

# Computational Methods for Geodynamics

---

Alik Ismail-Zadeh

Karlsruhe Institute of Technology (KIT)  
Moscow Institute of Mathematical Geophysics,  
Russian Academy of Sciences (MITPAN)  
Institute de Physique du Globe de Paris (IPGP)

Paul J. Tackley

Swiss Federal Institute of Technology Zurich (ETH)



**CAMBRIDGE**  
UNIVERSITY PRESS

# Contents

<i>Foreword by Gerald Schubert</i>	<i>page xi</i>
<i>Preface</i>	<i>xiii</i>
<i>Acknowledgements</i>	<i>xvii</i>
<b>1 Basic concepts of computational geodynamics</b>	<b>1</b>
1.1 Introduction to scientific computing and computational geodynamics	1
1.2 Mathematical models of geodynamic problems	2
1.3 Governing equations	3
1.4 Boundary and initial conditions	13
1.5 Analytical and numerical solutions	14
1.6 Rationale of numerical modelling	15
1.7 Numerical methods: possibilities and limitations	16
1.8 Components of numerical modelling	17
1.9 Properties of numerical methods	20
1.10 Concluding remarks	22
<b>2 Finite difference method</b>	<b>24</b>
2.1 Introduction: basic concepts	24
2.2 Convergence, accuracy and stability	29
2.3 Finite difference sweep method	30
2.4 Principle of the maximum	31
2.5 Application of a finite difference method to a two-dimensional heat equation	32
<b>3 Finite volume method</b>	<b>43</b>
3.1 Introduction	43
3.2 Grids and control volumes: structured and unstructured grids	43
3.3 Comparison to finite difference and finite element methods	44
3.4 Treatment of advection–diffusion problems	45
3.5 Treatment of momentum–continuity equations	49
3.6 Modelling convection and model extensions	60
<b>4 Finite element method</b>	<b>63</b>
4.1 Introduction	63
4.2 Lagrangian versus Eulerian description of motion	64

---

4.3	Mathematical preliminaries	65
4.4	Weighted residual methods: variational problem	66
4.5	Simple FE problem	69
4.6	The Petrov–Galerkin method for advection-dominated problems	71
4.7	Penalty-function formulation of Stokes flow	75
4.8	FE discretisation	75
4.9	High-order interpolation functions: cubic splines	76
4.10	Two- and three-dimensional FE problems	79
4.11	FE solution refinements	91
4.12	Concluding remarks	92
<b>5</b>	<b>Spectral methods</b>	<b>93</b>
5.1	Introduction	93
5.2	Basis functions and transforms	93
5.3	Solution methods	98
5.4	Modelling mantle convection	100
<b>6</b>	<b>Numerical methods for solving linear algebraic equations</b>	<b>109</b>
6.1	Introduction	109
6.2	Direct methods	109
6.3	Iterative methods	114
6.4	Multigrid methods	119
6.5	Iterative methods for the Stokes equations	126
6.6	Alternating direction implicit method	128
6.7	Coupled equations solving	130
6.8	Non-linear equation solving	131
6.9	Convergence and iteration errors	132
<b>7</b>	<b>Numerical methods for solving ordinary and partial differential equations</b>	<b>134</b>
7.1	Introduction	134
7.2	Euler method	134
7.3	Runge–Kutta methods	135
7.4	Multi-step methods	137
7.5	Crank–Nicolson method	139
7.6	Predictor–corrector methods	140
7.7	Method of characteristics	141
7.8	Semi-Lagrangian method	142
7.9	Total variation diminishing methods	144
7.10	Lagrangian methods	146
<b>8</b>	<b>Data assimilation methods</b>	<b>148</b>
8.1	Introduction	148
8.2	Data assimilation	151

---

8.3	Backward advection (BAD) method	152
8.4	Application of the BAD method: restoration of the evolution of salt diapirs	153
8.5	Variational (VAR) method	156
8.6	Application of the VAR method: restoration of mantle plume evolution	162
8.7	Challenges in VAR data assimilation	168
8.8	Quasi-reversibility (QRV) method	171
8.9	Application of the QRV method: restoration of mantle plume evolution	177
8.10	Application of the QRV method: restoration of descending lithosphere evolution	180
8.11	Comparison of data assimilation methods	192
8.12	Errors in forward and backward modelling	195
<b>9</b>	<b>Parallel computing</b>	<b>197</b>
9.1	Introduction	197
9.2	Parallel versus sequential processing	197
9.3	Terminology of parallel processing	199
9.4	Shared and distributed memory	201
9.5	Domain decomposition	203
9.6	Message passing	207
9.7	Basics of the Message Passing Interface	209
9.8	Cost of parallel processing	213
9.9	Concluding remarks	215
<b>10</b>	<b>Modelling of geodynamic problems</b>	<b>216</b>
10.1	Introduction and overview	216
10.2	Numerical methods used	217
10.3	Compressible flow	223
10.4	Phase transitions	227
10.5	Compositional variations	231
10.6	Complex rheologies	235
10.7	Continents and lithospheric plates in mantle convection models	238
10.8	Treatment of a free surface and surface processes	244
10.9	Porous flow through a deformable matrix	245
10.10	Geodynamo modelling	247
<b>Appendix A</b>	<b>Definitions and relations from vector and matrix algebra</b>	<b>250</b>
<b>Appendix B</b>	<b>Spherical coordinates</b>	<b>258</b>
<b>Appendix C</b>	<b>Freely available geodynamic modelling codes</b>	<b>264</b>

<i>References</i>	267
<i>Author index</i>	301
<i>Subject index</i>	307

*Colour plate section between pages 238 and 239.*