\mathbf{y}_{\frac{1}{2}}, \mathbf{y}_{\frac{3}{2}}, ....$ He therefore cannot learn from the other injurer's accidents, implying that his learning process is the same as before, with adaptations in his behavior possible only at periods $t = 1, 2, ....$

Now return to the case where negligence is the governing liability rule. Although our injurer does not observe the sequence $\mathbf{y}_{\frac{1}{2}}, \mathbf{y}_{\frac{3}{2}}, ...$, the court does. Consequently, the court's learning process is faster than before, with adaptations in the due care standard possible at periods $t = \frac{1}{2}, 1, \frac{3}{2}, 2, ....$ This accelerates our injurer's learning process, for he observes the sequence $\overline{\mathbf{x}}_{\frac{1}{2}}, \overline{\mathbf{x}}_1, \overline{\mathbf{x}}_{\frac{3}{2}}, \overline{\mathbf{x}}_2, ...$, making adaptations in his behavior more frequent.[31] In this way, negligence can move the injurer's behavior in the direction of the social optimum more rapidly than strict liability.

Admittedly the foregoing argument rests on strong assumptions. By challenging them we can construct potential counterarguments. Let me highlight three.

The first counterargument challenges the assumption that our injurer cannot observe the other injurer's accidents. Suppose instead that our injurer can observe the other injurer's accidents when they result in litigation. This assumption, together with the maintained assumption that accidents always result in litigation, would imply that strict liability generates the same amount of information as negligence, eliminating negligence's dynamic advantage.

The second counterargument disputes the assumption that accidents always result in litigation. Suppose instead that there is more litigation under strict liability than under negligence. Shavell (1987, p. 264) provides the standard justification for this assumption: "Under strict liability a victim will have an incentive to make a claim whenever his losses exceed the costs of making a claim (assuming that he can credibly

[31] Indeed, the more injurers we add to the model, the greater is the rate of this acceleration.

establish that the injurer was the cause of harm and that he was not contributorily negligent). Under the negligence rule a victim will not have an incentive to make a claim so often because he will also be concerned about establishing the injurer's negligence."[32] If there were more litigation under strict liability, then strict liability would generate more information than negligence.

The third counterargument modifies the assumption that the injurers are identical (aside from the periodicity of their accidents). Suppose instead that each injurer receives a private signal about which precautions to take. Assuming the signals are not too strong, we would expect the two injurers to take different precautions under strict liability, but to "herd" on the due care standard under negligence. There consequently would be greater experimentation under strict liability, implying that it would generate better information than negligence.

There is of course much more that could be said about these arguments and counterarguments. However, a deeper dive into the dynamics of tort deterrence is beyond the scope of this paper. I therefore leave this topic for future exploration.

# Appendix

## A Selected Definitions

In this section I define selected mathematical terms used in the paper. References include Boyd and Vandenberghe (2004) and Simchi-Levi, Bramel, and Chen (2005).

**Definition 1 (convex closure)** *The convex closure of a function* $f : \{0,1\}^n \to \mathbb{R}$ *is the greatest convex function* $f^- : [0,1]^n \to \mathbb{R}$ *that everywhere lowerbounds* $f$*.*

**Remark** If a function $f : \{0,1\}^n \to \mathbb{R}$ is submodular, then its convex closure is known as the Lovász extension (Lovász 1983).

**Definition 2 (convex hull)** *The convex hull of a discrete set* $X \subseteq \mathbb{R}^n$ *is the set of all convex combinations of points in* $X$*.*

[32]See also Miceli (2017, p. 45): "If a victim expects to lose, she will not file suit. Thus, under strict liability, the victim will file if (1) she can prove that the injurer caused her injuries, and (2) her losses exceed the cost of bringing suit. Under negligence, the preceding conditions for filing must be met, but in addition, the victim must prove that the injurer is at fault (that is, that he failed to meet the due standard of care). And since . . . the injurer has a powerful incentive to meet the due standard, victims will often be deterred from filing suit under negligence. . . . Thus, we expect fewer lawsuits under negligence as compared to strict liability."

**Definition 3 (convex set)** *A set $X \subseteq \mathbb{R}^n$ is convex iff $t\mathbf{x} + (1-t)\mathbf{y} \in X$ for all $\mathbf{x}, \mathbf{y} \in X$ and $t \in [0,1]$.*

**Definition 4 (convex function)** *A real-valued function $f$ on a convex set $X \subseteq \mathbb{R}^n$ is convex iff $f(t\mathbf{x} + (1-t)\mathbf{y}) \leq tf(\mathbf{x}) + (1-t)f(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in X$ and $t \in [0,1]$. It is strictly convex iff the inequality is strict.*

**Remark** Let $f$ be a real-valued function on a convex set $X \subseteq \mathbb{R}^n$. If $n = 1$ and $f$ is twice continuously differentiable, then $f$ is convex iff $f''(x) \geq 0$ and it is strictly convex iff $f''(x) > 0$. If $n > 1$ and $f$ is twice continuously differentiable, then $f$ is convex iff its Hessian is positive semidefinite and it is strictly convex iff its Hessian is positive definite.

**Definition 5 (discrete set)** *A set $X \subseteq \mathbb{R}^n$ is discrete iff for every point $\mathbf{x} \in X$ there exists an open ball around $\mathbf{x}$ that contains no other points in $X$. An open ball around a point $\mathbf{x} \in \mathbb{R}^n$ is a set $B_r(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^n : d(\mathbf{x}, \mathbf{y}) < r\}$ where $r > 0$ and $d(\mathbf{x}, \mathbf{y})$ is the Euclidean distance between $\mathbf{x}$ and $\mathbf{y}$.*

**Definition 6 (lower triangular matrix)** *A symmetric $n \times n$ real-valued matrix $A$ is lower triangular iff all of the entries above the main diagonal are zero.*

**Definition 7 (monotone increasing/decreasing function)** *A real-valued function $f$ on a set $X \subseteq \mathbb{R}^n$ is monotone increasing iff $f(\mathbf{x}) \leq f(\mathbf{y})$ for all $\mathbf{x} \leq \mathbf{y}$ in $X$. It is monotone decreasing if $f(\mathbf{x}) \geq f(\mathbf{y})$ for all $\mathbf{x} \leq \mathbf{y}$ in $X$.*

**Definition 8 (positive definite/semidefinite matrix)** *A symmetric $n \times n$ real-valued matrix $A$ is positive definite iff $\mathbf{x}^\intercal A\mathbf{x} > 0$ for all nonzero $\mathbf{x} \in \mathbb{R}^n$. It is positive semidefinite iff the inequality is weak.*

**Definition 9 (quadratic function)** *A real-valued quadratic function $f$ on a set $X \subseteq \mathbb{R}^n$ has the form $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\intercal A\mathbf{x} - \mathbf{b}^\intercal\mathbf{x} + c^2$ where $A$ is a symmetric $n \times n$ real-valued matrix, $\mathbf{b} \in \mathbb{R}^n$, and $c \in \mathbb{R}$.*

**Remark** Let $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\intercal A\mathbf{x} - \mathbf{b}^\intercal\mathbf{x} + c^2$ be a real-valued quadratic function on a set $X \subseteq \mathbb{R}^n$. If $X$ is convex and $f$ is twice continuously differentiable, then $A$ is the Hessian of $f(\mathbf{x})$.

**Definition 10 (supermodular/submodular function)** *A real-valued function $f$ on a set $X \subseteq \mathbb{R}^n$ is supermodular iff it has increasing differences, i.e., for all $\mathbf{x} \leq \mathbf{y}$ in $X$, $f(\mathbf{x}+\mathbf{t}) - f(\mathbf{x}) \leq f(\mathbf{y}+\mathbf{t}) - f(\mathbf{y})$ for all $\mathbf{t} \geq \mathbf{0}$ such that $\mathbf{x}+\mathbf{t}, \mathbf{y}+\mathbf{t} \in X$. It is submodular iff it has decreasing differences, i.e., for all $\mathbf{x} \leq \mathbf{y}$ in $X$, $f(\mathbf{x}+\mathbf{t}) - f(\mathbf{x}) \geq f(\mathbf{y}+\mathbf{t}) - f(\mathbf{y})$ for all $\mathbf{t} \geq \mathbf{0}$ such that $\mathbf{x}+\mathbf{t}, \mathbf{y}+\mathbf{t} \in X$.*

**Remark** Let $f$ be a real-valued function on a set $X \subseteq \mathbb{R}^n$. If $n = 1$ and $X$ is convex, then $f$ is supermodular (i.e., has increasing differences) if it is convex. If $X$ is convex and $f$ is twice continuously differentiable, then $f$ is supermodular (i.e., has increasing differences) iff $\frac{\partial^2 f}{\partial x_i \partial x_j} \geq 0$ for all $i \neq j$. If $f$ is supermodular then $-f$ is submodular (and vice versa).

**Definition 11 (symmetric matrix)** *An $n \times n$ real-valued matrix $A$ is symmetric iff $A = A^\intercal$, i.e., $a_{ij} = a_{ji}$ for all $i$ and $j$ where $a_{ij}$ denotes the entry in the $i$-th row and $j$-th column.*

# B Proof of Claim 1

Let $\mathbf{x} = (x_1, \ldots, x_n)$ where $n > 1$ is an integer. The max cut problem is: Choose $\mathbf{x} \in \{0,1\}^n$ to maximize $\sum_{i,j=1}^n \frac{1}{2} w_{ij}(x_i - x_j)^2$, where $w_{ij} > 0$ and $w_{ij} = w_{ji}$.[33] Let $\mathbf{x}^\dagger$ denote the solution. Note that $\mathbf{x}^\dagger$ is also the solution to: Choose $\mathbf{x} \in \{0,1\}^n$ to minimize $f(\mathbf{x}) = -\sum_{i,j=1}^n \frac{1}{2} w_{ij}(x_i - x_j)^2$. Observe that $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\intercal Q \mathbf{x}$ where

$$Q = \begin{bmatrix} \sum_{j=1:j\neq 1}^n - 2w_{1j} & 2w_{12} & 2w_{13} & \cdots & 2w_{1n} \\ 2w_{21} & \sum_{j=1:j\neq 2}^n - 2w_{2j} & 2w_{23} & \cdots & 2w_{2n} \\ 2w_{31} & 2w_{32} & \sum_{j=1:j\neq 3}^n - 2w_{3j} & \cdots & -2w_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2w_{n1} & 2w_{n2} & 2w_{n3} & \cdots & \sum_{j=1:j\neq n}^n - 2w_{nj} \end{bmatrix}.$$

The social problem is: Choose $\mathbf{x} \in \{0,1\}^n$ to minimize $s(\mathbf{x}) = \sum_{i=1}^n \alpha_i x_i^2 + (m - \sum_{i=1}^n \beta_i x_i)^2$, where $\alpha_i \geq 0$, $\beta_i > 0$, and $m \geq \sum_{i=1}^n \beta_i$. Let $\mathbf{x}^*$ denote the solution. Recall that $s(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\intercal H \mathbf{x} - \mathbf{b}^\intercal \mathbf{x} + m^2$ where $H$ and $\mathbf{b}$ are defined in

[33] A more general version of the max cut problem allows for $w_{ij} \geq 0$ (i.e., for an incomplete graph). Both versions are $\mathcal{NP}$-hard (Karp 1972; Garey and Johnson 1979).

Section 3.1. Because $x_i \in \{0,1\}$, $x_i^2 = x_i$ and hence $s(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\intercal K\mathbf{x} + m^2$ where

$$K = \begin{bmatrix} 2\alpha_1 + 2\beta_1^2 - 2m\beta_1 & 2\beta_1\beta_2 & 2\beta_1\beta_3 & \cdots & 2\beta_1\beta_n \\ 2\beta_2\beta_1 & 2\alpha_2 + 2\beta_2^2 - 2m\beta_2 & 2\beta_2\beta_3 & \cdots & 2\beta_2\beta_n \\ 2\beta_3\beta_1 & 2\beta_3\beta_2 & 2\alpha_3 + 2\beta_3^2 - 2m\beta_3 & \cdots & 2\beta_3\beta_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2\beta_n\beta_1 & 2\beta_n\beta_2 & 2\beta_n\beta_3 & \cdots & 2\alpha_n + 2\beta_n^2 - 2m\beta_n \end{bmatrix}.$$

Moreover, because $m^2$ is a constant, $\mathbf{x}^*$ is also the solution to: Choose $\mathbf{x} \in \{0,1\}^n$ to minimize $g(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\intercal K\mathbf{x}$.

Let $\beta_i\beta_j = w_{ij}$ and $\alpha_i = m\beta_i - \beta_i^2 - \sum_{j=1:j\neq i}^{n}\beta_i\beta_j$ for all $i$. (Note that $\alpha_i \geq 0$ because $m \geq \sum_{i=1}^{n}\beta_i$, $\beta_i > 0$ because $w_{ij} > 0$, and $\beta_i\beta_j = \beta_j\beta_i$ because $w_{ij} = w_{ji}$.) Then $K = Q$, which implies $g(\mathbf{x}) = f(\mathbf{x})$ and thus $\mathbf{x}^* = \mathbf{x}^\dagger$. This establishes that the max cut problem is reducible to the social problem. ■

## C General Model and Analysis

As in Section 3, I analyze tort deterrence using the UCFA model. In this section I present a general version of the model which maintains the shape restrictions on the social cost function that are implied by the diminishing returns assumption, but does not assume a specific functional form for the social cost function.

In the model there are two agents: an injurer and a victim. Both are risk neutral expected utility maximizers. The agents are strangers and not in any contractual relationship. Transaction costs are sufficiently high to preclude Coasian bargaining.

The injurer engages in a risky activity. In the event of an accident the victim incurs a loss. The injurer, but not the victim, can take precautions against an accident. The set of feasible safety precautions forms the injurer's choice set—the *precaution set*. More specifically, the injurer's precaution set is the Cartesian product of $n \geq 1$ sets, where each factor set represents a different type of precaution and the elements of each factor set represent the feasible levels of care within each type.

The governing liability rule determines whether the injurer is liable to the victim for her loss in the event of an accident. I consider the two basic liability rules of Anglo-American tort law: strict liability and negligence. Under negligence the injurer is liable to the victim if the injurer failed to exercise due care (a legal standard set by the court). Under strict liability the injurer is liable to the victim whether or not the injurer exercised due care.

A liability rule is *optimal* if the solution to the injurer's care-taking problem under that rule is *socially optimal* (i.e., coincides with the solution to the social problem). An optimal liability rule can provide *robust deterrence* if the injurer's problem under that rule is always easily solved (i.e., computationally tractable). After all, if the injurer's problem is computationally intractable, even an optimal liability rule cannot always induce the injurer to take socially optimal care.

### C.1 Convex Care

The standard "convex care" version of the model makes the following assumptions (cf. Shavell 1987).

(C1) The injurer's precaution set is a convex set $\mathcal{X} \subseteq \mathbb{R}^n_+$ where $n \geq 1$. The injurer chooses an array of care $\mathbf{x} = (x_1, \ldots, x_n) \in \mathcal{X}$.

(C2) The injurer's cost of care is $c(\mathbf{x}) \geq 0$, where $c : \mathcal{X} \to \mathbb{R}_+$ is monotone increasing, convex, and supermodular.

(C3) The victim's expected loss is $\ell(\mathbf{x}) \geq 0$, where $\ell : \mathcal{X} \to \mathbb{R}_+$ is monotone decreasing, convex, and supermodular.

(C4) The functions $c$ and $\ell$, and their respective subdifferentials, can be computed in polynomial time at all $\mathbf{x} \in \mathcal{X}$.

Assumption (C1) is the convex care assumption. Assuming $n = 1$ would constitute the unidimensional care assumption. Assumptions (C2)–(C3) comprise the diminishing returns assumption. Assumption (C4) is a minimal computability assumption. It is an implicit, unstated assumption in prior expositions of the UCFA model.

**The Social Problem** The social problem is to find $\mathbf{x} \in \mathcal{X}$ that minimizes $s(\mathbf{x}) \equiv c(\mathbf{x}) + \ell(\mathbf{x})$, where $s(\mathbf{x})$ is the social cost of the injurer's activity. I assume that the social problem has a unique interior solution $\mathbf{x}^*$. By definition $\mathbf{x}^*$ is socially optimal.

Given assumptions (C1)–(C4), the social problem is a convex optimization problem: it entails the minimization of a convex function on a convex choice set. It follows that the social problem is computationally tractable—whether care is unidimensional

($n = 1$) or multidimensional ($n > 1$)—for it can be efficiently solved using known polynomial-time algorithms for convex optimization problems. For instance, it can be efficiently solved using subgradient methods if the social cost function is non-differentiable (see, for example, Bertsekas 2016) or gradient or interior-point methods if the social cost function is differentiable (see, for example, Ben-Tal and Nemirovski 2001; Boyd and Vandenberghe 2004). In the non-differentiable case $\mathbf{x}^*$ satisfies the condition $0 \in \partial s(\mathbf{x}^*)$, where $\partial s(\mathbf{x})$ is the subdifferential of $s$ at $\mathbf{x}$. In the differentiable case $\mathbf{x}^*$ satisfies the condition $0 = \nabla s(\mathbf{x}^*)$, where $\nabla s(\mathbf{x})$ is the gradient of $s$ at $\mathbf{x}$.

**The Injurer's Problem** Under strict liability the injurer's problem is identical to the social problem. This is because strict liability forces the injurer to internalize the social cost of his activity. Under strict liability, therefore, the solution to injurer's problem is socially optimal and the injurer's problem is computationally tractable. Hence, strict liability is optimal and can provide robust deterrence.

Let $\overline{\mathbf{x}} \in \mathcal{X}$ denote the due care standard under negligence. Under negligence the injurer faces no liability if he chooses $\mathbf{x} \geq \overline{\mathbf{x}}$, and he effectively faces strict liability if he chooses $\mathbf{x} \ngeq \overline{\mathbf{x}}$. Accordingly, the injurer's problem under negligence has two steps. First, the injurer finds the solution to the social problem, $\mathbf{x}^*$, which is computationally tractable. Second, the injurer chooses $\mathbf{x} = \mathbf{x}^*$ if $\mathbf{x}^* \ngeq \overline{\mathbf{x}}$ and $s(\mathbf{x}^*) < c(\overline{\mathbf{x}})$, and he chooses $\mathbf{x} = \overline{\mathbf{x}}$ otherwise. This step is also computationally tractable because $s(\mathbf{x}^*)$ and $c(\overline{\mathbf{x}})$ are easily computed per assumption (C4). Moreover, if the court sets the due care standard equal to the social optimum, $\overline{\mathbf{x}} = \mathbf{x}^*$, then the injurer always chooses $\mathbf{x} = \mathbf{x}^*$. Under negligence, therefore, the injurer's problem is computationally tractable, and if $\overline{\mathbf{x}} = \mathbf{x}^*$ the solution is always socially optimal. Thus, negligence (with $\overline{\mathbf{x}} = \mathbf{x}^*$) is optimal and can provide robust deterrence.

The following proposition recaps the foregoing results.

**Proposition 3** *Suppose that care is convex. The injurer's problem, whether under strict liability or negligence, is computationally tractable whether care is unidimensional or multidimensional. Thus, because strict liability and negligence (with $\overline{x} = x^*$) are both optimal, either rule can provide robust deterrence when care is convex.*

### C.2 Discrete Care

The "discrete care" version of the model makes the following assumptions.

(D1) The injurer's precaution set is a discrete set $\mathcal{X} \subseteq \mathbb{N}_+^n$ where $n \geq 1$. The injurer chooses an array of care $\mathbf{x} = (x_1, \ldots, x_n) \in \mathcal{X}$.

(D2) The injurer's cost of care is $c(\mathbf{x}) \geq 0$, where $c : \mathcal{X} \to \mathbb{R}_+$ is monotone increasing and supermodular.

(D3) The victim's expected loss is $\ell(\mathbf{x}) \geq 0$, where $\ell : \mathcal{X} \to \mathbb{R}_+$ is monotone decreasing and supermodular.

(D4) The functions $c$ and $\ell$ can be computed in polynomial time at all $\mathbf{x} \in \mathcal{X}$.

Assumption (D1)–(D4) are the discrete analogs of assumptions (C1)–(C4). In particular, assumptions (D2)–(D3) comprise the diminishing returns assumption. Although we cannot assume that $c$ and $\ell$ are convex (because $\mathcal{X}$ is discrete), assuming they are supermodular entails that the marginal cost of care is increasing and the marginal benefit of care (i.e., the marginal reduction in expected loss) is decreasing.

As before, the injurer's problem under strict liability coincides with the social problem—find $\mathbf{x} \in \mathcal{X}$ to minimize $s(\mathbf{x})$—and his problem under negligence includes the social problem as a first step. And like before, the solution to the injurer's problem under strict liability and negligence (with $\overline{\mathbf{x}} = \mathbf{x}^*$) coincides with the solution to the social problem, implying that both rules are optimal. Unlike before, however, the injurer's problem is not computationally tractable in all cases. In particular, it is computationally intractable when care is multidimensional. I reach this conclusion on the basis of the following two theorems.

**Theorem 1** *When care is multidimensional and discrete, it is $\mathcal{NP}$-hard to approximate the injurer's problem, whether under strict liability or negligence, within any constant factor.*

**Proof** The proof is by reduction from the max cut problem. Let $f : \{0,1\}^n \to \mathbb{R}_+$, $n > 1$, denote the cut capacity function (i.e., the objective function) in the max cut problem. It is well-known that $f$ is submodular, non-negative, and not necessarily monotone (see, for example, Feige, Mirronki, and Vondrák 2011). In addition, let $\overline{w} = \sum_{i,j=1}^{n} \frac{1}{2} w_{ij}$ denote the total capacity of all edges in the graph.

For every cut $\mathbf{x} \in \{0,1\}^n$, let $g(\mathbf{x}) = \overline{w} - f(\mathbf{x}) + \eta$ where $\eta > 0$. Note that $g : \{0,1\}^n \to \mathbb{R}_+$ is a supermodular function.[34] Note further that $g$ is non-negative and not necessarily monotone.

It is $\mathcal{NP}$-hard to approximate the max cut problem within a factor of $\frac{17}{16}$ (Håstad 2001).[35] This implies that it is $\mathcal{NP}$-hard to distinguish between the following two mutually exclusive instances of the max cut problem (formulated as a decision problem):

1. There exists a cut $\mathbf{x} \in \{0,1\}^n$ such that $f(\mathbf{x}) = \overline{w}$.
2. There does not exist a cut $\mathbf{x} \in \{0,1\}^n$ such that $f(\mathbf{x}) > \frac{16}{17}\overline{w}$.

Note that in the first instance the minimum value of $g$ is $\eta$, while in the second instance the minimum value of $g$ exceeds $\frac{1}{17}\overline{w}$.

Suppose there exists an $\rho$-approximation algorithm for the injurer's problem—the minimization of a supermodular, non-negative, and not necessarily monotone function from a subset of $\mathbb{N}_+^n$ to $\mathbb{R}_+$. Then we could apply this algorithm to the problem $\max_{\mathbf{x}\in\{0,1\}^n} g(\mathbf{x})$. In the first instance the algorithm would return a cut $\widetilde{\mathbf{x}}$ such that $g(\widetilde{\mathbf{x}}) \leq \rho\eta$. In the second instance it would return a cut $\widetilde{\mathbf{x}}$ such that $g(\widetilde{\mathbf{x}}) > \frac{1}{17}\overline{w}$. Because $\eta$ is arbitrary, it can be chosen so that $\rho\eta < \frac{1}{17}\overline{w}$. This would make it possible to distinguish between the two instances, because in the first instance the algorithm would yield $g(\widetilde{\mathbf{x}}) \leq \rho\eta < \frac{1}{17}\overline{w}$, while in the second instance it would yield $\rho\eta < \frac{1}{17}\overline{w} < g(\widetilde{\mathbf{x}})$. This, however, contradicts the fact that it is $\mathcal{NP}$-hard to distinguish between the two instances. It follows, therefore, that there does not exist an $\rho$-approximation algorithm for the injurer's problem, which is equivalent to the statement that it is $\mathcal{NP}$-hard to approximate the injurer's problem within any constant factor. ■

**Remark** Mittal and Schulz (2013) prove a similar result for the minimization of an integer-valued supermodular function. Their proof is by reduction from the E4-set splitting problem.[36] The proof of Theorem 1, which is by reduction from the max cut problem, generalizes their result to a real-valued supermodular function.

[34]The sum of a supermodular function and a constant is supermodular. Note that $-f(\mathbf{x})$ is supermodular (because $f(\mathbf{x})$ is submodular) and that $\overline{w} + \eta$ is a constant.

[35]The result in Håstad (2001) is stated for the unweighted version of the max cut problem (i.e., the case where all $w_{ij} = 1$). However, Crescenzi, Silvestri, and Trevisan (2001) prove that the weighted and unweighted versions of the max cut problem have exacty the same approximation threshold.

[36]For a statement of the E4-set splitting problem, see, for example, Håstad (2001).

**Theorem 2** *When care is multidimensional and discrete, the injurer's problem, whether under strict liability or negligence, is $\mathcal{NP}$-hard.*

**Proof** The result follows immediately from Theorem 1. If it is $\mathcal{NP}$-hard to approximate the injurer's problem within any constant factor, then it is $\mathcal{NP}$-hard to approximate the injurer's problem within a factor of $1 + \epsilon$ for any $\epsilon > 0$, which is equivalent to the statement that the injurer's problem is $\mathcal{NP}$-hard. ■

**Remark** The foregoing results on the hardness of supermodular minimization stand in contrast to the fact that submodular minimization is easy (Grötschel, Lovász, and Schrijver 1981). The basic reason is that the convex closure of a submodular function has a closed form that is easy to compute, while this generally is not the case for supermodular functions (Bach 2013). Consequently, one can easily solve a submodular minimization problem by leveraging the fact that the minimum of a submodular function is equivalent to the minimum of its convex closure, while this generally is not the case for a supermodular minimization problem.

Theorems 1 and 2 establish that the injurer's problem, whether under strict liability or negligence, is not computationally tractable (or efficiently approximable) when care is multidimensional and discrete. Hence, neither rule can provide robust deterrence in this case. The reason is that brute-force is the only known solution algorithm for the social problem—the minimization of a supermodular function on a discrete choice set. When care is multidimensional (i.e., when there are multiple types of precaution) the size of the social problem is governed by the dimension of the precaution set (i.e., the number of different precautions), and the time complexity of brute-force increases exponentially with the dimension of the precaution set.

It is a different story, however, when care is unidimensional and discrete. In this case the injurer's problem is computationally tractable, and thus either strict liability or negligence (with the due care standard set equal to the social optimum) can provide robust deterrence. The difference is that when care is unidimensional (i.e., when there is only one type of precaution) the size of the social problem is governed by the cardinality of the precaution set (i.e., the number of feasible levels of care within the single type of precaution), and the time complexity of the brute-force method increases linearly with the cardinality of the precaution set.

The following proposition recaps the foregoing results.

**Proposition 4** *Suppose that care is discrete. Although it is computationally tractable when care is unidimensional, the injurer's problem, whether under strict liability or negligence, is computationally intractable ($\mathcal{NP}$-hard) when care is multidimensional. Thus, although strict liability and negligence (with $\overline{x} = x^*$) are both optimal, neither rule can provide robust deterrence when care is discrete.*

## D Proof of Claim 2

The decision problem that corresponds to the injurer's problem under strict liability is: Given $u \in \mathbb{R}_+$, is there an $\mathbf{x} \in \mathcal{X}$ such that $s(\mathbf{x}) \leq u$? The decision problem that corresponds to the injurer's problem under negligence is: Given $\overline{\mathbf{x}} \in \mathcal{X}$ and $c(\overline{\mathbf{x}}) \in \mathbb{R}_+$, is there an $\mathbf{x} \in \mathcal{X}$ such that $\mathbf{x} \ngeq \overline{\mathbf{x}}$ and $s(\mathbf{x}) < c(\overline{\mathbf{x}})$? Take either decision problem and suppose we are given a proposed solution $\mathbf{y} \in \mathcal{X}$. Per assumption (D4), it can be efficiently verified whether (i) $s(\mathbf{y}) \leq u$ or (ii) $\mathbf{y} \ngeq \overline{\mathbf{x}}$ and $s(\mathbf{y}) < c(\overline{\mathbf{x}})$, as the case may be. Hence, each decision problem is in $\mathcal{NP}$. ■

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