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COMPUTATIONAL METHODS IN NUCLEAR REACTOR STRUCTURAL DESIGN FOR HIGH-TEMPERATURE APPLICATIONS: AN INTERPRETIVE REPORT

R. H. Gallagher

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R. H. Gallagher
Cornell University

FEBRUARY 1973

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Preface

This document is one of a series of interpretive, or state-of-the-art, reports prepared as a part of the Oak Ridge National Laboratory program entitled High-Temperature Structural Design Methods for LMFBR Components. The goal of this program is to develop a verified design technology applicable to the high-temperature, long-term operating conditions expected for LMFBR vessels, components, and core structures.

In addition to contributing to the establishment of the overall state of the art of elevated-temperature design technology, these reports are intended to assist in identifying and interpreting near-term needs in their respective areas. They should also contribute to the identification and recommendation of potential paths of approach to some of the longer-range needs associated with the generation of a verified technology.

Professor R. H. Gallagher, Chairman of the Structural Engineering Department at Cornell University, was chosen to prepare this report because of his experience in the development of almost every aspect of the finite-element methods currently being employed by a majority of nuclear design organizations. His industrial background includes several years in the aerospace field, where many of the analysis methods available to the LMFBR structural designer were first developed and employed on similar problems.

J. M. Corum
Oak Ridge National Laboratory

Abstract

The status of the finite-element method as a basis for design analyses of nuclear reactor structures for elevated-temperature service is reviewed and assessed. First, a general picture is given of procedures in analyses of the complete structural system from the view of the finite-element method, and then reviews are presented of two classes of element important to nuclear reactor structures – solid elements and thin-shell elements. Questions of inelastic analysis are discussed in chapters dealing with time-independent plasticity and time-dependent viscoelasticity and creep respectively. Since the accuracy of design analysis for elevated-temperature conditions is dependent upon the analysis of heat transfer, the utilization of the finite-element method in this connection is also surveyed. Finally, an examination is made of the role of the finite-element method in the relevant, but more peripheral, topics of dynamic, finite displacement (stability), and fracture mechanics analyses. The report concludes with an assessment of the scope and limitations of finite-element analysis for the intended applications, based on the material covered in prior chapters.

1. Introduction

The issuance of interim design criteria for the FFTF/LMFBR along with recent and contemplated changes in segments of the ASME Boiler and Pressure Vessel Code have intensified interest of all members of the nuclear structural design community in computational methods in structural mechanics. These new criteria and changes seek a new level of realism in the prediction of structural behavior in design analysis that can only be reached through application of the most modern digital-computer-based methods of structural analysis. The modernity of these methods, however, immediately raises questions concerning both their validity and their feasibility for the solution of the specified engineering design problems.

Although many diverse analytical procedures may be classed as digital-computer-based (or numerical), this report focuses on the finite-element method. Justification for this restriction derives mainly from the adoption of the method by the majority of affected design organizations in response to the aforementioned code requirements. It might be contended that among the candidate numerical methods, only the finite-element method can cope with the complete range of structural geometries and behavior mechanisms, linear and nonlinear, in a single computer program. In a less controversial vein, it may alternatively be argued that since other methods (e.g., finite differences, harmonic series), being of early vintage, are already well tested within their range of applicability, they need no further evaluation.

The latter statement implies the question: In view of the lateness of development of the analytical capabilities represented by the finite-element method, are codification agencies realistic in demanding the performance of analysis to a degree of refinement not possible by use of elementary methods? In other words – as stated earlier – is the method valid in performance of the stipulated design analyses? Furthermore, if validity of theory can be proved, one must give serious consideration to the economics of the analysis.

The scope of the subject problem, and consequently of the related literature, is so great as to preclude any detailed developments from first principles.[1-13] The purpose of this report is to define the background and state-of-the-art assessments of the pertinent component aspects of finite-element analysis. To certain of these are appended correlations of the extensive alternative paths confronting the user and, insofar as is possible, the identification of recommended paths.

Turning now to a statement of the component aspect of the problem (and report), it is well to emphasize at the outset that the key design code requirements mentioned earlier are essentially those demanding the performance of *inelastic* analyses, both for time-independent plasticity and time-dependent viscoelasticity and creep. However, the technical areas encompassed by such requirements extend beyond inelasticity. The high degree of nonlinearity of these phenomena and the need for numerical integration in time in the case of creep make clear the importance of efficiency in the analysis of the complete *structural system*. The accuracy of the method in solution of the special (inelastic) problem can be no greater than the accuracy in linear analysis, which is governed in part by the *finite-element relationships* themselves. Also, one must consider *modes of failure*, whose prediction can be accomplished with use of the finite-element method, and *thermal analysis*.

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The above italicized component aspects of the problem are discussed in this report in the following order. Chapter 2, System Analysis Procedures, gives a view of the system analysis procedures in current application, describes the theoretical bases of existing alternatives which hold major potential for improvements in this area, and delineates accomplishments in the subsidiary problem of efficient equation-solving algorithms.

Element formulation considerations are discussed in Chapters 3 and 4. Two classes of elements are identified as being of primary importance in metal nuclear reactor structural design: solid elements (Chap. 3) and thin-shell elements (Chap. 4). In each case an extremely large number of alternative formulations confront the practitioner. Our objective here is to identify and classify these alternatives and, since no single element formulation is uniquely acceptable, to identify the characteristics of preferable formulations by review of published numerical results.

The central objective of the report is reached in Chapter 5, Inelastic Analysis – Time Independent. The theoretical bases of the two common approaches to finite-element inelastic analysis, the tangent stiffness and initial strain methods, are given in greater detail than are fundamentals of other topics in this report. The pitfalls in application of these alternatives are discussed, and an attempt is made to relate one to the other.

Creep analysis constitutes the subject matter of Chapter 6. Finite-element solutions to this problem have to date depended upon a single approach to inelastic analysis (the initial strain method), and contributions to the literature have been few. Nevertheless, numerical results of applications to problems with experimental data have yielded reasonable theory-test correspondence.

Problems of a dynamic nature are significant in the design of nuclear reactor structures. The most prominent of these arise due to response to earthquake excitations, although other environments and forms of excitation are occasionally of concern. Chapter 7 outlines various facets of finite-element analysis for dynamic response.

Accuracy of design analysis for thermal stress situations is clearly dependent upon the accuracy of the subsidiary conduction heat transfer analysis. The complexities that necessitate finite-element analysis for structural response are likewise present in analyses for thermal response. It would therefore be desirable to invoke finite-element capabilities for the latter as well. An additional advantage arises because the thermal and structural models may be defined identically and no costs are accrued in the transfer of data from one to the other. Finite-element thermal analysis has grown rapidly to the point where it is relatively complete and widely employed; this progress is reviewed in Chapter 8.

Chapter 9 treats two disparate modes of failure, elastic instability and fracture. As will be shown, finite-element analysis may play an important role in the prediction of each.

The final chapter (Chap. 10) draws upon the detailed perspectives of the prior sections to form an overall picture of the report subject matter. The major advantages and disabilities of current practice and capabilities are reasserted, and specific directions of future work are suggested.

Before proceeding into the aforementioned topics, note should be taken of recently published works that provide an invaluable supplement and elaboration of the contents of this document. The book edited by Gill[14] gives a detailed summary of the theoretical bases of modern pressure vessel analysis. The Proceedings of the 1971 Berlin Conference on Structural Mechanics in Reactor Technology[15] contain over 300 papers on various aspects of structural design of nuclear reactors, many of which deal with numerical methods. Reference 16 is the proceedings of a conference on large-scale general-purpose programs in structural analysis with particular reference to nuclear reactor structures.

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2. System Analysis Procedures

In recent years there has been a tendency to neglect consideration of the procedures for finite-element analysis of the total structure, it being assumed that this is a solved problem, and concentrate instead on the formulation of new element relationships or the extension of the method to cope with nonlinear problems. This has been the direct result of the fact that one approach to system analysis – the direct stiffness method of matrix displacement analysis – is almost universally adopted in practical applications. However, a continuing consideration of alternative system analysis procedures is of extreme importance, and this chapter is devoted to an examination of aspects of significance to these procedures.

The importance of continuing consideration of system analysis procedures stems from many factors. Of principal concern to the general area under examination is computational efficiency in the face of repetitive linearized analyses of nonlinear problems. Even in direct stiffness analysis there is not a single, generally accepted scheme, but rather many forms that differ in various ways from one organization to another. Clearly, the relative efficiency of these versions likewise differs. It must also be recognized that new theoretical developments have identified alternatives to direct stiffness analysis that are substantially more efficient for certain applications. In addition, these alternatives may be more suited to the conduct of analyses for certain phenomena, for example, viscoelastic deformation.

Motivated by the above issues, this chapter reviews developments in the three major alternative methods of system analysis: the stiffness, flexibility, and mixed methods. This review also constitutes an essential background for work to be cited in later chapters. It has been found that the alternative methods give rise to new operational requirements, specifically in substructuring and the handling of constraints; this is discussed in Section 2.2.

The past five years have seen a remarkable intensification of effort in procedures for the solution of the algebraic equations of finite-element analysis and in the assessment of errors in the overall solution process. Both are motivated by the coming of age of general-purpose programs and the emergence of the third generation of computers. Section 2.3 is therefore devoted to equation-solving methods, with a division into direct and iterative methods, and Section 2.4 discusses sources of error and methods for their identification.

2.1 BASIC METHODS

2.1.1 Potential Energy (Stiffness) Formulations

The displacement method of finite-element analysis consists in the formulation of algebraic equations relating known applied loads to the unknown parameters which define the displaced state of the structure. Once these equations are solved, the evaluated displacements are substituted into separate equations to determine internal forces, stresses, and support reactions. Although the method may be established by direct reasoning, it is common to invoke energy concepts in the form of the minimum potential energy principle. Thus the method is alternatively termed the stiffness, equilibrium, or potential energy method.

The most popular form of the displacement method is the "direct stiffness" scheme.[1,2] In this approach, a set of stiffness equations is calculated for each element in

the analytical model,

$$[k]\ \{\Delta^e\} = \{F\} + \{F^i\}\ , \tag{1}$$

where [k] is the element stiffness matrix, $\{\Delta^e\}$ are the element displacements, and $\{F\}$ and $\{F^i\}$ are joint internal and initial forces respectively. The initial forces, which are known quantities, result from initial strains and are essential to the analyses of phenomena such as thermal stress, creep, and time-independent plastic deformation. The stiffness equations for the complete structure are established simply by adding all terms with like subscripts; for example, for the system coefficient K_{ij},

$$K_{ij} = \sum_{e=1}^{e=r} k_{ij}^e \tag{2}$$

where r is the total number of elements with stiffness coefficients possessing the y subscripts. Thus the term "direct stiffness method" derives from the formation of the system stiffness equations by direct addition of the element stiffness coefficients. The system equations may now be written in the form

$$[K]\ \{\Delta\} = \{P\} + \{P^i\} \tag{3}$$

where now $\{\Delta\}$ lists all displacements, or degrees of freedom, in the system, $\{P\}$ lists the applied loads, and $\{P^i\}$ is the vector of system initial forces. The solution is

$$\{\Delta\} = [K]^{-1}\ \{\{P\} + \{P^i\}\}\ . \tag{4}$$

The designation above of the inverse is symbolic, any method of solution being acceptable. The solution for stresses and internal forces is obtained by first constructing a system stress matrix [S] defined by

$$\{\sigma\} = [S]\ \{\Delta\}\ . \tag{5}$$

where $\{\sigma\}$ lists the internal stresses and forces. The desired results are obtained after insertion of Eq. (4) into (5).

Other methods for establishing the stiffness equations in a displacement formulation include the "congruent transformation" and "energy search" methods. Both depend upon work and energy considerations, and their description gives the opportunity here to define the relationship between the stiffness-displacement method and potential energy concepts. According to the potential energy principle,[3] of all "admissible" displacement states of a deformed structure, the one that causes the potential energy (Π_p) to assume a stationary value is the correct equilibrium state. For stable equilibrium, the stationary value is a minimum, in which

$$\Pi_p = U - V\ . \tag{6}$$

and

$$U = \text{strain energy of deformation} = \frac{1}{2}\int_V \lfloor\epsilon\rfloor\ [E]\ \{\epsilon\}\, dV\ .$$

$$V = \text{potential of applied loads} = \int_{S_q} \bar{q}\cdot\Delta\, dS\ .$$

$\lfloor\epsilon\rfloor$ = row vector of strains $\epsilon_x, \ldots, \gamma_{xz}$.

[E] = matrix of elastic constants .

S_q = surface upon which the loads are prescribed .

$\bar{q}$ = prescribed loads on surface S_q .

Δ = displacement corresponding to $\bar{q}$.

An admissible displacement satisfies the boundary conditions on displacements and is differentiable to a degree governed by the specific form of the strain-energy expression. For finite-element representations it is necessary to satisfy the admissibility condition within the element and on the boundary of elements abutting actual physical boundaries. Along boundary lines with adjacent elements, it is necessary that the element displacement fields meet similar conditions for a rigorous potential energy formulation, for example, continuity of the displacements and in certain cases their normal derivatives, etc. This is termed the *interelement continuity* condition, and displacements that meet it are termed *conforming* displacement fields.

Much attention is given to establishing conforming displacement fields in the development of element formulations, but it must be emphasized that a system potential energy formulation can be constructed using nonconforming element formulations. Indeed, a variational statement of finite-element structural analysis which relaxes the interelement continuity requirement can be formulated[4] and various levels of satisfaction of the requirement can be constructed within this approach. The solutions obtained with nonconforming displacement fields may prove to be of satisfactory numerical accuracy but will not constitute a rigorous application of the stationary potential energy principle.

In the finite-element potential energy procedure, the assumed displacement fields are differentiated in accordance with the pertinent strain-displacement equations, and, after appropriate substitutions, integrations,

and other operations, the following discretized functional is established:

$$\Pi_p = \frac{\lfloor \Delta \rfloor}{2} [a]^T \lceil k^e \rfloor [a] \{\Delta\} + \lfloor \Delta \rfloor [a]^T \{F^i\} - \lfloor \Delta \rfloor \{P\} , \quad (7)$$

where

$$\lceil k^e \rfloor$$

represents the "unassembled" master stiffness matrix and [a] is the system "kinematical" or "compatibility" matrix. The latter serves to tie together the elements to form the complete analytical model. By imposing the condition of stationary potential energy, one obtains

$$[a]^T \lceil k^e \rfloor [a] \{\Delta\} = \{P\} - [a]^T \{F^i\} , \quad (8)$$

so that

$$[K] = [a]^T \lceil k^e \rfloor [a] , \quad (9)$$

and

$$\{P^i\} = [a]^T \{F^i\} . \quad (10)$$

Now $\{F^i\}$ represents a list of all element initial force vectors.

The procedure represented by Eq. (8) is termed here the conjugate transformation potential energy method.[5] Although this method appears to be less efficient than direct stiffness analysis, the effort in constructing the unassembled stiffness matrix is minimal and there are advantages resulting from the exclusion of rigid-body degrees of freedom from the element stiffness matrices. The terms of [a] are unit values, direction cosines, or structural dimensions. The algorithm for effecting the matrix product need not be the conventional algorithm for this purpose, but may exploit the advantages of the structural system format, as indicated below.

It is useful, at this point, to illustrate the formulative procedure for element relationships established by means of the minimum potential energy principle, since this procedure is referred to directly or by implication in the chapters to follow. The cornerstone of the formulative process is the choice of a functional representation for the element displacement field,

$$\Delta = \sum_{i=1}^{n} N_i \Delta_i = \lfloor N \rfloor \{\Delta\} , \quad (11)$$

where Δ symbolizes a displacement (e.g., u, v, or w for direct stress problems in cartesian coordinates), Δ_i is the ith displacement parameter, the specific value of a displacement component at a joint of the element or alternatively a "generalized" value, and N_i is a function of the spatial coordinates which gives the spatial variation of Δ resulting from a unit value of Δ_i with all other specific displacement components held fixed. The vector of such functions, $\lfloor N \rfloor$, is termed the "shape function" vector.

The strains are related to the displacements through the appropriate strain-displacement differential equations (e.g., $\epsilon_x = \partial u/\partial x$), and by differentiation of Eq. (11) in accordance with these conditions, one obtains

$$\{\epsilon\} = [D] \{\Delta\} . \quad (12)$$

Note that the strain-displacement equations require differentiation with respect to spatial variables and that only the shape functions $\lfloor N \rfloor$ involve these variables. Thus [D] derives exclusively from operations on the terms of $\lfloor N \rfloor$.

The constitutive relationships of linear elasticity, including initial strains ($\{\epsilon^i\}$), are of the form

$$\{\sigma\} = [E] \{\epsilon\} - [E] \{\epsilon^i\} , \quad (13)$$

where all terms are as defined below Eq. (6). By substitution of Eqs. (12) and (13) into Eq. (6), we have for the potential energy of an element (with appropriate modification of the expression for strain energy of deformation to account for a distinction between the total strains $\{\epsilon\}$ and initial strains $\{\epsilon^i\}$):

$$\Pi_p^e = \frac{\lfloor \Delta \rfloor}{2} [k^e] \{\Delta\} - \lfloor \Delta \rfloor \{F^i\} - \lfloor \Delta \rfloor \{F\} , \quad (14)$$

where

$$[k^e] = \int_v [D]^T [E] [D] \, dV , \quad (15)$$

element stiffness matrix, and

$$\{F^i\} = \int_v [D]^T [E] \{\epsilon^i\} \, dV , \quad (16)$$

element initial force matrix. The vector {F} represents

the force parameters corresponding to the displacement parameters $\{\Delta\}$, and the volume V represents here the element volume.

The potential energy is a scalar value, so that the potential energy of the system is simply the sum of the element potential energies supplemented by the potential of the applied loads,

$$\Pi_p = \sum_{e=1}^{p} \Pi_p^e - \lfloor\Delta\rfloor \{P\}$$

$$= \frac{\lfloor\Delta^e\rfloor}{2} \sum_{e=1}^{p} [a^e]^T [k^e] [a^e] \{\Delta^e\}$$

$$+ \lfloor\Delta^e\rfloor \sum_{e=1}^{p} [a^e]^T \{F^i\} - \lfloor\Delta\rfloor \{P\} \ . \quad (17)$$

for a system composed of p elements, where now all quantities with the superscript e refer to individual elements. The matrices $[a^e]$ are the kinematical matrices defined on an element basis and serve to connect the respective elements to the remainder of the system. In this form, which is alternative to Eq. (8), it is feasible to apply one of the numerous available methods for the minimizztion of a function of many variables. A large share of such methods are in the class of "search" schemes, where a search for the minimum point is conducted through a "hyperspace" whose coordinate axes represent the respective variables. Thus, in application to finite-element potential energy problems, these methods are termed "energy search" procedures. Fox and Stanton,[6] for example, exploit the conjugate gradients search algorithm[7] in structural analysis applications.

A major advantage of the energy search approach is, apparently, the avoidance of formalized assembly of the system equations and a special utility in the treatment of nonlinear problems. On the other hand, experience in use of the approach is needed to insure numerical accuracy and convergence.

2.1.2 Complementary Energy (Flexibility) Formulations

The "classic" approach to finite-element complementary energy analysis is through the "redundant force method,"[8-10] which differs markedly in form from the direct stiffness method. However, the complementary energy counterpart of the "direct flexibility" method has recently gained clarification and is especially significant in inelastic analysis. Therefore we first describe the classic redundant force procedure, in order to identify its shortcomings, and then outline the procedure for direct flexibility analysis in a direct flexibility format.

We define, at the outset, the element flexibility matrix $[\delta^e]$ as the inverse of the element stiffness matrix with imposed statically determinate stable support conditions. The unassembled flexibility matrix for the system is denoted as $\lceil\kappa\, \delta\, \kappa\rfloor$ and is merely a diagonal matrix of submatrices, each of which is an element flexibility; all element flexibilities of the complete structure are included herein.

All member forces and support reactions are listed in the vector $\{S\}$, and it is assumed that these values can be obtained as the sum of two systems of forces, $\{S_x\}$ and $\{S_p\}$. The forces $\{S_x\}$ derive from redundant force systems, whose amplitudes are listed in the vector $\{x\}$. Hence,

$$\{S_x\} = [b_1] \{X\} , \quad (18)$$

where each column of $[b_1]$ represents a self-equilibrating system of forces. The redundants $\{X\}$ are the basic problem unknowns. The forces $\{S_p\}$ are in equilibrium with the applied loads $\{P\}$ but are computed without regard to conditions of compatibility. Thus

$$\{S_p\} = [b_0] \{P\} , \quad (19)$$

where each column of $[b_0]$ represents a system of forces in equilibrium under the application of the corresponding component of $\{P\}$. We therefore write for $\{S\}$,

$$\{S\} = \{S_p\} + \{S_x\} = [b_0] \{P\} + [b_1] \{X\} . \quad (20)$$

With this representation, and with introduction of the complementary energy integral and the imposition of the principle of least work, it can be shown that the solution for $\{X\}$ is given by

$$\{X\} = -[G]^{-1} [b_1]^T \lceil\kappa\, \delta\, \kappa\rfloor [b_0] \{P\} . \quad (21)$$

and the element forces and support reactions are

$$\{S\} = \left[[b_0] - [G]^{-1} [b_1]^T \lceil\kappa\, \delta\, \kappa\rfloor [b_0] \right] \{P\} , \quad (22)$$

where

$$[G] = [b_1]^T \lceil \kappa \delta \rfloor [b_1] \ . \qquad (23)$$

Clearly this approach requires more extensive matrix operations than direct stiffness analysis. Far more serious, however, is a need for computer-based procedures for the construction of the $[b_0]$ and $[b_1]$ matrices from the basic data of a problem. Procedures for the automation of this operation, termed "structure cutting," have been established[11,12] but are expensive in application and there is no agreement on a "best method." Another problem in classical redundant force method analysis is the scarcity of element formulations: nearly all types of structures must be described by means of axial force members and shear panels rather than continuum elements.

In forming a method of direct flexibility analysis, the most widely used scheme is to employ stress functions, rather than redundant force systems, as the problem unknowns.[13-15] Analogies exist in certain forms of elastic behavior (e.g., plate bending vs plate stretching) between the homogeneous differential equations of equilibrium in one form, where displacements are unknowns, and the compatibility differential equations of another form, where stress functions are unknowns. The Airy stress function for plate stretching and Southwell's stress functions for plate bending exemplify these situations. For other cases there is no analogy, but stress functions are nevertheless definable (e.g., Finzi's stress functions[16] for three-dimensional stress analysis).

One difficulty in the direct flexibility method of complementary energy analysis is the presence of constraint equations. Complementary energy, Π_c, is defined as

$$\Pi_c = \frac{1}{2}\int_V \lfloor \sigma \rfloor [E]^{-1} \{\sigma\} \, dV - \int_{S_\Delta} q \cdot \bar{\Delta} \, dS \ , \qquad (24)$$

where now the surface integral is taken over the portion of the surface on which the displacements $\bar{\Delta}$ are prescribed. Thus no account is taken of the applied loads, and to do so one must append constraint conditions. Methods of accounting for constraints are explored later in this chapter.

The introduction of stress functions is not essential to the proper formulation of a direct flexibility complementary energy analysis. Stresses have recently been used for this purpose by Hodge and Belytschko[17] for plane stress and Anderheggen[18] for bending.

2.1.3 Mixed Methods

Mixed methods of system analysis are as old as the topic of matrix structural analysis itself, and although they have found only limited use in the intervening years, they are presently the subject of considerable research interest. The reasons for this growth of interest include the expansion of mixed variational principles, a special suitability of mixed methods for shell analysis problems, and the desirability in certain situations of an output that gives directly both the stresses and displacements.

The Reissner variational principle[19] is perhaps the most widely employed basis for mixed formulations. Nevertheless, alternative forms of this variational principle are readily constructed[20,21] and other, more general, "two field" principles[22,23] have been delineated.

The earliest popular approaches to mixed analysis were the transfer matrix method[24] and Klein's[25] simple matrix method of structural analysis. Both are based upon direct reasoning pertinent to equilibrium and displacement continuity conditions, neither employs variational or energy concepts. The transfer matrix method was devised principally for framed structures and for attempts to exploit their topology so as to permit analysis with operations on small-order matrices. Klein's approach is devised principally with continuum structures in mind and in final format is readily identified with the procedures based on variational principles.

Reissner's variational principle[20] in finite-element analysis gained attention in 1965. The special advantages of the method come to the fore in plate bending, where the more common displacement-based potential energy method faces considerable difficulty in the establishment of interelement-compatible representations and it is essential to use complicated displacement fields. Simple linear fields are employed in representation of both stress and displacement when a mixed variational principle is adopted. More recently, in shell analysis,[21] advantage has been taken of a mixed representation.

It should be noted that the basic form of the element and system equations in the mixed approach is as

follows:

$$\left[\begin{array}{c:c} R & L^T \\ \hdashline L & O \end{array}\right] \left\{\begin{array}{c} \sigma \\ \Delta \end{array}\right\} = \left\{\begin{array}{c} a \\ b \end{array}\right\} , \qquad (25)$$

where $\{\sigma\}$ and $\{\Delta\}$ are the pertinent stress and displacement parameters. Due to the presence of zeros in the lower right portion of this partition, care must be exercised in the selection of an equation-solving algorithm.

2.2 SPECIAL SYSTEM OPERATIONS

Certain aspects of finite-element analysis may require that operations of a special type be performed on the complete system of equations representing the finite-element model of the total structure. Two of the more important such operations concern the treatment of constraint equations and substructuring. Constraint equations must be taken into account in complementary energy analysis, as noted above. Constraint equations appear in numerous other situations of importance to finite-element analysis of nuclear reactor structures. As shown in Fig. 1, it is sometimes necessary to form an analytical model composed of shell and solid elements. In order to form the juncture of these two types of elements, the joints of the solid elements must be constrained to obey the condition of linear variation of displacement across the juncture. Constraint equations also arise when dealing with incompressible materials.[26]

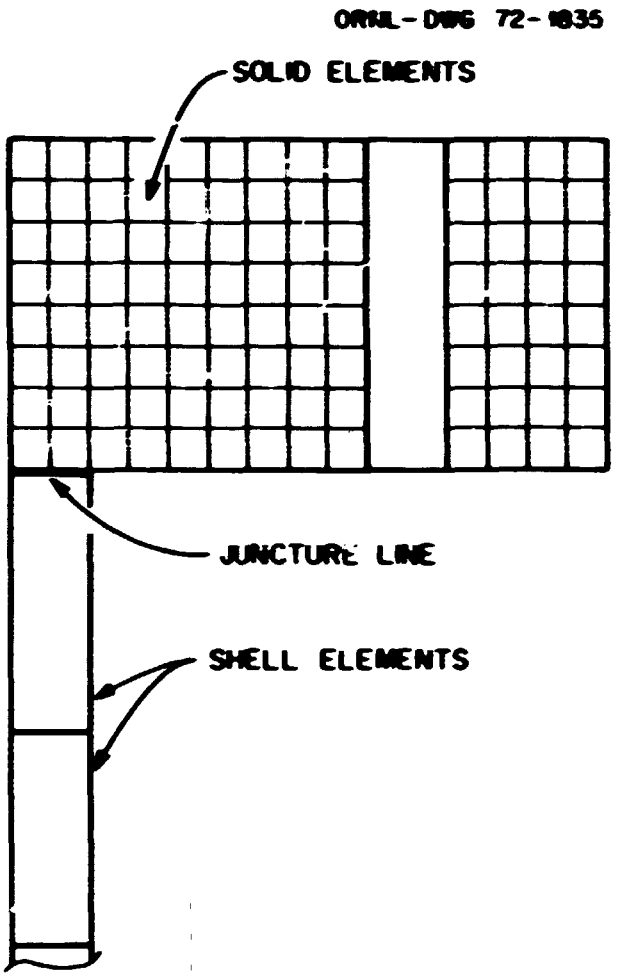

Fig. 1. Analytical model composed of shell and solid elements.

Methods for handling constraints include:

1. elimination of certain degrees of freedom in terms of others on the basis of the constraining equations;
2. a method of matrix manipulation, devised by Morley;[14]
3. the method of Lagrange multipliers.

Method 1 is easily described algebraically. Thus, if m constraint equations exist in an (n + m) order system and the constraint equations are of the form

$$[C]\ \{\Delta\} = \left[\underset{n\times m}{C_1} : \underset{m\times m}{C_2}\right] \left\{\begin{array}{c} \Delta_1 \\ \Delta_2 \end{array}\right\} = \{b\} . \qquad (26)$$

we can solve for $\{\Delta_2\}$.

$$\{\Delta_2\} = -[C_2]^{-1}\,[C_1]\ \{\Delta_1\} + [C_2]^{-1}\ \{b\}. \qquad (27)$$

and write the transformation of degrees of freedom,

$$\left\{\begin{array}{c} \Delta_1 \\ \Delta_2 \end{array}\right\} = \left[\begin{array}{c} I \\ \hdashline -C_2^{-1}C_1 \end{array}\right] \{\Delta_1\} + \left[\begin{array}{c} O \\ \hdashline C_2^{-1} \end{array}\right] \{b\} . \qquad (28)$$

This expression must be substituted into the relevant energy expression before the variation is taken. It is clear that each constraint equation represents an opportunity to eliminate a degree of freedom from the problem. However, a very great amount of matrix manipulation is involved; so although the method is highly advantageous because of the consequent reduction in sizes of the system of equations to be solved, it is ill-suited to most of the well-established finite-element analysis programs. A discussion of this approach is given in Ref. 27.

Morley's approach[14] also requires a very great amount of matrix manipulation and results in a system of equations that is unchanged in size relative to the original system. The details of the method are relatively complex and are beyond the scope of this report. However, it can be concluded that this method is confronted with the same difficulties as the approach discussed above.

The method of Lagrange multipliers is the classical procedure for accounting for constraints. In this method, one simply multiplies each constraint equation by a parameter λ_i, the Lagrange multiplier, and adds the result to the functional to be minimized, resulting in a new "augmented" functional. Hence in view of Eqs. (7)–(10) and (26), the augmented potential energy functional Π_p is

$$\Pi_p = \frac{\lfloor\Delta\rfloor}{2}[K]\{\Delta\} - \lfloor\Delta\rfloor\{P^i\} - \lfloor\Delta\rfloor\{P\} + \lfloor\lambda\rfloor[C]\{\Delta\} - \lfloor\lambda\rfloor\{b\} = 0 \ , \quad (29)$$

where $\lfloor\lambda\rfloor$ lists the Lagrange multipliers, and the functional is varied with respect to both the degrees of freedom and the λ_i's. The resulting algebraic equations, in the case of a potential energy formulation, are of the form

$$\begin{bmatrix} K & C^T \\ C & 0 \end{bmatrix} \begin{Bmatrix} \Delta \\ \lambda \end{Bmatrix} = \begin{Bmatrix} P + P^i \\ b \end{Bmatrix} , \quad (30)$$

which are solved directly to yield all values of Δ and λ.

Substructuring refers to a process in which the total structure is divided into a relatively small number of component structures, each of which is generally composed of many finite elements. Solutions are first obtained for each substructure independently supported at the joints which connect to adjacent substructures. Then the substructures are analytically tied together at these joints.

Evidence points toward a major role for substructuring in very large-scale finite-element analysis. The significance of substructing in aerospace design is described quite clearly by Grisham[28] and for ship structure design by Rorem.[29] The efficiency of substructuring versus one-pass analysis is the subject of much debate, but the importance of substructuring as a means of keeping track of data and identifying sources of analysis errors is widely accepted. Also, in inelastic analysis, the region of plastic deformation may be quite localized and, by isolating this region in a predominantly elastic structure, the iterative portion of the solution process can be reduced to deal only with the degrees of freedom in the inelastic region. Finally, as will be noted in Chapters 7 and 8, substructuring may play a subsidiary role in the formation of efficient solution processes for time-dependent phenomena.

The theoretical basis of substructuring is quite simple. Consider the situation shown in Fig. 2 where substructures A, B, and C are adjacent to each other. In

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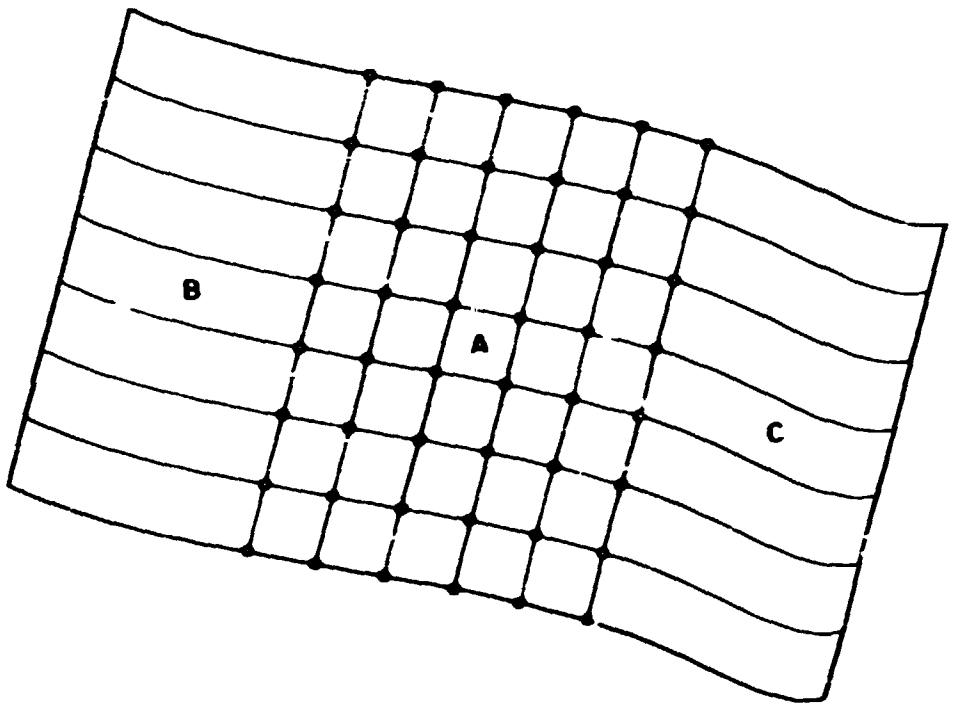

- POINTS c (POINTS OF CONNECTION OF SUBSTRUCTURE A WITH SUBSTRUCTURES B AND C).
- POINTS o (POINTS INTERIOR TO SUBSTRUCTURE A).

Fig. 2. Representative arrangement for substructuring.

developing the necessary relationships for substructure A, we designate all degrees of freedom on A which attach to B and C by the subscript c and all remaining ones by a. We can then write the stiffness equations for A in the partitioned form,

$$\begin{bmatrix} k_{aa} & k_{ac} \\ k_{ca} & k_{cc} \end{bmatrix} \begin{Bmatrix} \Delta_a \\ \Delta_c \end{Bmatrix} = \begin{Bmatrix} P_a \\ F_c \end{Bmatrix}. \tag{31}$$

The forces at degrees of freedom c are as yet internal forces and are designated by F_c, while those at a are external and are denoted by P_a. Solving the top partition for Δ_a and substituting into the lower partition, we obtain

$$[\bar{k}_{cc}] \{\Delta_c\} = \{F_c\} - \{F_c^a\}, \tag{32}$$

where

$$[\bar{k}_{cc}] = [k_{cc} - k_{ca} k_{aa}^{-1} k_{ac}] \tag{33}$$

and

$$\{F_c^a\} = [k_{ca} k_{aa}^{-1}] \{P_a\}. \tag{34}$$

$[\bar{k}_{cc}]$ is now a stiffness matrix which applies only to the degrees of freedom of the connection points of A, and $\{F_c^a\}$ is a vector of known values. Such stiffness matrices are constructed for each substructure, formed to define a system stiffness matrix which refers only to substructure juncture points, and this is solved to give the displacements at these points. The displacements at all other points are obtained by back-substitution in Eq. (31) for each substructure.

2.3 SOLUTION ALGORITHMS

2.3.1 Direct Methods

Direct methods are those that yield a solution to a system of simultaneous equations in a fixed number of operations. In discussing the most efficient procedures in this class, it should first be observed that the central efficiencies derive from accounting for the sparseness and symmetry found in finite-element equations. All effective schemes, whether direct or iterative, must incorporate recognition of these factors. Second, it may prove desirable to integrate the process of forming the equations with the solution process. In this approach there are generally questions of hardware configuration to be taken into account.

The basic form of direct solution is the Gaussian elimination process, which first triangularizes the coefficient matrix of the system of equations to be solved. Melosh and Bamford[30] have observed that Gaussian elimination is both reliable and efficient but that decomposition methods (e.g., Choleski decomposition) result in more efficient computer storage. They contend, however, that Choleski decomposition is a source of error and therefore introduce a special form of decomposition.

"Wave front" processing is also a feature of the Melosh-Bamford paper, and of the paper by Irons[31] as well. In this approach, a single element is formulated and all exterior degrees of freedom are immediately eliminated. The contiguous elements are then formed, added to the reduced stiffness, and any resulting exterior degrees of freedom are eliminated. The process continues in the manner of a wave progressing across the structure, and it is clear that full account is taken of the sparsity of the system of equations being solved and that the matrix-assembly and solution processes have been integrated.

Another group of direct methods, based essentially on Gaussian elimination but pursuing more traditional paths, is represented by the papers of Gatewood and Ohanian,[32] Whetstone,[33] Jensen and Parks,[34] and others. These authors divide the joint or element stiffness relationships into "blocks" which are operated upon in the manner of single terms. An efficient triangularization scheme is described by McCormick, and also by other authors, in the proceedings of a conference on the solution of sparse matrices.[35]

Finally, note should be taken of the Choleski decomposition approach, which directly calculates upper and lower triangular matrices. The report by Klein[36] represents the most recent work in this direction. It should be observed that one study of solution accuracy[37] contends that the square-root operation in Choleski decomposition is a source of significant error, for certain machines, in large-scale applications.

2.3.2 Iterative Methods

Two noteworthy iterative methods are the conjugate gradient procedure described by Fox and Stanton[6] and the alternating iterative component method of Rashid.[38]

The basis of the conjugate gradient method was discussed in Section 2.1. One can identify the structural analysis problem when cast in potential or complementary energy forms for stable structures as requiring

the minimization of a quadratic function. Given this objective, the analyst may employ any one of a large number of available algorithms pertinent to the minimization of a quadratic function of many variables; one of these is the conjugate gradient method introduced by Hestenes and Stiefel.[7] The method had been little used until recent years because of the influence of rounding errors on solution accuracy, but Fox and Stanton[6] show that these difficulties can be neutralized through proper operations on the system stiffness coefficients.

2.4 SOURCES OF ERROR

The problem of error prediction and elimination in finite-element analysis looms as one of the most important topics in this field because, if the method is capable of producing a solution to problems for which no alternative solutions exist, procedures for establishing confidence in the computed results must be established. Although a substantial number of recent studies will be cited in the following, this area of investigation is in its infancy and is only beginning to have impact in practical applications.

The sources of error in finite-element analysis can be grouped as follows:

1. **Input data errors.** These result both from outright mistakes in the definition of problem data and from random errors or uncertainty in the input data. The former have been reduced considerably by the checking capabilities associated with computer graphics devices. Errors due to uncertainty in the input data may prove significant when the analytical model is itself numerically ill-conditioned, but the case for this has not been well established as yet.

2. **Discretization errors.** These are due to the replacement of the continuum by the finite-element model with its attendant assumed displacement and stress fields. Each type of element is usually given close study in the developmental phase, insuring that the associated discretization errors are acceptably small for realistic grid refinements. Discussion of this aspect of the topic appears directly or by reference in the chapters to follow. (See also Ref. 38.)

3. **Numerical manipulation errors.** These errors are represented by truncation and round-off. Studies related to this topic are the principal concern of this section and are delineated below.

4. **Output interpretation errors.** These constitute the approximations made in representing and interpreting the solved-for stresses and displacements. There is no unique basis for these operations, and the problem is especially acute for triangular element fields. The reader is advised to consult Refs. 39 through 41 for alternative views of this topic.

The basis of studies of numerical error is the concept of examining the change in the solution $\delta\Delta$ due to a change in the stiffness δK:

$$\Big[\,[K] + [\delta K]\,\Big]\ \Big\{\{\Delta\} + \{\delta\Delta\}\Big\} = \{P\}\ . \qquad (35)$$

Then, by taking norms of both sides and assuming $\|\delta K\|/\|K\| \ll 1$, we find

$$\frac{\|\delta\Delta\|}{\|\Delta\|} = N(K)\,\frac{\|\delta K\|}{\|K\|}\ , \qquad (36)$$

where $N(K)$ is the "condition number" of $[K]$. The norm of a vector or matrix is a scalar value obtained through the performance of simple operations on the coefficients of the vector or matrix. The Δ Euclidian norm is the square root of the sum of the squares of the matrix coefficients, and for this choice the condition number is the ratio of the maximum to minimum eigenvalues of $[K]$.

It should be noted that the existing studies are directed toward two subsidiary sources of error: the conditioning of the system stiffness matrix and the accuracy of equation-solving algorithms. Studies of the latter have been pursued by Rosanoff,[42,43] Rashid,[44] and Melosh and Palacol.[37] The contribution of the role of the element stiffness matrices to this problem has been studied by Tong,[45] Fried,[46] and Kelsey et al.[47] More general studies have been published by Roy[48] and by Shah.[49]

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3. Solid Elements

Although solid, or three-dimensional, elements are not feasible for representation of the major parts, or regions, of metallic reactor structures, they have assumed considerable importance in the solution of problems in local regions. Problems of pipe-cylinder intersections that have drawn attention[1-3] are intrinsically three dimensional, as are problems relating to flanges and certain thick-walled pipes. It is therefore of interest to examine, in brief outline, progress to date in three-dimensional elements.

The original solid elements were merely three-dimensional generalizations of the triangle and rectangle in plane stress, in the forms of the tetrahedron (Fig. 3) and the hexahedron[2] or rectangular parallelepiped (Fig. 4). Alternative forms, such as pentagonal- or wedge-shaped elements, have emerged; but the predominant practice has concentrated upon the tetrahedron and hexahedron, and our attention will be fixed on these in Section 3.1.

An important specialization of the 3-D element is the axisymmetric solid, or annulus, which is mainly triangular or quadrilateral in cross section, as would be expected of the generalization of planar behavior. This element is reviewed in Section 3.2.

Questions of analysis efficiency are of critical importance in 3-D finite-element analysis, since when general elements are employed problems in this class suffer from the "curse of dimensionality," a consequence of the expansion from two to three dimensions. This problem has furnished the largest share of motivation for improvements in equation-solving algorithms (discussed in the previous chapter) and for automation of the input data procedures. Efforts in improvement of equation-solving algorithms have reflected upon the element formulation considerations, since it can be shown that certain of the many alternative forms of solid elements are more suited to efficient equation solving than others.

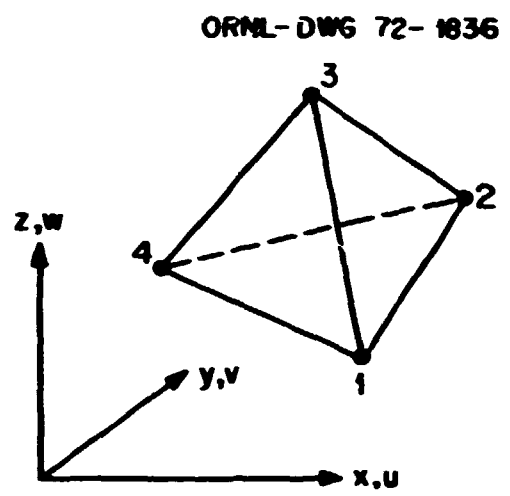


Fig. 3. Regular tetrahedronal geometry.

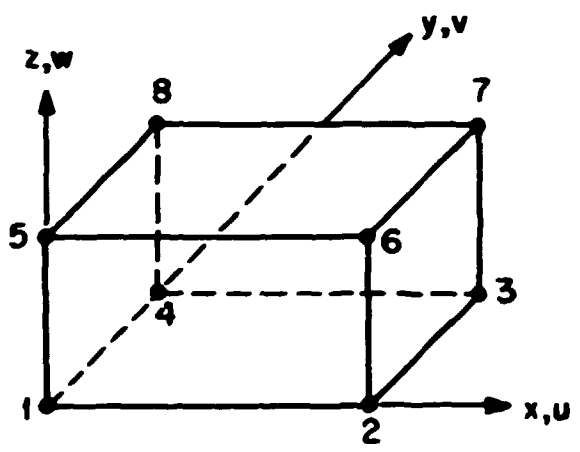


Fig. 4. Regular hexahedronal geometry.

A remarkably large number of papers have surveyed 3-D finite-element analysis (excluding axisymmetric representations) in recent years.[4–8] These papers combine examinations of alternative publications with details of individual element formulations, comparisons of accuracy, demonstrations of practical applications, and discussions of key aspects of practical application. Since the cited surveys do not reach unified conclusions with reference to optimal practice, a consultation of their aggregate contents is essential to an understanding of appropriate directions to be taken under specific circumstances.

3.1 GENERAL 3-D ELEMENTS

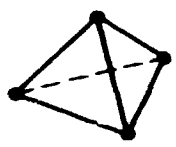
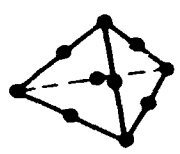
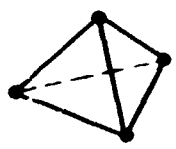
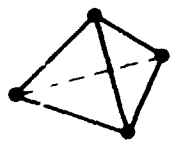
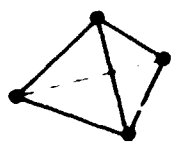
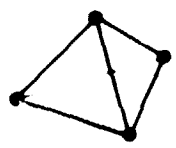
3.1.1 Tetrahedra

Tetrahedronal elements in common practice, representing work described in Refs. 1, 3, 4, and 6–10, are summarized in Table 1.*

The fundamental element (entry a) is the three-dimensional generalization of the basic plane stress

*For convenience, the table includes element acronyms employed by various authors.

Table 1. Tetrahedronal finite elements

	Representation	Element	Number of joints	Degrees of freedom: Per joint	Degrees of freedom: Per element	Remarks
a		TETRA 4[a] CSTh[b] T12[c]	4	3	12	Complete linear displacement fields (constant strain); for details see Refs. 1, 3, 4, and 8; u, v, w degrees of freedom at each node
b		TETRA 10[a] LSTh[b] T30[c]	10	3	30	Complete quadratic displacement fields (linear strain variation); for details see Ref. 4; u, v, w degrees of freedom at each node
c		TET 20[a] QSTh[b]	20	3	60	Complete cubic displacement fields; for details see Ref. 10
d		TEA 8[a]	8		60	Complete cubic displacement fields; for details see Ref. 10; 12 degrees of freedom at vertices and along edges (u, v, w, u_x, v_x, w_x, u_y, v_y, w_y, u_z, v_z, w_z) and 3 degrees of freedom at centroid of faces (u, v, w)
e		TET 16[a]	16	3	48	Incomplete cubic displacement field; for details see Refs. 6, 9, and 10
f		TEA 3[a] T 48[c]	4	12	48	Incomplete (conforming) cubic displacement fields; for details see Refs. 5, 8, and 10; displacements and displacement derivatives as degrees of freedom (u, v, w, u_x, v_x, w_x, u_y, v_y, w_y, u_z, v_z, w_z)

[a]Argyris notation. [b]Clough notation. [c]Fjeld notation.

triangle, possessing linear displacement and constant strain fields. The requirements of practical analysis, in which considerable grid refinement is needed for geometric and stress field representation, have submerged the value of this element, with preference being given to "higher-order" elements.

Two directions are apparent in "higher-order" element representation: the use of the direct displacement component (u, v, w) degrees of freedom, which carries the requirement for many node points, and the use of displacement derivatives as degrees of freedom, with these being collected principally at the vertices.

Entries b, c, and e are representative of higher-order tetrahedronal element formulations with only direct displacement components. Entry b represents use of quadratic displacement fields, but this scheme has been indicated to be less efficient than the complete cubic (entry c) or incomplete cubic (entry e). In theory, it is possible to continue the expansion in degree of the displacement fields simply by adding tiers of joints and interior node points. In fact, there are no numerical results available for elements of higher order than entries c and e.

The two elements with displacement derivatives as joint degrees of freedom are portrayed as entries d and f. Entry d is based on complete cubic polynomials in representation of the displacement fields, while entry f is incomplete in this regard. Three formulative approaches have been taken in establishment of the stiffness matrix for entry f, and these are not identical. Reference 5 chooses complete cubic polynomials that are subsequently contracted by application of constraint equations. References 8 and 11 develop displacement fields directly in terms of tetrahedronal coordinates. General formulas for shape functions in tetrahedronal coordinates and for the integrals of these coordinates in a tetrahedron are given by Sylvester.[12]

The important advantage of the use of derivatives as element vertex degrees of freedom lies in the minimization of the number of physical joints to be specified and in good bandwidth properties. On the debit side, however, certain negative factors may counterbalance this advantage. Continuity of displacement derivatives is synonymous with continuity of strain, a situation that might not always prevail, especially for nonhomogeneous materials and discontinuous geometry. The full set of derivatives is relatively large at each vertex in the general case, and the mere construction of appropriate transformations from the element to system levels and the imposition of continuity requirements may prove complex. Also, limitations are placed on the combination of constant strain and higher-order elements in a single representation, although it would appear that such limitations can be removed without great difficulty when special constraint conditions are imposed on the higher-order elements. These same comments apply also to the hexahedronal elements.

It is generally agreed that it is unfeasible to employ tetrahedronal elements without special provision for the input specification of the analytical model. An essential capability is the generation of fundamental elements (e.g., hexahedrons) from the tetrahedrons, with the former being the basis for the specification of the analytical model.

3.1.2 Hexahedra

Hexahedronal elements, representing work described in Refs. 2, 4, 7, 8, 11, 13, and 14, are shown in Table 2. Here the fundamental element (entry a) is based upon simple Lagrangian interpolation between vertex points (a linear displacement field) and involves 24 degrees of freedom. In contrast with the basic tetrahedronal element, this formulation is of great utility, and studies reported by Clough[4] assert that this is the preferable form of element for practical applications.

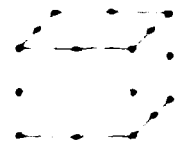
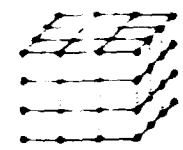
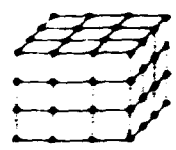
Lagrangian interpolation permits the expansion of the hexahedronal representation to any order of polynomial, and although this expansion can be progressed in an orderly manner with interior nodes, preference has been given to functional representations which exclude these interior nodes. Entries b and c are in this class for quadratic- and cubic-order displacement fields respectively. The HEXE 64 element (Argyris notation) represents a complete third-order Lagrangian interpolation in three dimensions.

As in the case of the tetrahedron, attempts have been made to formulate a higher-order element with degrees of freedom exclusively at the vertices. This representation, the H96 or HERMES 8 element (entry e), has been applied successfully in the solution of a classical problem, but no practical applications have as yet been reported.

3.1.3 Isoparametric Representation

The concept of isoparametric representation of element geometry, that is, the use of displacement shape functions for the modeling of element boundaries, assumes special importance in the case of solid elements. It has already been noted that a fundamental problem in 3-D finite-element analysis is economic feasibility in the face of the need to employ large

Table 2. Hexahedronal finite elements

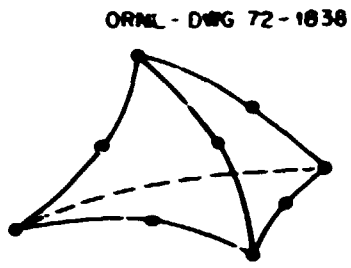
	Representation	Element	Number of joints	Degrees of freedom: Per joint	Degrees of freedom: Per element	Remarks
a		H24[a] HEXE 8[b] Z1B 8[c] DPLS[d]	8	3	24	Linear displacement fields; u, v, w degrees of freedom at each node (for details see Refs. 2 and 4)
b		Z1B 20[c] H60[a]	20	3	60	Quadratic displacement fields; u, v, w degrees of freedom at each node; external nodes only (for details see Refs. 2 and 4)
c		Z1B 32[c] 96D[a]	32	3	96	Incomplete cubic displacement fields; u, v, w degrees of freedom at each node (see Refs. 4 and 14)
d		HEXE 64[b]	64	3	192	Complete cubic displacement fields with internal nodes; u, v, w degrees of freedom at each node; lumina element when isoparametric (see Ref. 13)
e		H96[a] HERMES 8[b]	8	12	96	Incomplete quintic displacement fields (Ref. 8) or Hermitian polynomial interpolation (Ref. 11); displacements and derivatives as degrees of freedom (u, v, w, u_x, v_x, w_x, u_y, v_y, w_y, u_z, v_z, w_z); isoparametric representations discussed in Refs. 7 and 11

[a]Fjeld notation.[8]
[b]Argyris notation.[11]
[c]Clough notation.
[d]STRUDL-II notation.[38]

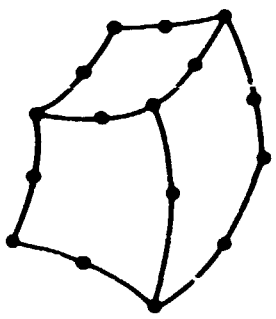
numbers of degrees of freedom. When curved boundaries are present and straight-sided elements are employed, it is necessary to use many elements for geometric representation. Thus a large share of the degrees of freedom merely represent the curved sides without any gain in representation of the stress and deformational behavior. The isoparametric representation concept eliminates the expenditure of degrees of freedom merely for geometric modeling purposes and thereby helps to achieve a given level of solution accuracy with a minimum number of degrees of freedom. This is done at the expense of increased complexity in element formulation.

Isoparametric representation concepts were introduced by Ergatoudis, Irons, and Zienkiewicz.[14] Zienkiewicz et al.[7,14] discuss at length the application of these concepts to solid hexahedronal elements (Fig. 5b). The hexahedron in curvilinear (isoparametric) coordinates is also described by Argyris[10] and Clough.[4] Hughes and Allik[5] emphasize the isoparametric representation of the tetrahedronal element (Fig. 5a).

It is important to recognize that isoparametric element concepts do not obligate the user to the same degree of shape function representation for geometry as for displacements. Zienkiewicz et al.[7] term a representation where the geometric shape function is of lesser order than the behavior shape function as "subparametric." Moreover, based on studies of computational costs, these authors conclude that a refinement beyond the quadratic representation of element edges is not



(a) TETRAHEDRON



(b) HEXAHEDRON

Fig. 5. Isoparametric solid elements.

justified, whatever the level of refinement in description of the element displacement field.

In practice, the complexity of isoparametric element concepts obligates the user to numerical integration in the numerical evaluation of the element stiffness matrix, and the costs of this operation must be minimized. The most recent summary of progress in the formulation of quadrature expressions for this operation is given in Ref. 15.

3.2 AXISYMMETRIC SOLIDS

3.2.1 Triangular Cross Section

The triangular cross-section ring element was introduced in papers by Wilson[16] and Clough and Rashid[17] in the form of a generalization of the plane stress triangle. Thus all considerations of higher-order representation and choices for degrees of freedom (i.e., linear displacements alone vs displacement derivatives as degrees of freedom) are applicable here. The major differences between this element and the plane stress triangle arise due to integration in cylindrical coordinates, in which a variable (the radius) often appears in the denominator of the function to be integrated, and in treatment of elements with joints on the axis of symmetry (zero radius).

Exact integration of the stiffness coefficients for the triangular element is given by Utku.[18] This work includes representation of the zero-radius case. Similar results are given in Refs. 19 and 20. However, if the element is remote from the axis of symmetry then the approximate integrals presented in Ref. 21 are applicable.

For the zero-radius case, apart from the use of exact formulations, approaches that have proved satisfactory are the use of a small nonzero core radius in the analytical representation[16] and the use of specially formulated core elements.[19]

With respect to nonaxisymmetric loadings, an approach based upon representation of the loading in Fourier series is feasible, and such an approach was indeed introduced in the original paper by Wilson.[16] A more complete approach to this representation, which admits both asymmetric and symmetric components of loading, is detailed by Argyris et al.[22] It is important to observe that each harmonic is representable in terms of an uncoupled, independent element stiffness matrix. Thus, existing general-purpose finite-element analysis programs may be employed for such analysis, with a different element for each needed harmonic.

The above type of representation is, of course, limited by nonaxisymmetric geometry and circumferential temperature-dependent material properties and also by the number of harmonics needed for accurate representation of load. At some point in the latter it is preferable to revert to a general 3-D element representation, in the class described earlier in this chapter. An interesting alternative, which is midway between the axisymmetric solid and the general 3-D element, is the "sector" element described by Argyris et al.[22]

It should be noted that higher-order triangular elements, in the form of the conventional higher-order plane-stress triangles, are described by Argyris et al. This work extends to the use of isoparametric concepts in representation of curved boundaries for the elements and also to the representation in series of circumferential variations of loading.

3.2.2 Quadrilateral Cross Section

Very little explicit attention has been given in the literature to the quadrilateral axisymmetric solid element. The derivation for an element with linear edge displacements and with the shape of an arbitrary quadrilateral through use of isoparametric formulation concepts is easily established. Reference 23 gives formulations for higher-order quadrilateral elements.

3.3 APPLICATIONS EXPERIENCE

Published descriptions of experience in application of 3-D elements to practical problems are concentrated upon the problems of dams, piping intersections, and nuclear reactor structures. Reference 24 contains a number of papers relating to dam analysis with 3-D finite elements. The pipe-cylinder intersection problem is discussed in Refs. 22 and 25. Analyses of nuclear reactor structures are described in Refs. 26 and 27, and applications to other areas are discussed in Refs. 28 and 29. Reference 26 is noteworthy for the inclusion of analysis-test comparisons. A variety of applications of the axisymmetric solid elements is given in Refs. 30 through 32.

Experience in application of 3-D elements to special phenomena, such as thermal stress, inelasticity, and heat transfer, is quite limited. Thermal stress was the subject of the initial paper relating to 3-D elements,[1] but few analyses of this type of problem have since been discussed. A 3-D inelastic analysis, employing constant strain tetrahedra and the tangent stiffness approach of Yamada,[24] has been published by Miyamoto.[33] Levy and Marcal[34] have performed fracture mechanics analyses of solids using a distorted cubic element and the tangent stiffness approach to inelastic analysis.

Zienkiewicz and Parikh[35] have developed conduction heat transfer formulations for isoparametric solid elements and have performed analyses of turbine blades. As will be emphasized in Chapter 8, a thermal analysis situation in which structural design is influenced by thermal stress requires integrated thermostructural analysis capabilities.

3.4 SUMMARY

There are numerous, significantly different approaches to element representation for 3-D stress analysis, each with demonstrated advantages. The comparison data and viewpoints of the respective authors are contradictory, as would be expected in a topic of maximum complexity. Regardless of the form of representation chosen, success in application is critically dependent upon a high degree of efficiency in solution of the formulated equations.

Development and application of 3-D finite-element representations for other than linear static analysis is in its early stages. This probably is due to the preoccupation of analysts up to the present with the establishment of economic feasibility for linear static analysis. A similar condition exists with respect to use of other than displacement functions and the principle of minimum potential energy as the basis for element formulation.

Many of the widely available general-purpose analysis programs have been implemented with solid elements: SAFE-3D,[36] MARC-2,[37] STRUDL-II,[38] ASKA,[39] and FESS, ANSYS, and ELAS etc. Implementation is in progress for other systems, such as NASTRAN.[40]

An important factor in achieving economic feasibility in 3-D finite-element stress analysis is the maximum utilization of computer graphics capabilities, both in the automatic generation of meshes from basic problem data and in output data interpretation. Discussion of this topic is beyond the scope of this report, but the interested reader will find useful information in Refs. 27 and 41.

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4. Sheli Elements

We have already emphasized in Chapter 3 that the major influence on solution accuracy in finite-element analysis of any type of problem is likely to be the adequacy of the element formulation. The formulation depends upon the underlying theoretical relationships and the merit of the chosen geometric and assumed behavior representations. Only the assumed behavior representations were of major importance for solid elements, but for shell elements, all the above factors play a significant role in solution accuracy.

Pressure vessels are in whole or in part doubly curved, and the difficulties of geometric representation of such forms often give rise to approximations. These structures are generally thin shells, and the appropriate form of underlying theoretical relationships continues to generate controversy; many alternative relationships have been proposed through the years. As will be shown, the difficulties in defining behavior representations, especially displacement fields, for doubly curved shell elements are more severe than for other types of elements.

With these complicating factors in mind, it is readily accepted that the literature of finite-element shell analysis is quite extensive. Fortunately, a major demarcation of activity can be identified with respect to axisymmetric vs general thin-shell elements, and this division is followed here.

4.1 AXISYMMETRIC SHELL ELEMENTS

4.1.1 Background

The developmental history of axisymmetric thin-shell elements was reviewed in the papers by Brombolich and Gould[1] and by Dawe.[2] For present purposes, it is convenient to consider a division of developmental efforts into conic elements (Fig. 6) and the more general meridionally curved elements (Fig. 7).

Published formulations for conic segments representing work described in Refs. 3–8 are summarized in Table 3. The initial formulations for axisymmetric thin shells[3–6] were for conic segments. It was anticipated in these studies that satisfactory representations of general meridionally curved shells could be obtained by representation as a series of conic segments. Subsequently, numerical results[9] demonstrated that this approach was not reliable, and many separate efforts have been devoted to the formulation of meridionally curved elements. The causes for deficiencies in the representation by conic elements and the difficulties in accomplishing satisfactory curved elements are related to questions of geometric and displacement field representations. These topics are discussed below, and an outline is given of the pertinent application experience using axisymmetric thin-shell elements.

4.1.2 Geometric Representation

As noted above, conic elements are generally unsatisfactory for geometric representation of meridionally curved axisymmetric shells. If a shell is actually composed of a sequence of conic segments (as in Fig. 8), meridional bending moments (discontinuity stresses) exist at the juncture lines of the respective conic elements; the finite-element representation will predict these stresses. These stresses do not exist, however, if the shell is continuously curved and in a membrane state of stress, and they are generally small except at the juncture of different geometric forms.

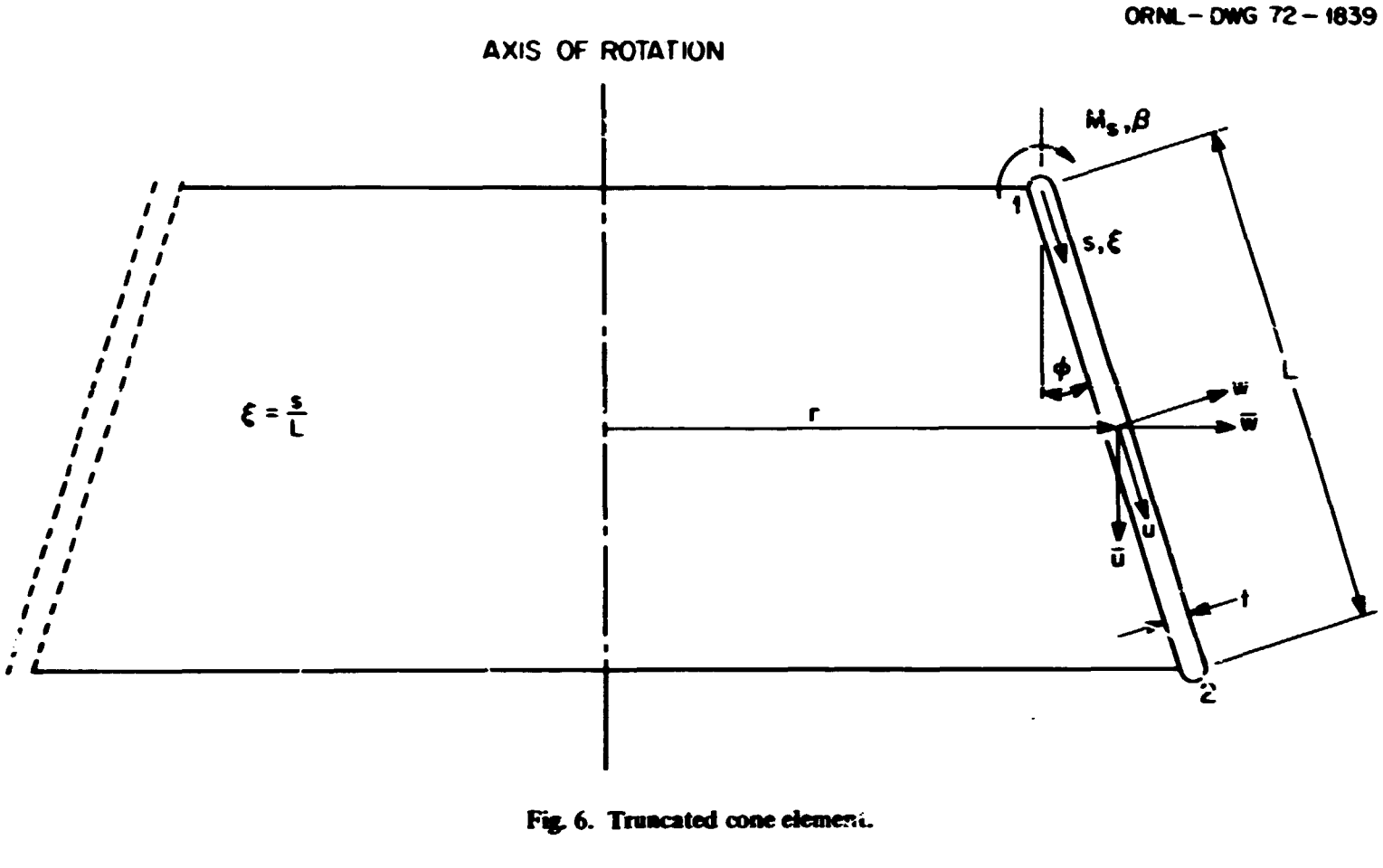


Fig. 6. Truncated cone element.

Table 3. Axisymmetric conic thin-shell elements

See Fig. 6 for element geometry

Reference	Displacement function	Remarks
3,4	Exact solution to governing differential equation	Stiffness coefficients are in terms of Bessel functions
5	$\bar{u}$ is basically linear and $\bar{w}$ is basically cubic, but both are coupled via trigonometric functions to satisfy rigid-body motion	$\bar{u}$ and $\bar{w}$ are defined as radial and axial system displacements
6	$u = a_{11} + a_{12}s + a_{13}s^2 + a_{14}s^3 + \ldots$ $w = a_{21} + a_{22}s + a_{23}s^2 + a_{24}s^3 + \ldots$	$a_{11}, \ldots a_{24}$ are found in usual way in terms of the six nodal displacements; additional coefficients are evaluated by minimizing potential energy
7	Linear variations of u, w, β, and M_s	Mixed-method approach (definition of independent displacement and moment fields)
8	Linear variations of u, w, and β	Transverse shear strain is made to vanish at the midpoint of each element, thus defining relationships between the coefficients of w and β

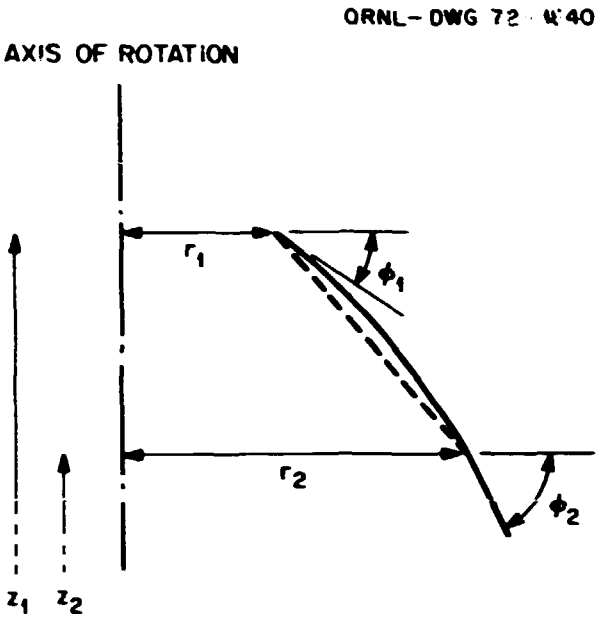


Fig. 7. Meridionally curved element.

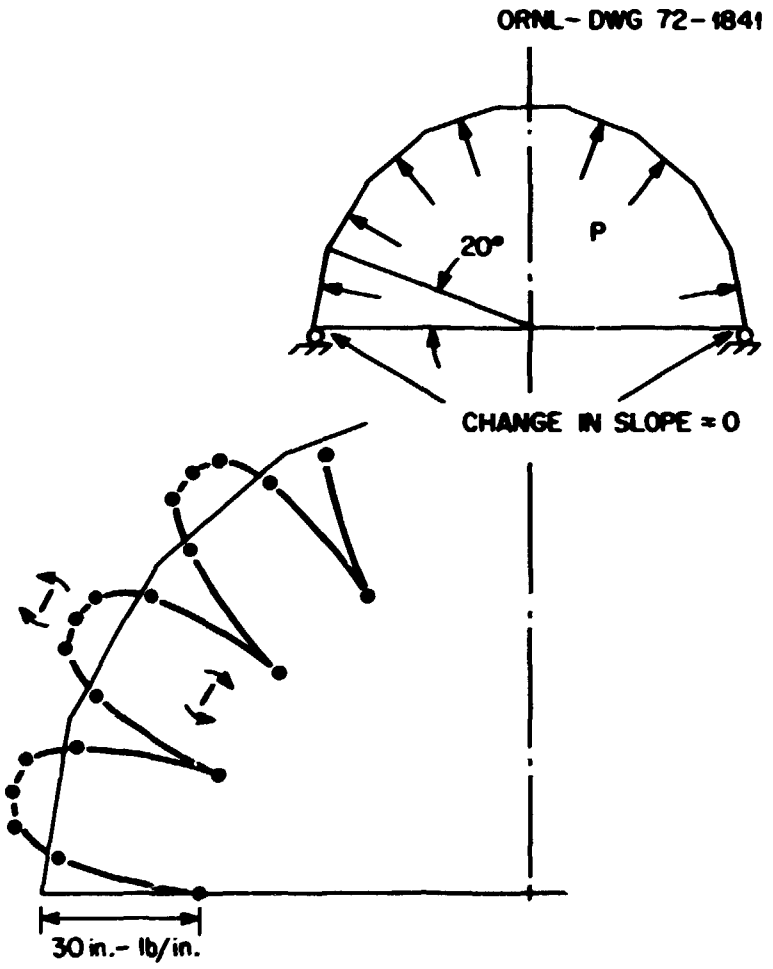


Fig. 8. Meridional moments in shell modeled with conic elements. (From Ref. 9.)

It should be noted that the above difficulty can be circumvented if the solved-for displacements are substituted into stress-displacement equations written for the curved shell (e.g., by use of finite differences).[10] This approach may be awkward, however, and recent developments in axisymmetric thin-shell finite-element analysis have given attention to the representation of meridional curvature.

Table 4, comprising work described in Refs. 11 to 20, summarizes these efforts under the heading "geometric representation." Two categories of geometric representation can be identified. One group, the final three entries,[17-20] anticipates availability of an exact functional representation of shell geometry and defines the stiffness formulations in integral form. Thus numerical integration is invoked in the evaluation of element stiffness coefficients. In the second group (all other entries), it is assumed that the given geometric data must be transformed into a single type of functional representation. This enables formulation of explicit matrix coefficients, although the product of a series of such matrices may be needed to construct the complete element stiffness matrix. As indicated in Table 4, the levels of sophistication vary widely among the respective developments.

4.1.3 Displacement Representations

Since the axisymmetric thin-shell element is "one dimensional" in the meridional coordinate, no difficulties are encountered in the construction of inter-element-compatible displacement fields. Application of such fields to conic elements is straightforward. However, difficulties arise in the case of meridionally curved elements in satisfaction of rigid-body motion requirements, as discussed below.

For conic elements, reference is made to Table 3, which shows that two types of displacement fields are employed. In one, the fields represent the exact solution of the homogeneous forms of the governing differential equations. Such solutions are feasible but involve Bessel functions that add considerably to the complexity of programming and cost of stiffness coefficient evaluation. Also the situation to which they apply is the line-loading of the edges of the element; this is not the usual situation in practice, where the structure is under pressure loadings. The other type of conic element displacement field is the counterpart of the axial-flexure behavior of prismatic members.

Use of the uncoupled radial and meridional displacement fields, of the form employed for conic elements, may introduce errors due to deficiencies under rigid-body motion. The situation is described in Fig. 9, which shows the arc of an axisymmetric shell. It is clear that if the displacements are described in curvilinear coordinates, a rigid-body motion in the axial

Table 4. Meridionally curved axisymmetric thin-shell elements

See Fig. 7 for element geometry

Reference	Geometric representation	Displacement fields
11	Quadratic for ϕ $(\phi = c_1 + c_2\xi + c_3\xi^2)$	$u = a_1 + a_2\xi$ $w = a_3 + a_4\xi + a_5\xi^2 + a_6\xi^3$
12	Quadratic for ϕ	u = above + $a_7 \sin\phi + a_8 \cos\phi + a_9[(x - \bar{x})\cos\phi - (r - \bar{r})\sin\phi]$ w = above $- a_7 \cos\phi + a_8 \sin\phi + a_9[(x - \bar{x})\sin\phi + (r - \bar{r})\cos\phi]$
13	$d = \xi(1 - \xi)(c_1 + c_2\xi + c_3\xi^2 + c_4\xi^3)$ (d = distance from chord)	$u = a_1 + a_2\xi$ $w = a_3 + a_4\xi + a_5\xi^2 + a_6\xi^3$
14	Torispherical segment	$u = a_1 + a_2\xi + a_3\xi^2 + a_4\xi^3$ $w = a_5 + a_6\xi + a_7\xi^2 + a_8\xi^3 + a_9\xi^4 + a_{10}\xi^5$
15	Cubic for r and z (4-point interpolation)	Same shape functions as for geometry
16	Cubic for r and z (beam shape functions)	$u = a_1 \sin\phi + a_2 r\cos\phi + a_3\xi + \Sigma b_i\xi^{i+1}$ $w = a_1 \cos\phi + a_2 r\sin\phi + a_4\xi + a_5\xi^2 + a_6\xi^3 + \Sigma c_i\xi^{i+3}$
17	Exact (numerical integration)	$u = \Sigma a_i\xi^{i-1}$ v = same as for u
18	Exact (numerical integration)	$u = a_1 + a_2\xi + a_3\xi^2 + a_4\xi^3$ v and w of corresponding form
19,20	Exact (numerical integration)	Same as Ref. 17

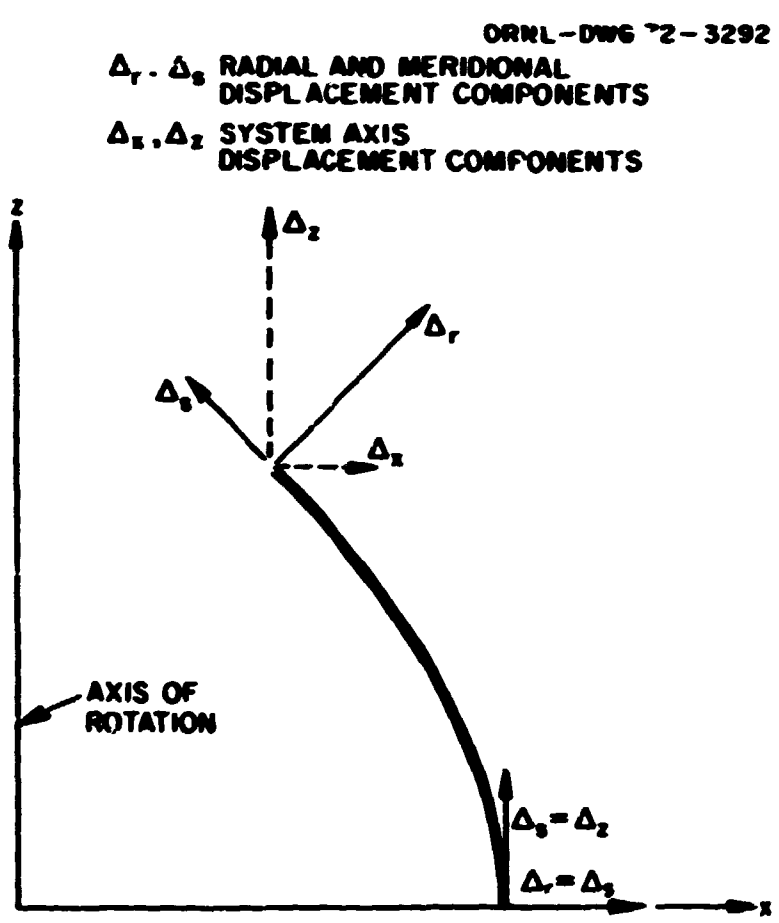


Fig. 9. Segment of axisymmetric shell showing displacement components.

direction requires that the radial and meridional displacements be coupled. Thus, certain formulations[12,14,16] modify the beam-type displacement fields to accomplish this coupling and the satisfaction of the no-strain-under-rigid-body-motion condition. The condition can also be met if the uncoupled displacement fields are written in terms of a local rectilinear coordinate system.[13] Other formulations,[1,11,18-20] however, disregard satisfaction of this condition and define the uncoupled displacement fields in curvilinear coordinates.

Haisler and Stricklin[21] examine the strain energy content during rigid-body motion of an element that sustains strain during rigid-body motion. They demonstrate numerically that the correct result is obtained in the limit; these results are confirmed in Ref. 22. The convergence is relatively slow, however, and a more recent note[23] demonstrates that the addition of a single term to the membrane displacement field greatly enhances the convergence characteristics.

The significance of additional terms in the displacement expansions has been demonstrated clearly in numerical results presented in Refs. 17, 18, and 20. The

approaches in Refs. 17 and 20 define the displacement expansions in infinite series and employ various numbers of terms in comparison studies of solution accuracy.

In conclusion, it should be noted that the strain-displacement equations presented by Novozhilov[24] are generally accepted as the basis for construction of the strain-energy expression for meridionally curved axisymmetric, thin-shell elements. Special developments use different theories (e.g., Ref. 17 employs the strain-displacement equations of Washizu[25]), while other formulations establish pertinent strain-displacement equations from basic considerations.

4.1.4 Applications Experience

Experience in application of axisymmetric thin-shell elements is extensive and is described adequately in the review by Brombolich and Gould.[1] A large amount of solution data has been correlated with classical solutions and with test data for problems of linear elastic analysis. The attention of this section is restricted to elastoplastic analysis experience.

Elastoplastic finite-element analyses of axisymmetric thin shells have been reported only recently. Fowler[26] and Witmer and Kotanchik[27] describe the use of the initial strain approach with incremental plasticity theory. This work is noteworthy because it treats relatively complex shell cross sections, including bonded double layers with foam cores, and because it correlates analysis results with test data for an asymmetrically loaded flat plate.

In another group of papers and reports, Khojasteh-Bakht[13,28] and Popov et al.[29] present tangent-stiffness elastoplastic analyses of axisymmetric plates and shells, in which incremental relationships are based upon the von Mises yield condition and the associated flow rule. Reference 29 is of particular interest in that 16 ASME standard torispherical heads, attached to cylinders, were analyzed for elastoplastic behavior under internal pressure and the results compared with the provisions of the ASME Code.[30] The results of the calculation indicate that the maximum elastic stresses predicted by the code are underestimated, a considerable redistribution of stresses occurs during the customary water pressure test, and high compressive stresses may exist in certain regions. A computer program is given in Ref. 28.

The work of Zudans,[15] which has not been published as of this writing, appears to be the most complete theoretical exposition to date. In his work, an incremental virtual work principle is established, enabling the treatment of both finite displacements and plastic deformation, and constitutive relationships are defined for both isotropic and kinematic strain hardening. Isoparametric function concepts are applied in representation of the geometry and displacements, and, with the ensuing requirement for numerical integration of the element stiffness matrix, variation of shell thickness along the meridian is easily handled. The numerical results presented include analyses of a torispherical shell studied by Khojasteh-Bakht.[28]

4.2 GENERAL SHELL ELEMENTS

4.2.1 Basic Considerations

A comprehensive review of finite-element formulations for flat plates and curved shell elements was presented by the author;[31] other reviews are given in Refs. 2 and 32. The following is largely a restatement of the portion of Ref. 30 dealing with curved shell elements, with additional commentary regarding work published in the intervening period of time.

Well-established flat-plate element formulations are capable of representing shell structures simply by superposition of membrane and flexural behavior and application of coordinate transformations. This approach has been described extensively[33-36] and used by design offices for some time. However, this method has various difficulties and shortcomings, as follows.

1. The slope discontinuities between adjacent elements may produce large calculated bending moments in regions where these are small or nonexistent. This effect was described above for axisymmetric shell problems and must be anticipated as a source of difficulty for general shell problems.
2. The coupling of membrane and flexural behavior within the individual elements lacks representation.
3. The difficulty in achieving interelement continuity is heightened.

For these reasons, the preferential approach to analysis of general curved shell structures is by use of general curved-shell finite elements. This approach is not flawless, however, and includes problems related to: (1) choice of an appropriate shell theory, (2) description of element geometry, and (3) selection of element displacement fields.

The basic considerations and implications of these three items is discussed more fully below, after which a review of published formulations is presented for quadrilateral and triangular elements.

4.2.2 Choice of Shell Theory

In discussing shell theories for finite-element analysis it is necessary to distinguish between "nonshallow" and "shallow" formulations.

The usual approach in nonshallow shell theory is to describe behavior with reference to curvilinear coordinates (α_1, α_2) in the middle surface of the shell. One may then describe the stress and deformational behavior in terms of "membrane" stress resultants and strains (N_{11}, N_{22}, N_{12}, ϵ_{11}, ϵ_{22}, ϵ_{12}) and moment-curvature parameters (M_{11}, M_{22}, M_{12}, k_{11}, k_{22}, k_{12}), which are defined with respect to the curvilinear coordinates.

The basic complexity of the shell analysis problem led early theorists to establish strain-displacement relationships with different types of approximations, and to date no one of these formulations has received exclusive acceptance. The significant differences between the respective shell theories for linear conditions arise in conjunction with the strain-displacement expression for the twisting curvature k_{12}. A thorough examination by Koiter[37] of available theories discloses that certain formulations do not properly account for the condition of zero strain under rigid-body motion in the representation of this term. These formulations include the work of Love[38] and certain forms published by Donnell[39] and Flugge.[40] The same question is studied in the context of finite-element analysis by Cantin[41] for the restricted case of the circular cylindrical shell.

It is essential that the strain-displacement equations for finite-element shell analysis meet all conditions related to rigid-body motion. This is a requirement apart from that associated with rigid-body motion existing in the choice of displacement fields, and errors in the satisfaction of one will reinforce errors in the other. Hence, theories that satisfy these requirements should be chosen. Popular choices have been those due to Novozhilov,[24] and Koiter,[37] or Budiansky and Sanders.[42] It should also be noted that the designation "best first-order" or "consistent" shell theory has been applied to the latter formulation because it is consistent with respect to the basic Love-Kirchoff hypotheses (i.e., normals remain normal, etc.). Koiter[37] shows, however, that the "inconsistency" of other formulations is manifest in terms of order t/R (t = shell thickness, R = radius of curvature) which have no significance on numerical results. Kraus[43] presents a lucid development and summary of these and other aspects of thin-shell theory.

The situation in shallow-shell analysis parallels that of deep-shell theory. Alternative formulations of the strain-displacement equations have appeared which differ in the expression for twisting curvature. Again certain theories (e.g., Marguerre[44]) are not free of strain under rigid-body motion, but this does not appear to be as significant as in deep-shell theory if the element is indeed shallow. Brebbia and DebNath[32] give a discussion of finite-element shell analysis exclusively from the view of shallow-shell theory.

4.2.3 Geometric Representation

The problem of geometric representation in finite-element shell analysis is one of its most important aspects and yet has drawn very little attention to date. Many shell element formulations pertain to specific geometric configurations (e.g., cylindrical shell elements) where no considerations arise with respect to definition of geometric parameters or the transformations needed to connect elements with differing curvatures, etc. In the general case, however, the problem does arise of defining not only the x, y, z coordinates at the element nodes but also the physical slopes and curvature parameters.

This problem has been examined by Key and Beisinger[45,46] and Lien.[47] Both developments employ bicubic polynomial interpolation and apply to the arbitrary quadrilateral shell element. Key and Beisinger's work is formulated in the context of isoparametric element concepts and utilizes bicubic interpolation in representation of geometric parameters. Lien's work[47] is similar but is phrased in terms of Coons' "surface patch" concepts[48] and presents the formulation of all necessary relationships and coordinate transformations in considerable detail. It should be noted that problems of curved surface representation via computer methods first drew importance in the early 1960s in conjunction with computer graphics (cathode-ray-tube applications), where the most extensive published work is due to Coons and his associates.

Isoparametric element concepts have also been applied to a triangular thin-shell finite element by Dupuis.[49,50]

Problems related to geometric representation are reduced when the element formulation is based upon shallow-shell theory. This eliminates difficulty in the establishment of curvilinear coordinate transformations. Nevertheless, care must be taken in the definition of proper coordinate transformations so that deep-shell behavior is reflected by the complete finite-element representation (see Ref. 51 for details), and the question of specifying slope and curvature parameters

on the basis of given problem data remains as it is in deep-shell analysis.

4.2.4 Quadrilateral Shell Elements

Four-sided shell elements have been formulated under a variety of restrictions, ranging from rectangular cylindrical segments to general shapes, as portrayed in Tables 5 and 6. A discussion of these tables, which pertain to developments in Ref. 45 to 47 and 52 to 61, is given below.

The cylindrical shell elements are described in Table 5. Olson and Lindberg[52] formulate the stiffness matrix for such an element using Love's strain-displacement equations. A 12-term polynomial is chosen to describe the radial displacement, while the membrane (tangential) displacements are each described by 8-term polynomials. The degrees of freedom at each joint are u, v, w, $\partial w/\partial x$, $\partial w/\partial y$, $\partial u/\partial x$, $\partial v/\partial y$, a total of 28 for the entire element. The rigid-body mode requirements are satisfied, but the displacement compatibility conditions are not.

The 12-term representation of radial displacement, together with bilinear functions for tangential displacements, is employed by Megard[53] in the formulation of a series of cylindrical elements. Strain-displacement relationships due to Donnell[39] are invoked. The deficiency of the latter, with respect to the condition of zero strain under rigid-body motion, has already been observed.

In order to achieve interelement compatibility of displacement for flat-plate elements together with inclusion of all constant strain states, a 16 degrees-of-freedom transverse displacement field is required. This function, which is a complete Hermitian polynomial expansion in two directions and is termed "bicubic," has been applied to the definition of radial displacements for cylindrical shell elements in Refs. 47, 54, 55, and 60–61.

Reference 54 formulates a cylindrical element with functions that are essentially linear in the axial and circumferential coordinates. The latter are modified in such a way, however, as to satisfy the requirement of zero strain under rigid-body motion for the element as a whole. Compatibility of displacement is not satisfied due to the disparity in the orders of polynomials used for radial and membrane displacement respectively. A careful numerical study is given to the significance of the rigid-body mode requirement, and the results show that element formulations that do not meet this requirement demonstrate poor convergence characteristics, although eventual convergence to the correct solution is demonstrated. This conclusion was recently substantiated by study of curved-element representations in arch analysis.

Bogner et al.[55] employed the complete bicubic expansion for all three displacement components, resulting in a 48 degrees-of-freedom formulation. This element comes close to meeting all rigid-body motion requirements and achieves excellent results in numerical comparisons. A further development of these concepts for sandwich cylinders appears in Ref. 62.

General quadrilateral shell-element formulations are summarized in Table 6. References 60 and 61 describe the development of a doubly curved shell element, lacking Gaussian curvature, using Novozhilov's strain-displacement equations and a bicubic field for radial displacement and a bilinear field for tangential displacements. Neither interelement displacement continuity nor the rigid-body motion requirements are met. More recently,[47] this work was extended to the choice of bicubic fields for all displacements (48 degrees of freedom as in Ref. 55) and to a general geometric form. The results obtained in numerical comparisons appear to be the most accurate yet achieved by quadrilateral thin-shell elements for a given grid refinement.

Connor and Brebbia[56] employed a 12-term polynomial to represent transverse displacements and bilinear representations for the tangential displacements. Reissner's shallow-shell theory[63] is invoked.

Wempner, Oden, and Kross[57] derived a linear theory for the deformation of thin shells, including transverse shear deformation, in terms of middle surface displacements and the rotations of normals to the middle surface. These displacement parameters, for a curvilinear quadrilateral element, are then represented by simple bilinear polynomials in the curvilinear surface coordinates. Continuity of displacement is imposed by matching the displacements and rotations at four points on the boundary of the element, while a discrete equivalent of the Kirchoff hypothesis is introduced to assure that the Kirchoff theory is approached in the limit. Thus the method has been termed a "discrete-Kirchoff" approach.

Another approach to the general quadrilateral thin-shell element is due to Key and Beisinger,[45,46] who derive a potential energy functional that includes transverse shear effects. In Ref. 46 they chose 12-term polynomials in representation of radial and tangential displacements and a biquadratic polynomial in description of fiber rotations; conditions of interelement displacement continuity and on rigid-body motions are satisfied.

Table 5. Circular cylindrical shell elements

Reference	Joint degrees of freedom	Assumed functions	Remarks
Olson and Lindberg[52]	u, v, w, w_x, w_y, u_x, v_y (28 terms)	w – 12-term polynomial	Includes vibrational frequency analysis
Megard[53]	u, v, w, w_x, w_y (20 terms)	Basically u, v – bilinear, w – 12-term polynomial, but coupled to satisfy rigid-body motion conditions	
Cantin and Clough[54]	u, v, w, w_x, w_y, w_{xy} (24 terms)	w – bicubic polynomial, u, v – bilinear polynomials (plus coupling terms)	Satisfies rigid-body motion requirements
Bogner et al.[55]	u, v, w, u_x, u_y, u_{xy}, u_x, v_v, v_{xy}, w_x, w_y, w_{xy} (48 terms)	w, u, v } bicubic polynomials	

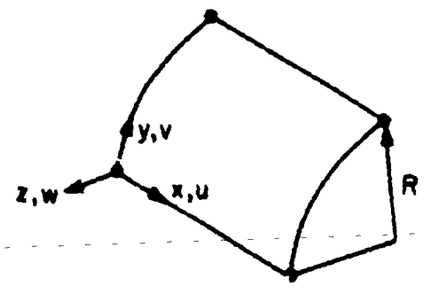
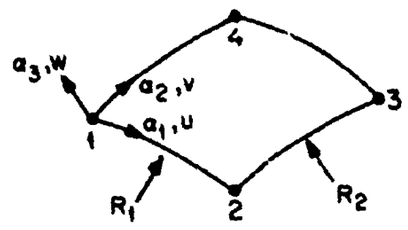


Table 6. Quadrilateral shell elements

Reference	Degrees of freedom	Assumed functions	Remarks
Connor and Brebbia[56]	u, v, w, w_x, w_y (20 terms)	w – 12-term polynomial u, v – bilinear polynomial	Rectangular (includes R_x, R_y, and R_{xy}); Reissner shallow-shell theory[63]
Wempner et al.[57]	u, v, w, w_x, w_y (20 terms)	u, v, w – bilinear polynomials	Develops shell theory to include transverse shear
Ahmad et al.[58] Pawsey and Clough[59]	Numerous representations are given	Dependent upon chosen representation	Isoparametric element concept; obtains shell element by applying special conditions to solid elements
Key and Beisinger[45,46]		u, v, w – 12-term polynomials plus biquadratic angular displacement fields	Washizu[25] shell equations (includes transverse shear)
Gallagher,[60] Gallagher and Yang[61]	u, v, w, w_x, w_y, w_{xy} (24 terms)	w – bicubic polynomial, u v – bilinear polynomial	Principal curvatures (R_x and R_y) only; Ref. 61 covers linear instability
Lien[47]	(48 terms)	u, v, w – bicubic polynomials	Includes all curvatures



Greene, Jones, and Strome[64] describe an approach to shell analysis based on the "generalized variational principle."[65] Strain-displacement equations are taken from Novozhilov's[24] nonshallow-shell theory, and the displacements are represented by cubic polynomials in two variables.

The isoparametric element concept in shell analysis is exploited fully by Ahmad et al.[58] and also by Pawsey and Clough.[59] This is accomplished by degenerating the isoparametric solid element formulations by introduction of assumptions that original normals to the middle surface are inextensible and straight and that the elastic modulus in the normal direction is zero. These assumptions allow retention of a measure of the transverse shear deformation. Since prior developments for solid elements are drawn upon, a wide variety of functional representations for displacement (with appropriate definitions of joints at the corners and along the sides) are immediately available. The approach adopted in Ref. 58 yields poor results for thin shells. Recent work,[59,66] however, demonstrates that this problem may be resolved via a scheme of "reduced order" integration in the evaluation of the element stiffness coefficients.

The above element formulations are based exclusively upon assumed displacement fields. Since interelement displacement continuity, which is a principal motivation for a displacement-based (potential energy) formulation, is rarely achieved, a "mixed method" of element formulation is attractive. Mixed methods permit the selection of simple forms of displacement representation and introduce assumed stress fields as well.

A variational principle for mixed formulation has been defined by Herrmann,[67] who subsequently employed it for a doubly curved shell element representation.[68] Similar developments have been advanced by Eatcock-Taylor[69] and by Atluri.[70]

4.2.5 Triangular Shell Elements

Triangular shell element formulations are summarized in Table 7 and described as follows. Utku[71] incorporates representations of linear displacement fields for the coordinate displacements u, v, and w and for two angular displacements as well. Shallow-shell theory is employed, and transverse shear deformations are taken into account. The resulting stiffness matrix is algebraically simple, and extensive numerical results are presented.

The displacement field originally proposed in Ref. 84 for flat-plate flexure is employed by Strickland and Loden[72] for a description of the radial displacements in a shallow-shell theory[24] formulation of a triangle. The tangential displacements are described with use of linear displacement fields.

Cowper et al.,[51,73] using both shallow- and deep-shell theory, describe the transverse displacement with use of an incomplete quintic, and the tangential displacement fields by complete cubic, polynomials. The element features 36 degrees of freedom, is conforming, and includes all rigid-body modes and fundamental strain states. Transformation of the relationships to cylindrical shell theory is presented in Ref. 74.

The complete quintic polynomial is adopted in representation of all three displacement fields in the SHEBA-6 element of Argyris and Scharpf.[75] Nonshallow-shell theory and a special "natural" coordinate system is employed. The element is presented in terms of 63 degrees of freedom – 18 at each vertex (the function and all of its first and second derivatives for each of the three functions) and 3 at the midpoint of each side (the angular displacement in the direction of the normal to the side for each function). The formulation is quite general with respect to the description of geometry.

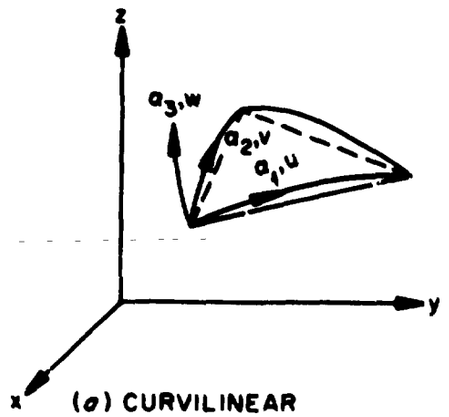
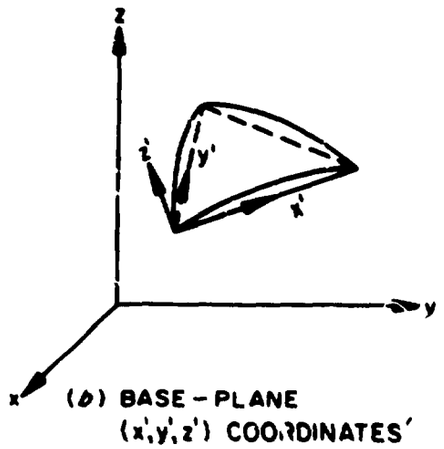
The approach to the formulation of a triangular shell element that incorporates transverse shear deformation via the "discrete-Kirchoff" hypothesis is presented by Dhatt.[76,77]

The formulation of Bonnes et al.,[78] based on shallow-shell theory, divides the element into three triangular subregions and describes the radial and tangential displacement fields with complete cubic polynomials within the subregions. This approach is a generalization of the procedure employed by Clough and Felippa[79] for the flat triangle. Upon elimination of all interior degrees of freedom, the element stiffness matrix contains 36 degrees of freedom – 9 at each corner and 3 at the midpoints of each side. Neither interelement displacement compatibility nor rigid-body mode requirements are satisfied completely.

A triangular element that meets the requirements associated with interelement continuity of displacement and rigid-body motion is presented by Dupuis and Goel[49] and by Dupuis.[50] This development is based on the relatively sophisticated Koiter-Sanders shell theory[37,42] and, together with the SHEBA element formulation of Argyris and Scharpf,[75] is generally regarded as the most sophisticated triangular shell element formulation.

Hermann and Campbell[80] employed flat triangular elements, drawing the flexural behavior representation from Ref. 67. An interesting feature is the description

Table 7. Triangular shell elements

	Reference	Joint degrees of freedom	Assumed function	Remarks
 (a) CURVILINEAR COORDINATES	Utku[71]	u, v, w, θ_x, θ_y	Linear u, v, w and linear θ_x, θ_y	Marguerre[44] shallow-shell theory
	Strickland and Loden[72]	u, v, w, w_x, w_y	w -- same as Bazeley et al.[54] for flat plate; u, v, linear	Novozhilov[24] shallow-shell theory
	Cowper et al.[51,73,74]	u, v, w, u_x, u_y, v_x, v_y, w_x, w_y, w_{xx}, w_{xy}, w_{yy}	w -- complete quintic polynomial; u, v -- cubic functions	Two-shell theories are available as options: Koiter[37] (general), Novozhilov[24] (shallow)
	Argyris and Scharpf[75]	u, v, w, and all first and second derivatives at vertices plus angular displacement at midpoint of sides	Complete quintics	
	Dhatt[76,77]	u, u_x, u_y, v, v_x, v_y, w, β_x, β_y	Variety of assumptions in respective papers in framework of discrete Kirchoff[57] approach	Shallow-shell theory
 (b) BASE-PLANE (x',y',z') COORDINATES	Bonnes et al.[78]	u, v, w and all first derivatives at vertices plus u_x and v_y at midpoints of sides	Complete cubics for u and v and subregion[79] representation	Reissner[63] shallow-shell theory
	Dupuis[49,50]	u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, v, v_x, v_y, v_{xx}, v_{xy}, v_{yy}, w, w_x, w_y, w_{xx}, w_{xy}, w_{yy}	Option of rational functions or quintic polynomial	Koiter-shell theory;[37] rigid-body motion conditions are satisfied exactly
	Connor and Will[82]	u, v, w, m_x, m_y, m_{xy}	Various combinations of linear and quadratic functions for u, v, and w	Hellinger-Reissner variational principle and Reissner shallow-shell theory[63]
	Visser[83]	u, v, w, m_x, m_y, m_{xy}	Quadratic u, v, w; linear m	Hellinger-Reissner variational principle and Koiter-shell theory[37]

of applications to complicated practical problems. Prato[81] effects a modification of Herrmann's variational principle[67] and treats a triangle based on linear expansions in all variables. Connor and Will[82] utilize shallow-shell theory[44] and generate five different triangular elements based on different assumed stress and displacement fields.

Another mixed formulation has been presented by Visser,[83] based on Koiter's strain-displacement relationships[37] reduced to shallow-shell form. Parabolically varying displacement fields are combined with linearly varying moment distributions.

4.2.6 Summary

The requirements for valid minimum potential energy solutions in thin-shell finite-element analysis are extremely difficult to satisfy. For this reason the development period has been very long and has involved many contributions, and the satisfactory formulations are relatively complicated. The most reliable and sophisticated triangular shell element formulations are those due to Dupuis,[50] Cowper et al.,[51] and Argyris and Scharpf,[75] but it should be noted that not all of these adhere completely to the principle of minimum potential energy. It should also be observed that recent work[35,36] gives greater confidence in the usefulness of the well-established flat elements for thin-shell representation, and still other recent work[59,66] has delineated the manner in which isoparametric solid elements can be adapted to thin-shell analysis.

Because of the relative sophistication of displacement-based shell representations of any type – curved thin elements, flat elements, or isoparametric solids – a great deal of attention has been drawn to mixed variational principles, which permit the use of simpler assumed fields in the form of either stresses alone or both stresses and displacements. The accuracy and reliability of these schemes have also been verified, but a principal difficulty in practical application at this point is their amenability to incorporation in existing large-scale general-purpose programs.

There are a number of unfulfilled requirements in finite-element shell analysis, quite apart from the matter of basic theoretical formulation of elements, as follows:

1. Continued study is needed of geometric representation for curved-shell elements, although there has been substantial progress in this respect as outlined in Section 4.2.3.

2. The problem of representation of discrete stiffeners, including eccentricity effects, has not yet been given sufficient attention. Experience in numerical applications is particularly important here, since there are numerous alternative approaches and simplifications.

3. The need for experimental data for verification of finite-element analysis continues to grow, with only small progress toward the satisfaction of such requirements. Finite-element formulations are traditionally verified by comparisons with classical solutions. It is presumed, nevertheless, that the method is a tool for solution of complex problems where no alternative solution procedures are applicable. Verification must therefore be accomplished in application to such complex structures and the test procedures and arrangements must be of a form appropriate to the finite-element solution format.

4. Tradeoffs between element formulative effort (sophistication in the definition of element behavior and geometry) and the number of equations to be solved (simplicity of element formulation) are not yet clearly established. Comparisons of these alternatives, to be valid, must include not only the operational costs to reach a desired level of solution accuracy, but must also reflect an amortization of costs to develop the associated software.

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5. Inelastic Analysis - Time Independent

Inelastic analysis, which during the formative years of the finite-element method drew little attention, has emerged as one of its foremost capabilities. This results from the recognition that the method, represented by large-scale general-purpose programs, is the only workable procedure for inelastic analysis of practical structures. Coincidentally, design requirements have emerged for such analyses, especially in the subject area of nuclear reactor structures.

The principal challenges in the extension of linear finite-element analysis to inelastic problems have concerned the establishment of a proper relationship with classical plasticity theory and the formulation of efficient and reliable algorithms for solution of the associated large-order systems of algebraic equations which are at least implicitly nonlinear. Explicit consideration of incremental plasticity theory in finite-element analysis dates from only 1965. Algorithms for solution of the associated nonlinear algebraic equations are still being developed and refined.

Since the constitutive relationships are basic to finite-element elastoplastic analysis, the matrix format of such relationships and their transformation to element equations will be discussed first. The element formulations are discussed only from the view of stiffness (displacement method) relationships, since all but the most recent work and all experience in practical application are phrased in terms of this approach.

The general approaches (system algorithms) for finite-element elastoplastic analysis are discussed next. Two principal approaches have enjoyed independent development and application: (1) the tangent stiffness method and (2) the initial strain procedure. The tangent stiffness method involves a redefinition of the system stiffness matrix in each analysis cycle. The initial strain method preserves the system stiffness matrix (if temperature remains constant as load is changed) and represents the inelastic effects in the manner of initial strains.

A special section is devoted to finite-element elastoplastic analysis via flexibility (complementary energy) concepts. Only research work has been published in this direction, but the approach appears to have considerable promise for a number of reasons. The approach makes direct reference to as-measured constitutive relationships, is simpler in formulation for plate flexure, and possesses a potential for upper bound solutions of the limit analysis type.

The final section of this chapter examines applications experience. The accompanying tabular representations are divided into considerations of tangent stiffness and initial strain applications, respectively, which indicate a somewhat even division of popularity in practice.

General reviews of finite-element elastoplastic analysis have been published by Oden,[1] Marcal,[2] and Argyris et al.[3] A more general review of computational considerations in elastoplastic analysis of pressure vessels is found in the text edited by Gill.[4] Hodge[5] also discusses plastic behavior with reference to pressure vessel design, but without consideration of computational approaches.

5.1 CONSTITUTIVE RELATIONSHIPS

Basic relationships in plasticity theory are developed in detail in many texts, including Refs. 6 through 8. A particularly good basic treatment is presented by Johnson and Mellor,[9] and readable development has also been prepared by Merkle.[10] A

detailed description of constitutive equations to be used on an interim basis for analysis of liquid-metal fast breeder reactor components has been presented by Pugh et al.[11] This section merely outlines constitutive relation considerations to an extent necessary for identification of related facets in element stiffness formulation.

In constructing the relationships between stress and strain in the inelastic range for multiaxial states of stress, one must at least establish the condition for yielding the general form of the desired stress-strain law and a criterion for work hardening.

Consider first the yield criterion. Although alternative forms are available and have been employed in finite-element analysis, the nearly universal choice is the relationships due to von Mises for the increment in plastic strains $\{d\epsilon_p\}$ in relation to the yield surface. Based on the flow rule of von Mises, equivalent stress and strain concepts, and the use of the von Mises formulation for the yield, or plastic potential, function, the following relationship is obtained:

$$\{d\epsilon_p\} = d\bar{\epsilon}_p \left\{\frac{\partial\bar{\sigma}}{\partial\{\sigma\}}\right\} . \tag{1}$$

where $\bar{\epsilon}_p$ is the equivalent plastic strain and $\bar{\sigma}$ is the equivalent stress. We note also the incremental form of the von Mises yield criterion.

$$d\bar{\sigma} = \left|\frac{\partial\bar{\sigma}}{\partial\{\sigma\}}\right| \{d\sigma\} = H' \, d\bar{\epsilon}_p , \tag{2}$$

where H' is the slope of the equivalent stress–equivalent plastic strain ($\bar{\sigma} - \bar{\epsilon}_p$) curve. This curve, established by testing of a uniaxial specimen, is the link to multiaxial behavior. The relation between increments of stress and strain can always be written in the form

$$\{d\sigma\} = [E] \left\{\{d\epsilon\} - \{d\epsilon_p\}\right\} \tag{3}$$

and by suitable operations and combination of (1) through (3), one obtains

$$\{d\sigma\} = \left[[E] - \frac{[E]\left\{\frac{\partial\bar{\sigma}}{\partial\sigma}\right\}\left|\frac{\partial\bar{\sigma}}{\partial\sigma}\right|[E]}{H' + \left|\frac{\partial\bar{\sigma}}{\partial\sigma}\right|[E]\left\{\frac{\partial\bar{\sigma}}{\partial\sigma}\right\}}\right]\{d\epsilon\} , \tag{4a}$$

or

$$\{d\sigma\} = \left[[E] - [E_p]\right]\{d\epsilon\} = [E]\{d\epsilon\} - [E_p]\{d\epsilon\} , \tag{4b}$$

$$= [E_{ep}]\{d\epsilon\} , \tag{4c}$$

with

$$[E_p] = \frac{[E]\left\{\frac{\partial\bar{\sigma}}{\partial\sigma}\right\}\left|\frac{\partial\bar{\sigma}}{\partial\sigma}\right|[E]}{H' + \left|\frac{\partial\bar{\sigma}}{\partial\sigma}\right|[E]\left\{\frac{\partial\bar{\sigma}}{\partial\sigma}\right\}} . \tag{5}$$

The matrix $[E_{ep}]$ is termed the elastoplastic, or tangent, material stiffness matrix. It will be noted that the symmetry property of a material stiffness matrix is preserved and that perfect plasticity ($H' = 0$) presents no special difficulty. It is pertinent to note that original efforts in formulating a valid incremental plasticity finite-element procedure[12] established the strain vs stress form of the constitutive relationships (their basic form). The inverse form, required in any displacement-based procedure, was then obtained via numerical inversion.

In cases where a function other than the effective stress defines yield, Eq. (5) is generalized simply by replacing $\bar{\sigma}$ by the appropriate yield function f. As in the case of elasticity, specific forms of Eq. (5) are established for specific circumstances such as plane stress and plane strain. Detailed representations of these are given in Refs. 13 through 15, which also include specific developments for orthotropic materials in accordance with Hill's postulate.[8]

More recently, attention has been given to the extension of the above concepts to a continuum representation of the influence of temperature on yield. If the yield function is now taken to be a function of both the accumulated plastic strain and temperature T, we have, for the differential of the yield function,

$$df = \frac{\partial\bar{\sigma}}{\partial\epsilon_p} d\epsilon_p + \frac{\partial\bar{\sigma}}{\partial T} dT , \tag{6}$$

and the conventional procedure for establishment of incremental stress-strain equations leads to a relationship with a redefined H'. This development is

presented by Yamada.[16] Another treatment of this topic has been published by Radaj[17] and by Argyris et al.[3]

The problem of representation of strain hardening is a key aspect of the constitutive law. Two forms of strain hardening law are commonly employed: the isotropic hardening law, wherein the loading surface expands during loading, and the kinematic hardening law, where it translates as a rigid body. The foregoing treatment is referenced to isotropic hardening. Incremental stress-strain representations for kinematic hardening are detailed by Kamel and Sack,[18] Armen et al.,[19,20] Zudans,[21] and Jordan.[22]

Kinematic hardening, introduced by Prager,[23] has been modified by Ziegler,[24] and it is the latter form that has been adopted by finite-element theorists cited above. It would appear that if a hardening representation which is referenced to the concept of a yield potential is at all realistic, then actual material behavior will lie somewhere between the limits of isotropic and kinematic hardening. A combined hardening law can be constructed for finite-element analysis, but to the writer's knowledge, no numerical application of such a law has been published.

The objective in constructing the material tangent stiffness matrix is to form an element stiffness matrix. As outlined in Chapter 2, this follows directly, in displacement-based element stiffness formulations, via the expression

$$[k_{ep}] = \int_v [D]^T [E_{ep}] [D] \, dV \, , \qquad (7)$$

where $[k_{ep}]$ is the element elastoplastic stiffness matrix and $[D]$ is the appropriate strain-node displacement transformation matrix.

To establish the form of element equations for initial strain analysis, reference is made to Eq. (4b), from which $[E_{ep}] = [E] - [E_p]$. Substituting this expression into Eq. (7), we have

$$[k_{ep}] = \int_v [D]^T [E] [D] \, dV - \int_v [D]^T [E_p] [D] \, dV \, . \qquad (7a)$$

The first term on the right-hand side is the linear *elastic* stiffness matrix $[k]$ (see Eq. 15, Chap. 2). The second term on the right-hand side gives what may be termed the element *plastic* stiffness matrix,

$$[k_p] = - \int_v [D]^T [E_p] [D] \, dV \, , \qquad (7b)$$

and the product of $[k_p]$ by the element displacements gives a vector of forces

$$\{F_p\} = [k_p] \{\Delta\} \, . \qquad (8)$$

In the initial strain approach an estimated value of $\{\Delta\}$ (or, synonymously, of plastic strains, since $\{\epsilon_p\} = [D] \{\Delta\}$) is employed to construct the vector $\{F_p\}$, and this vector is treated as if it were the set of initial forces arising from "initial strains" $\{\epsilon_p\}$.

An interesting question that arises in conjunction with Eq. (8) is the approach to evaluation for complex elements. In the case of the basic triangle in plane stress, the analyst will adopt the concept of a fully plastified element due to the assumed constant strain state, and the evaluation of Eq. (8) presents no difficulty. For complex elements, for example, isoparametric solids and higher-order elements, the proper approach to evaluation is not so clearly defined. In such cases the analyst is confronted with the problem of partially plastified elements. These considerations have been discussed in detail by Argyris et al.[3] and by Zienkiewicz and Nayak.[25]

In the same vein, the distribution of inelastic stresses throughout plate and shell bending elements represents a complicated numerical evaluation problem. Displacement formulations for such elements are inherently higher order (see Chap. 4), and, in addition, one must account for variation of plastic deformation across the thickness. Some authors (e.g., Refs. 2, 26, 27) prefer to divide the thickness into a number of layers, each of which acts as a plane in plane stress with respect to the growth of plasticity. Others (e.g., Refs. 3, 28) assume a variation of the plastic zone throughout the thickness and across the surface of the element.

The foregoing comments mainly apply to formalized descriptions of the material constitutive relationships. The manner of representation of the uniaxial data for the purpose of defining the effective stress-strain law must also be considered. The Ramberg-Osgood[29] functional representation has enjoyed utilization (Refs. 30–32), while other authors deal with piecewise[33] or other representations of tabulated data. Actually, only a bilinear stress-strain representation can lead to a theoretically consistent kinematic hardening formulation.[11]

5.2 FORMULATIVE APPROACHES

5.2.1 Tangent Stiffness

The most apparent form of finite-element analysis for elastoplastic analysis would seem to be an approach that revises the elastic element stiffness matrix to account for the inelastic material properties. Such a revision is possible if inelastic constitutive relationships are available which define a linearized stress-strain behavior at any point in the loading history. As shown in the previous section, these relationships are available in classical plasticity theory, and their direct transformation to finite-element analysis comprises the tangent stiffness approach.

Since the tangent stiffness approach is a direct utilization of classical plasticity concepts, it is somewhat surprising that the approach appeared for the first time only in 1965. Related concepts were introduced by Pope,[34] amplified and improved upon in 1967 by Marcal and King[12] and by Yamada,[13] and have since been expanded and applied by many analysts whose contributions will be discussed in Section 5.4.

The basic scheme in tangent stiffness analysis proceeds as follows:

1. An elastic analysis is performed for an arbitrary load intensity, and this intensity is scaled to the level at which the yield criterion is satisfied.
2. An increment of loading is selected for the first passage into the inelastic range.
3. An estimate is made of the strain increments caused by the load increment of step 2, plastified elements are identified, and tangent stiffnesses are calculated for such elements and incorporated in the system stiffness matrix.
4. The load increment of step 2 is applied to the revised system stiffness matrix of step 3, and stress and strain increments are calculated.
5. Stresses and accumulated plastic strains are updated consistent with the results of the prior step.
6. Another increment of load is selected and steps 2 through 5 are repeated. The process of load incrementation is continued until the maximum value of load is reached or until limited by collapse or some similar phenomena.

The determination of a collapse situation has been interpreted by some authors[12,13] to occur when the change in effective plastic strain is negative.

In the case of the general nonlinear hardening situation, it is in theory necessary to iterate within a given load increment (steps 3 and 4) to establish a consistency of the tangential stiffness of yielded elements. It is normally sufficient, however, to work with an estimated tangent stiffness for the interval. This bears a relationship to the chosen size of load increment, which is discussed below.

A major aspect in the definition of load increments is the manner in which new plastified elements are introduced into the stiffness matrix. Yamada et al.[13] present a careful procedure in which the load increment is adjusted to bring in plastified elements one at a time. Marcal and King,[12] on the other hand, apply relatively large increments and delineate an approximate way of accounting for the "transitional" elements, those which enter the plastic range during a load increment.

Although nearly all the theoretical developments and applications of the tangent stiffness approach have favored the von Mises yield condition and isotropic hardening, departures from these have been recorded. The Tresca yield condition has been invoked by Anand et al.[33] As previously noted, kinematic hardening has been employed by the authors of Refs. 18–22, among others.

5.2.2 Initial Strain

Mendelson and Manson[35] were perhaps the first to characterize the inelastic analysis of multi-degree-of-freedom problems as initial strain procedures. Their analytical model emerged from finite-difference approximations to the governing differential equations. Subsequently, Gallagher, Padlog, and Bijlaard[36] defined an approach to finite-element analysis through initial strain concepts. This approach, which did not incorporate considerations of incremental plasticity and implied only a single iteration within each load increment, received acceptance among practitioners in the period 1963 to 1967 and has been the subject of careful convergence studies by Lansing et al.,[30] Mentel,[37] and Argyris and Scharpf.[38] The appearance of the tangent stiffness method, phrased in terms of classical plasticity concepts, drew attention away from the initial strain method, although it was at first assumed that the latter was incapable of accommodating such concepts.

The basic concept of "initial strain" procedures is to define a reference elastic modulus, with reference elastic strains, and to treat the departures from linearity as initial strains. The essentials of this

concept in the format of the constitutive relationships and element force-displacement equations were discussed previously. We now consider the algorithm for the complete system. The most elementary form of initial strain analysis, mentioned above, proceeds as follows (see Fig. 10).

1. Consider the application of a load $\{P_k\}$ of an intensity such that an initial excursion is made into the plastic range. From Eq. (4), Chapter 2, $\{\Delta_k\} = [K]^{-1} \{P_k\}$, with elastic strains and stresses $\{\epsilon_k\} = [D] \{\Delta_k\}$ and $\{\sigma_k\} = [S] \{\Delta_k\}$ [Eqs. (12) and (5), Chap. 2].

2. By reference to tabular information regarding the inelastic stress-strain law, the Ramberg-Osgood format,[29] or any other analytical representation, compute the stress $\{\sigma_k'\}$ corresponding to the strain $\{\epsilon_k\}$.

3. Compute, elastically, the strain $\{\epsilon_k'\}$ corresponding to $\{\sigma_k'\}$ through application of the linear constitutive relationships, $\{\epsilon_k'\} = [E]^{-1} \{\sigma_k'\}$.

4. Determine initial strains as $\{\epsilon_k^i\} = \{\epsilon_k\} - \{\epsilon_k'\}$.

5. Increase load to $\{P_{k+1}\}$, compute initial force $\{P_k^i\}$ based on $\{\epsilon_k^i\}$, and compute the associated displacements, stresses, and strains:

$$\{\Delta_{k+1}\} = [K]^{-1} \{P_{k+1}\} + [K]^{-1} \{P_k^i\} ,$$

$$\{\sigma_{k+1}\} = [S] \{\Delta_{k+1}\} - [E] \{\epsilon_k^i\} ,$$

$$\{\epsilon_{k+1}\} = [E]^{-1} \{\sigma_{k+1}\} + \{\epsilon_k^i\} .$$

6. Referring again to constitutive equations, the stress for $\{\epsilon_{k+1}\}$ is determined as $\{\sigma_{k+1}'\}$. The new accumulated plastic strain is $\{\epsilon_{k+1}^i\} = \{\epsilon_{k+1}\} - [E]^{-1} \{\sigma_{k+1}'\} = \{\epsilon_{k+1}\} - \{\epsilon_{k+1}'\}$. The process continues via repetition of steps 5 and 6.

A key operation in the above procedure is the calculation of the stress level for a given strain, the so-called "constant strain" form of initial strain analysis. Alternatively, one may calculate the strain for a given stress, furnishing the "constant stress" method that is shown schematically in Fig. 11 and described in detail in Ref. 30.

The constant stress and constant strain alternatives were originally published[36] without recommendation of one in preference to the other. Subsequently applications in practice disclosed serious difficulties in the constant stress method in achieving convergent solutions. This is not unexpected, as seen in Fig. 11, since the nature of real materials in the inelastic range is such that very large initial strains result from small load increments.

A simple computer program for elastoplastic plane stress analysis based on the above has been published by Salmon et al.[31] This program operates in the initial strain mode but refers to incremental plasticity constitutive relationships, employs a Ramberg-Osgood[29] representation of the effective stress-strain law, and iterates within each load increment.

A more sophisticated form of initial strain procedure, which accounts for incremental plasticity theory directly and provides for iteration within the load

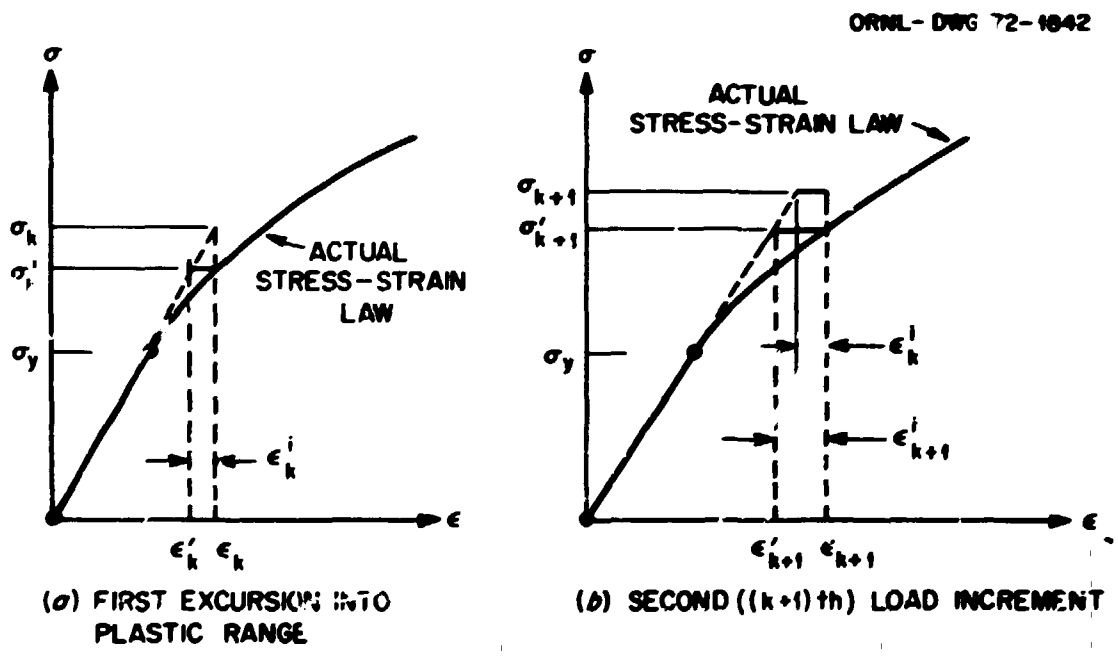

Fig. 10. Constant strain algorithm.

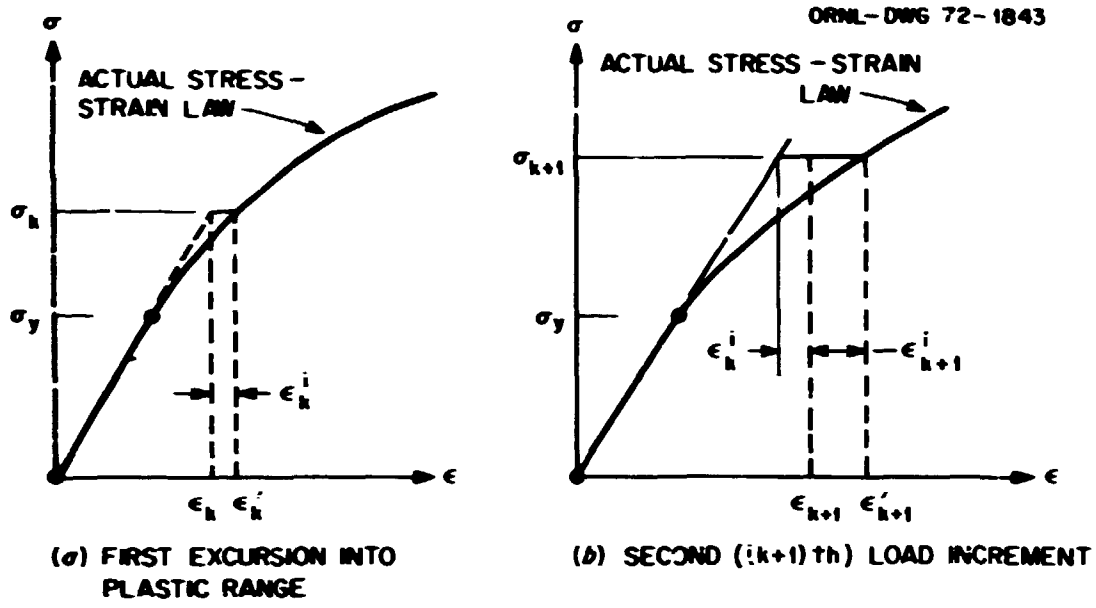

Fig. 11. Constant stress algorithm.

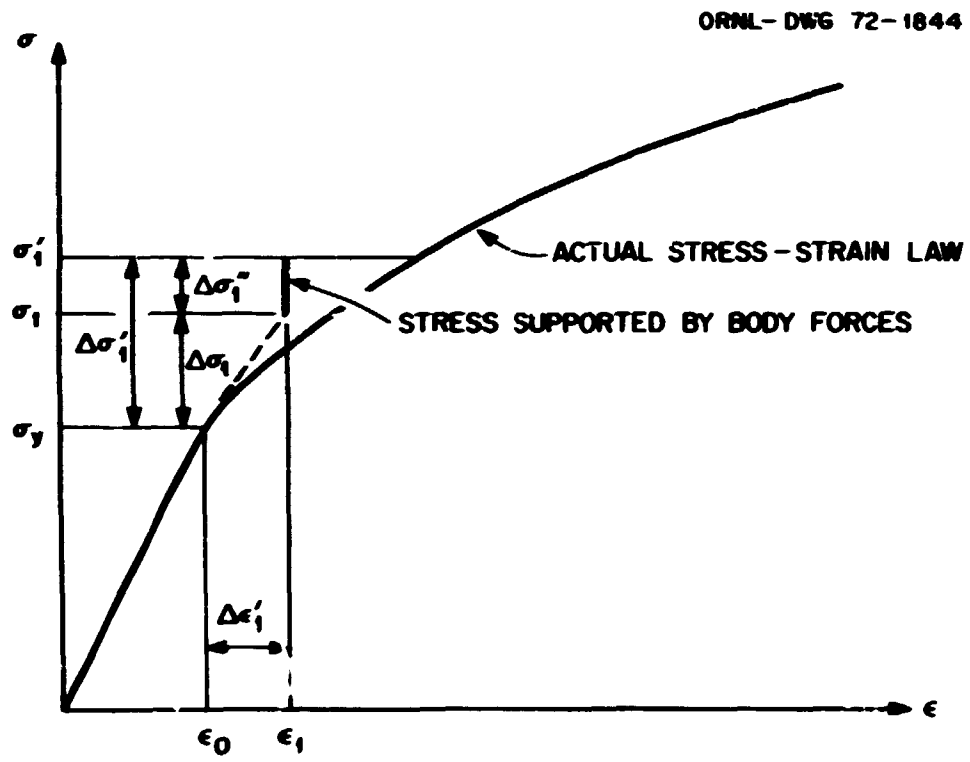

Fig. 12. Initial stress algorithm.

increment, is given below (see Fig. 12). It is assumed that the load level at which inelastic deformation is initiated has been identified and that an increment of load has been selected for the first excursion into the plastic range.

1. Apply load increment and determine $\{\Delta\sigma'\}_1$ and $\{\Delta\epsilon'\}_1$ elastically (in the following, primes denote an elastic computation).

2. Add $\{\Delta\sigma'\}$ to stresses at start of interval ($\{\sigma_0\}$) to form $\{\sigma'\}_1$, where $\{\sigma'\}_1 = \{\sigma_0\} + \{\Delta\sigma'\}_1$.

3. Calculate a first estimate of the stress change due to elastoplastic behavior within the interval, using a representative elastoplastic material stiffness matrix $[E_{ep}]$ and the already-calculated strain increment $\{\Delta\sigma\}_1 = [E_{ep}]\ \{\Delta\epsilon'\}$.

4. Correct the prior solution to account for the discrepancy between the elastic stress and the stress estimate of step 3. Thus, evaluate stress that can be regarded as being supported by "body forces," $\{\Delta\sigma''\}_1 = \{\Delta\sigma'\}_1 - \{\Delta\sigma\}_1$. Also define the current stress and

strain:

$$\{\sigma\}_1 = \{\sigma'\}_1 - \{\Delta\sigma''\}_1 = \{\sigma\}_0 + \{\Delta\sigma\}_1 ,$$

$$\{\epsilon\}_2 = \{\epsilon\}_0 + \{\Delta\epsilon'\}_1 .$$

5. Using Eqs. (7b) and (8) and noting that for this computation $\{\Delta\sigma''\} = [E_p]\,[D]\,\{\Delta\}$, compute the element initial forces $\{F^i\}_1$ due to the stress supported by body forces,

$$\{F^i\}_1 = \int_v [D]^T \{\Delta\sigma''\}\, dV .$$

Form a "system" vector of such forces, $\{P^i\}_1$.

6. Calculate the changes in displacement, stress, and strain due to $\{P^i\}_1$:

$$\{\Delta\}_1 = [K]^{-1} \{P^i\}_1 ,$$

$$\{\Delta\sigma'\}_2 = [S]\,\{\Delta\}_1 - \{\Delta\sigma''\}_1 ,$$

$$\{\Delta\epsilon'\}_2 = [D]\,\{\Delta\}_1 .$$

Steps 2 through 6 are repeated until the stress change computed in step 6 is acceptably small, and the load is then incremented again.

The above concepts, presented in Ref. 39 by Zienkiewicz et al. (a simpler, one-dimensional form appears in Ref. 40) have revitalized the initial strain concept as an approach to finite-element elastoplastic analysis and have assisted in the establishment of the approximate balance between tangent stiffness and initial strain applications that now exists. Reference 39 terms this scheme an "initial stress" procedure, since it is referenced to the correlation of an unequilibrated stress distribution and because the initial forces (step 5) are computed directly from stress parameters.

Care must be exercised in the above to establish $[E_{ep}]$ in step 3 on a realistic basis. Reference 39 specifies a choice based upon the elastic stress, $\{\sigma'\}_1$. Clearly, this is an overestimate, and it is desirable to base $[E_{ep}]$ on an average value within the interval. This cannot be done in closed form, but an approximate scheme can readily be devised.[41]

No difficulty is occasioned in the above procedure due to cyclic plasticity. Reference 39 describes a cyclic analysis, and the work of Armen et al.[19] describes further applications to such phenomena.

Many attempts have been made to correlate the initial strain approach and the tangent stiffness method. Marcal[42] succeeded in establishing the relationship between the simpler form of initial strain analysis and the tangent stiffness approach; further comments on his work are offered by Salmon.[43] Argyris and Scharpf[38] and Zienkiewicz and Nayak[25] also examine this question.

5.3 COMPLEMENTARY PROCEDURES

Complementary procedures in finite-element analysis, that is, procedures that are founded principally in assumed stress fields, have not made significant inroads into the practice of elastic finite-element analysis. (Their theoretical base is discussed in Chap. 2.) Nevertheless, since these procedures hold certain promise for inelastic finite-element analysis, they have recently drawn considerable interest.

As noted in Chapter 2, a complementary energy formulation in which stress functions are chosen as primary (joint) unknowns has special advantage because of the correspondence of the assumed functions and resulting equation coefficients with particular aspects of displacement-based formulations. Rybicki and Schmit[44] are apparently the first to have applied these ideas to elastoplastic analysis. They deal with orthotropic plane stress and employ the Prandtl-Reuss incremental stress-strain relations in an initial strain format.

The element represented in Rybicki and Schmit's development is a rectangle with a 36 degrees-of-freedom representation of the Airy stress function (fifth-order Hermitian polynomial interpolation). Clearly, the strain field in any such element can vary from elastic to plastic in a complex manner and must be treated via numerical integration of the element initial forces. The use of an element with this many degrees of freedom enables direct treatment of all boundary conditions but is likely to be more sophisticated than is required by overall structural idealization requirements.

A simpler representation, defined also for plane stress and in terms of the Airy stress function, has been presented by the writer and Dhalla.[41] This formulation of the rectangular element, with 16 degrees of freedom, requires careful attention in the treatment of stress (force) boundary conditions.

An alternative complementary approach, based upon direct representation of stress parameters, has been explored by Belytschko et al.[45-47] in a series of papers pertaining to both plane stress and flexure. The advantage of element matrices that are identical to those in conventional stiffness analysis are lost in this scheme, but the matrices required are nevertheless simple in form.

5.4 APPLICATIONS EXPERIENCE

In this section analyses of simple problems with known solutions are given for comparison purposes and for use in conducting practical analyses.

Very few comparison solutions of sufficient accuracy and detail are available for verification purposes. The test data of Theokaris and Marketos[48] for a rectangular, uniformly loaded plate with central circular hole, of Swedlow and Yang[49] for a similar problem, and of Percy et al.[50] for a plate with nonuniform stiffener subjected to a concentrated load are the most significant and most widely employed such comparison bases. Among classical solutions, the cantilever beam problem[51] and the simply supported beam problem[26] furnish substantive data. These problems are employed continually in the investigations summarized in this section.

In discussing applications experience it is useful to summarize available references through the medium of a tabular representation. Tables 8 and 9 give this information for publications related to the tangent stiffness and initial strain methods respectively. The types of elements employed, the constitutive law invoked, and the problems solved are delineated. It is not possible to represent each entry with the same detail. In the work of Argyris et al., referred to in Table 9, for example, the full range of elements, methods of approach, and applications is explored, but the interested reader must consult the cited Refs. 3, 35, 55, and 56 for relevant details.

Table 8. Finite-element elastoplastic analysis – tangent stiffness method

Authors and reference	Types of element	Constitutive law and solution algorithm	Problems solved – remarks
Pope[34]	Plane stress triangle	von Mises yield, Prandtl-Reuss flow, isotropic hardening	Edge-stiffened rectangular plate under nonproportional loading
Marcal and King[12]	Plane and axisymmetric triangles	Same as above	Notched plate, thick cylinder
Yamada et al.[13,15] and Miyamoto[52]	Plane and axisymmetric triangles, tetrahedron	Same as above	Notched plate, represented in plane stress (Ref. 13) and as solid (Ref. 52)
Swedlow and Yang[49]	Plane stress triangle	Same as above	Cracked plate
Richard and Blacklock[32]	Plane stress triangle	von Mises yield, Prandtl-Reuss flow, use of Ramberg-Osgood for effective stress-strain law, Runge-Kutta numerical integration	Comparison with plate test data of Ref. 51
Kamel and Sack[13]	Axial (also describes formulation of triangular and tetrahedronal elements)	Wide range of yield conditions and flow rules	Five- and ten-bar trusses (hypothetical); discusses substructuring to isolate region at plasticity
Zudans[21]	Axisymmetric thin shell	von Mises yield, isotropic and kinematic hardening	Attachment of two cylinders and torispherical head
Hofmeister et al.[53]	Plane and axisymmetric triangles (linear strain fields), triangle in bending	von Mises yield, Prandtl-Reuss flow, isotropic hardening	Introduces variational principle for tangent stiffness models, includes finite strains; numerous problems
Anand et al.[33]	Plane stress triangles	Tresca yield	Deep beam (Ref. 51) and notched tensile specimen (Ref. 48)
Ayres[54]	Isoparametric quadrilateral ring	Same as Ref. 12, MARC-H program	Pressure vessel nozzle for pressure and thermal strain
Popov and Khojastch-Bakht[27]	Axisymmetric thin shell	Incremental plasticity theory	Pressure vessel heads; discusses initial strain procedure

Table 9. Finite-element elastoplastic analysis – initial strain method

Authors and reference	Types of element	Constitutive law and solution algorithm	Problems solved – remarks
Gallagher et al.[36]	Triangle, tetrahedron	Experimental data for stress-strain law; simplified analytical representation	Plate with circular hole
Lansing et al.[30]	Bar and shear panel elements	Ramberg-Osgood stress-strain representation; methods of Ref. 36 studied in detail	Compares data of Ref. 50
Percy et al.[50]	Plane stress triangles	Same forms as Ref. 36	Develops significant test data for finite-element elastoplastic analysis
Whang[26]	Rectangular shell element	von Mises yield criterion, Prandtl-Reuss flow, isotropic hardening	Work applies also to tangent stiffness (Table 8), cantilever beam, flat plate, curved shell
Argyris et al.[3,38,55,56]	Wide range of elements	Extensive review of alternative representations	Ref. 38 gives comparison of initial strain and tangent stiffness
Zienkiewicz et al.[39]	Plane stress and plane strain triangles	General form of yield condition and flow rules, detailed for isotropic hardening	Plates with hole, notch; tunnel cross section, strip foundation
Zienkiewicz and Nayak[25]	Wide range of isoparametric elements	Discusses variety of flow rules and constitutive relationships	Bellows, thick cylinder, axisymmetric extrusion
Schultz and van Fossen[57]	Axisymmetric quadrilateral	von Mises yield, Prandtl-Reuss flow, isotropic hardening	Pressure vessel head and nozzle; discusses heat transfer and creep analyses
Salmon et al.[31]	Plane stress triangle, bar	Same as above, with three choices for stress-strain law	Truss and comparison with Ref. 50 test data; includes computer program listing

It is of interest to note from these tables that nearly the full range of element types is represented. On the other hand, with respect to constitutive relationships, use of the von Mises yield condition and associated flow rule predominates. Representation of isotropic hardening also predominates, although the number of efforts employing kinematic hardening is increasing. Actual performance of analyses for cyclic plasticity is similarly limited.

Comparison studies of the tangent stiffness and initial strain methods are included among the above. The most detailed of those appear to be the papers by Argyris and Scharpf,[38] Zienkiewicz and Nayak,[25] Marcal,[2] Kamel and Sack,[18] and Whang.[26] In his examination of relative efficiency, Whang[26] concluded that the tangent stiffness method accomplished a given level of accuracy with slightly less computational effort than the initial strain method. It is certain that differences in coding strategy and in the detailed aspects of the respective approaches could have reversed this conclusion. Thus, although comparison efficiency studies are badly needed, it appears that the choice of an approach will be governed by a bias born of experience and the specific circumstances of the problem at hand.

As a final note, it should be observed that certain recent finite-element papers have explored the use of the simpler deformational theory of plasticity, which is valid under conditions of proportional loading. These papers include the theoretically oriented studies of Egeland[58] and Stanton and Schmit[23] and the practical application described by Chan et al.[59]

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6. Time-Dependent Viscoelastic and Creep Analysis

The distinction between linear and nonlinear forms of time-dependent material behavior is crucial in any discussion of the background of related analysis capabilities. The linear form of time-dependent material behavior, which is conventionally termed viscoelasticity, has been studied analytically for more than a century, and the development of related analysis tools has progressed continuously to the present high level of capability. On the other hand, nonlinear time-dependent material behavior, or nonlinear creep, has been identified in analytical form only since 1910 (Ref. 1), and progress toward general analysis capabilities is measured from the late 1950s.

The time-dependent behavior of metals is characterized by nonlinear creep. This does not entirely discount an interest in finite-element-based solutions of viscoelastic deformation, since procedures formulated for viscoelasticity form a basis for creep analysis procedures. Certain other viscoelastic analysis procedures may prove useful for future creep analysis developments.

The following section is devoted to viscoelastic analysis, after which the problem of creep analysis is examined. Both sections are divided into reviews of the available constitutive equations and outlines of specific finite-element analysis procedures. A final section describes applications of the finite-element method to creep analysis.

6.1 LINEAR VISCOELASTICITY

6.1.1 Constitutive Relationships

Linear viscoelasticity attempts to deal with time-dependent material behavior by establishing mathematical forms of the constitutive relationships involving time (t) and derivatives with respect to time that are linear in the stresses and strains. Many thorough accounts of this topic, including both the representation of the constitutive relationships and methods of structural analysis based on these relationships, are available.[2-6] The following is a brief outline of considerations in representation of viscoelastic constitutive equations, which are important to the methods of viscoelastic finite-element analysis to be described subsequently.

Before discussing specific forms of viscoelastic constitutive relationships, it is essential to distinguish between such expressions when written for stress in terms of strain, and vice versa. When stress is written in terms of strain, strain rates, and time, the variation in stress with respect to time for constant strain can be established. This variation, portrayed schematically in Fig. 13a, is termed *relaxation*, so that the terms relating stress to strain are collectively designated as the *relaxation modulus*. Conversely, when strain is expressed in terms of stress,

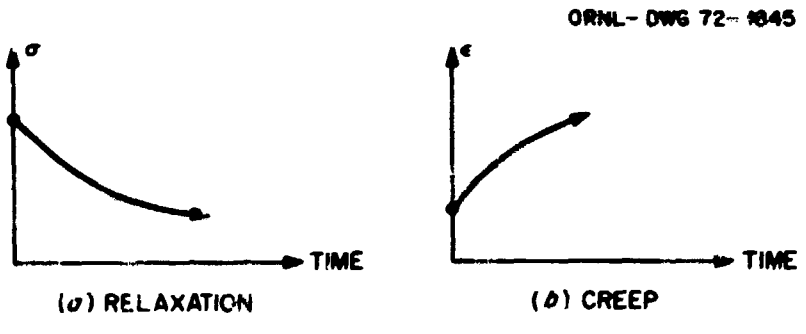

Fig. 13. Schematic form of basic representation of time-dependent material behavior.

derivatives of stress with respect to time, and time, the case of constant applied stress (Fig. 13b) produces *creep* behavior and the constitutive relationships define the *creep compliance*.

Measured (experimental) data are usually obtained in the creep compliance format. The principle of minimum potential energy, the almost universally employed approach to finite-element analysis, involves the relaxation modulus format for the constitutive relationships. This presents no practical difficulty for the usual approach to finite-element viscoelastic analysis (the time increment–initial strain method), since the coefficients of the constitutive relationships are defined "instantaneously" as numerical values determined separately by reference to the functional (differential or integral) form of these relationships.

Viscoelastic constitutive relationships may be assumed to be represented by mechanical models consisting of springs and dashpots. This leads directly to the differential form of the constitutive relationship. The simplest representations are those due to Maxwell and Kelvin, respectively (see Fig. 14).

The Maxwell model consists of a spring and dashpot in series, representing the following analytical expression for strain vs stress and time (Fig. 14a):

$$\epsilon = \frac{\sigma}{E} + \frac{\sigma}{\eta} t \ . \qquad (1)$$

Deficiencies of this representation include the linearity of the strain vs time variation and a failure to represent any "recovery" of viscoelastic strain upon removal of load. Recovery is an experimentally observed phenomenon. By combining a spring and dashpot in parallel (Fig. 14b), the behavior is represented as

$$\sigma = E\epsilon + \eta \frac{d\epsilon}{dt} \ . \qquad (2)$$

This representation is also deficient because it does not account for the initial elastic strain, as was done in the Maxwell model. Thus, to obtain features of

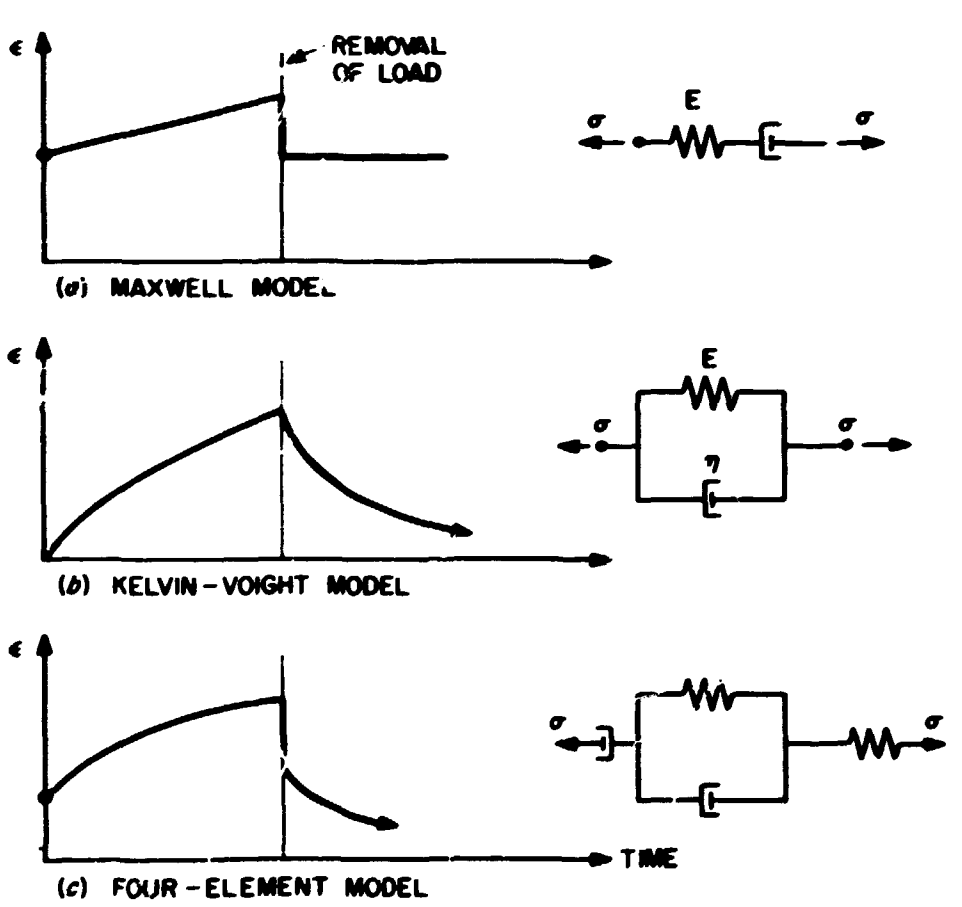

Fig. 14. Spring-dashpot models of viscoelastic behavior.

both models it is feasible to tie together four elements (Fig. 14c), to yield

$$\sigma + p_1 \frac{d\sigma}{dt} + p_2 \frac{d^2\sigma}{dt^2} = q_1 \frac{d\epsilon}{dt} + q_2 \frac{d^2\epsilon}{dt^2} \quad . \qquad (3)$$

where p_1, ... q_2 are material constants.

Successively more sophisticated and realistic viscoelastic constitutive relationships can be formed by combining still more springs and dashpots. The most general viscoelastic relationship is then of the form:

$$\sum_0^m p_k \frac{d^k\sigma}{dt^k} = \sum_0^n q_k \frac{d^k\epsilon}{dt^k} \quad , \qquad (4)$$

where p_0, ... p_k, ... p_m, q_0, ... q_k, ... q_n are material constants.

For the purposes of analysis one seeks a direct viscoelastic relationship in either the creep compliance [$\epsilon(t) = J(t)\sigma_0$] or relaxation modulus [$\sigma(t) = Y(t)\epsilon_0$] format, each referring to an initially applied stress (σ_0) or strain which is held constant, rather than the differential form of Eq. (4). This is conveniently accomplished by use of Laplace transform techniques. The analysis problem requires consideration of a time history of stress and strain intensities, however, and for these cases the concept of the hereditary integral is introduced. As in the case of constant stress or strain, either creep or relaxation formats of this integral may be written. For creep we have

$$\epsilon(t) = \sigma(t)\, J(0) + \int_0^t \sigma(t') \frac{d\, J(t - t')}{d\,(t - t')} dt' \quad . \qquad (5)$$

where t' is the time parameter to measure the stress variation and t measures time from the start of viscoelastic deformation.

Any attempt to introduce computational economies by direct use of the functional form of the constitutive relationships in the stiffness equations and integration thereof in time presents formidable difficulties. Inversion of the creep compliance to define the relaxation modulus is extremely costly. Procedures for this are given in Refs. 7 to 10, and an illustration of the related computational expense is presented by White.[11]

To avoid the difficulties of the formalized representation of viscoelastic constitutive equations and still retain the hereditary nature of the phenomenon, while dealing with a form appropriate to finite-element analysis, the scheme employed in Ref. 12 can be considered. With reference to the Kelvin model [Eq. (2)] for a single element (element i), we have

$$\frac{d\epsilon^c}{dt} = \frac{\sigma}{\eta_i} - \frac{E_i}{\eta_i} \epsilon^c \quad . \qquad (2a)$$

Here, since the elastic strain is not represented, we have set $\epsilon = \epsilon^c$. For a series of Kelvin models (1, ... i, ... r) and a finite increment of time (Δt), the increment of creep strain is

$$\Delta\epsilon^c = \left[\sum_{i=1}^{r} \left(\frac{\sigma}{\eta_i} - \frac{E_i}{\eta_i} \epsilon_i^c\right)\right] \Delta t \quad , \qquad (2b)$$

and the values of σ and ϵ_i^c for this interval are taken as those existing at the start of the interval.

The generalization of the above to multiaxial states of stress is straightforward, adopting the assumption that linear creep occurs only for the deviatoric components of stress, so that Poisson's ratio equals 0.5 (see Refs. 1–6, 12).

6.1.2 Analysis Procedures

Published finite-element viscoelastic analysis procedures are in each case cast in the form of the stiffness approach, but a variety of distinct treatments are represented. The treatments discussed here are (1) the time increment–initial strain method, (2) the correspondence principle, and (3) the integral approach.

The time increment–initial strain method is applicable to all classes of finite-element analyses for time-dependent material response and is almost exclusively the basis for nonlinear creep analysis. Thus it is given close attention in the following. This procedure was first introduced for the nonlinear creep problem[13] and was subsequently employed by Zienkiewicz et al.[12] for viscoelasticity.

In this procedure the time history of loading (and temperature, if this also varies with time) is represented by a series of constant load intervals, as shown in Fig. 15. The viscoelastic deformation accumulated at the close of a given interval is treated as an initial strain in a determination of the stress prevailing in the subsequent interval.

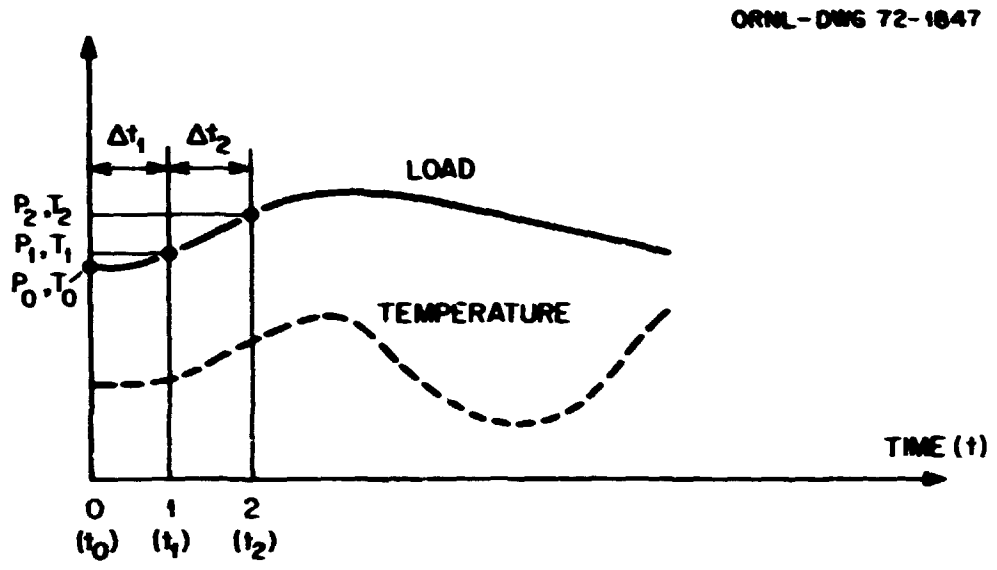

Fig. 15. Representation of time-varying loads and temperatures.

The complete algorithm, from time zero (t_0), proceeds as follows:

(a) Calculate the elastic stress distribution $\{\sigma\}_0$ at t_0, based on $\{P\}_0 = [K]\ \{\Delta\}$, and any initial strain due to temperature, if present.

(b) Refer to the constitutive relationships [e.g., Eq. (2b)] and calculate the change in time-dependent strain $\{\Delta\epsilon^c\}_1$ in the first interval Δt_1 assuming $\{\sigma\}_0$ to be constant within this interval.

(c) Solve the elastic problem at the close of the first interval, at $t_1 = t_0 + \Delta t_1$, using $\{\Delta\epsilon^c\}_1$ to form an initial force vector $\{F^i\}$ (and for temperatures T_1, if present):

$$\{P_1\}_1 = [K]\ \{\Delta\}_1 - \{P^i\}_0 \ . \qquad (6)$$

Calculate the elastic stresses $\{\sigma\}_1$ from this solution.

(d) Calculate the increment in time-dependent strain $\{\Delta\epsilon^c\}_2$ for the second time interval Δt_2 by reference to the constitutive relationships and add to the already sustained time-dependent strain.

(e) Employ step (c) for the close of the second interval, using the total time-dependent strain to calculate $\{P^i\}_2$.

(f) Repeat steps (d) and (e) for succeeding intervals.

It is important to note the significance of the selection of the time interval with respect to solution accuracy, even for a viscoelastic analysis. The above procedure implies that stress does not change during the selected interval. One may approximate the change in stress, however, and form an average stress for the interval in order to minimize the error. Questions of interval selection will again be taken up in Section 6.2.2.

The viscoelastic constitutive relationship represented by Eq. (2b) is the basis for steps (b) and (d) in Ref. 12. The problems of a prism and a reinforced concrete cylinder, for which exact solutions are available, and the more practical cases of a solid propellant rocket engine and a tunnel lining, for which no comparison solutions are available, are solved in Ref. 12.

The correspondence principle of viscoelasticity is employed by Webber[14] in the finite-element analysis of rectangular plates. In accordance with this principle, a Laplace transform is taken of the elastic solution and the elastic constants are replaced by certain combinations of the elastic and viscoelastic constants that are dependent upon the chosen form of the viscoelastic constitutive law. The Laplace inverse of the resulting relationships gives the viscoelastic solution.

The success of this approach depends upon a special form of the element relationships, which in Ref. 14 comprise the conforming rectangle in plane stress. The constitutive relationships are of the Maxwell form. Thus the approach is of limited value, especially in view of the inapplicability of the correspondence principle for nonhomogeneous transient temperature distributions. Reference 14 compares the numerical solution with the results of a rectangular plate test performed especially for this purpose.

Integral forms of finite-element viscoelastic analysis, as defined here, operate upon the integral form of the constitutive relationships. The latter are in

the class of Eq. (5). References 15 to 19 develop procedures based upon constitutive relationships of the specific form of Eq. (5) or its inverse but differing principally in the manner of approximation of such integrals. Taylor and Chang[16] and Lynch[18] describe analyses that attempt to account for the complete history represented by the integral. The papers by Taylor, Pister, and Goodreau,[17] Zienkiewicz and Watson,[15] and White[11] each introduce major simplifications of the evaluation of the integral in time.

6.2 CREEP

6.2.1 Creep Laws

Although many efforts have been launched by material scientists in recent years to gain a more complete understanding of creep behavior in metals,[20] many questions remain unanswered and reliable theoretical procedures are not yet available for the calculation of creep representations from more fundamental physical properties. Thus dependence is placed upon mathematical representations drawn from test data.

Comprehensive studies of both the creep response of materials and of procedures for structural analysis in the presence of this response are given in Refs. 21 to 29. In the latter context these references deal almost exclusively with classical analysis procedures whose results are of extremely limited applicability. Nevertheless, common considerations underlie both classical and numerical methods, and dependence must be placed by the reader on these references for detailed development of that which will be sketched in the following.

Three questions require study of the definition of appropriate constitutive relationships for creep analysis: (1) the form of uniaxial creep data and its dependence upon such factors as time and temperature, (2) the generalization of uniaxial creep data to multiaxial states of stress, and (3) the manner in which creep strains are accumulated under varying stress and temperature histories.

In discussing item 1, it is useful to refer to the original representation of Costa de Andrade,[1] shown in Fig. 16, where the creep strain is plotted as a function of time for a given stress level and temperature. This behavior is approximated by an expression of the form

$$\epsilon^c = \beta t^m + \gamma t \,. \tag{7}$$

where β, γ, and m are material constants. By differentiation with respect to time,

$$\frac{d\epsilon^c}{dt} = \dot{\epsilon}^c = m\beta t^{m-1} + \gamma = \dot{\epsilon}_1^c + \dot{\epsilon}_2^c \,. \tag{8}$$

The exponent m is less than 1, so that for short times the first expression predominates while for long times the second term, γ, is of greatest value. The early portion, governed by βt^m, is termed the primary phase of creep. The portion governed by γt is characterized as secondary creep. The third, or tertiary phase, leading to

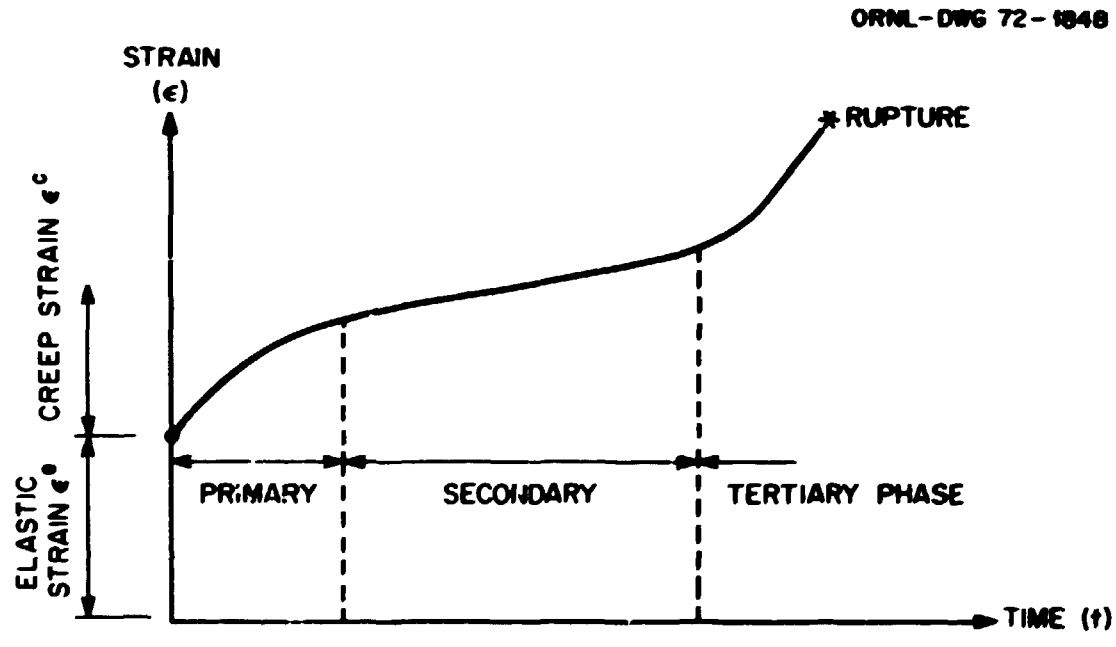

Fig. 16. Andrade representation of constant-stress creep behavior.

creep rupture generally represents a highly nonlinear form of behavior in which the usual engineering definition of strain does not suffice for valid characterization. Due to the high cost of creep analysis it is usually not feasible to conduct analyses into the tertiary phase with representation of phenomena encountered in this phase.

For analysis purposes it is necessary to express the material constants m, β, and γ as functions of stress and, if possible, temperature. This is desirable even for finite-element analysis, which does not require such functional representation, because of the extensive data tabulations needed to account for all ranges of behavior. One of the most popular basic forms of the steady-state creep rate $\dot{\epsilon}_1^c$, is that due to Norton,[30]

$$\dot{\epsilon}_2^c = B\sigma^n \ , \tag{9}$$

where B and n are material parameters. (If this equation is employed as a representation of primary creep, then B is a function of time.)

Bailey generalized Eq. (9) to account for temperature dependence by defining B as follows (see Refs. 26–28 for discussion of this work):

$$B = De^{bT}(\epsilon^c)^{-p} \ , \tag{10}$$

where e is the base of natural logarithms, T is the absolute temperature, and D, b, and p are material constants. Generally, the information available for representation of temperature variation has been so limited that the form of Eq. (9) has been used.

A number of alternatives to the above have been proposed and will continue to be explored. A long-standing alternative is Soderberg's formula,[31]

$$\dot{\epsilon}^c = A[e^{d\sigma} - 1] \ , \tag{11}$$

while other authors have employed

$$\dot{\epsilon}^c = A_3 \sin h \, d\sigma \ , \tag{12}$$

where again A and d are material constants (different for the two formulas).

Although terms such as B in Eq. (10) are defined in a form that has physical significance, the actual functional dependence upon stress must at the present time be established from experimental data. The information available is extremely limited. Thus the more complicated creep laws [Eqs. (10–12)] have often lost favor to the simpler expression given by Eq. (9). It should be noted, however, that any expression is theoretically acceptable for finite-element numerical analysis, so that new developments in material representation may be incorporated as they appear.

The commonly accepted procedure for dealing with item 2, the generalization of uniaxial creep data to multiaxial states of stress, is in direct analogy to the procedure used for time-independent plastic deformation, that is, the concepts of effective stress ($\bar{\sigma}$) and effective creep strain ($\bar{\epsilon}^c$) are introduced. The following assumptions are made in establishing stress-strain relationships phrased in terms of these parameters:

1. Creep deformation occurs under constant volume.
2. Hydrostatic stress has no influence on creep rate.
3. The principal directions of the stress and creep strain rate tensors coincide in an isotropic medium.

Under these conditions, the desired relationships are, in indicial notation,

$$\Delta\epsilon_{ij}^c = \frac{3}{2}\frac{\Delta\bar{\epsilon}^c}{\bar{\sigma}} S_{ij} \ , \tag{13}$$

where

$$(\bar{\sigma})^2 = \frac{3}{2} S_{ij}S_{ij} \ . \tag{14}$$

The incremental effective creep strain ($\Delta\bar{\epsilon}^c$) is

$$(\Delta\bar{\epsilon}^c)^2 = \frac{2}{3}\Delta\epsilon_{ij}^c\Delta\epsilon_{ij}^c \ , \tag{15}$$

and the deviatoric stresses S_{ij} are

$$S_{ij} = \sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk} \ , \tag{16}$$

where δ_{ij} is the Kronecker delta. In the special case of axisymmetric solids, for example, these relationships take the specific forms

$$\begin{aligned}
\Delta\epsilon_r^c &= \frac{\Delta\bar{\epsilon}^c}{2\bar{\sigma}}(2\sigma_r - \sigma_\theta - \sigma_z) \ , \\
\Delta\epsilon_\theta^c &= \frac{\Delta\bar{\epsilon}^c}{2\bar{\sigma}}(2\sigma_\theta - \sigma_r - \sigma_z) \ , \\
\Delta\epsilon_z^c &= \frac{\Delta\bar{\epsilon}^c}{2\bar{\sigma}}(2\sigma_z - \sigma_r - \sigma_\theta) \ , \\
\Delta\epsilon_{rz}^c &= \frac{3}{2}\frac{\Delta\bar{\epsilon}^c}{\bar{\sigma}}\sigma_{rz} \ .
\end{aligned} \tag{17}$$

Finally, to consider item 3, the manner in which creep strains are accumulated under varying stress and temperature histories, we examine Fig. 17, which plots the creep strain vs time for various stress levels in the primary phase of creep. Assume that the stress history is as described in Fig. 17b. After progressing along the σ_1 curve for time t_1, a question arises regarding the point on the σ_2 curve at which accumulation of creep strains should be initiated for the second time interval. If one moves directly at constant time (path A) from the σ_1 to σ_2 curves, the accumulation procedure is termed "time hardening"; if the move is made at constant creep strain (path B), it is termed "strain hardening."

Other possibilities have been explored (see Rabatnov[25]), and more recently a formulation has been proposed[32] that includes features of both time and strain hardening. The limited amount of experimental data in this area and the even more limited data relating to the significance of the respective rules of numerical solution results place the more sophisticated rules in the academic realm for the present. Finite-element analysis programs may easily accommodate either strain-hardening or time-hardening rules, with preference usually being given to the former because of superior accuracy recorded in limited past studies.

The question of a hardening rule does not arise in the secondary creep phase. As Fig. 17c demonstrates, application of either rule produces the same creep rate.

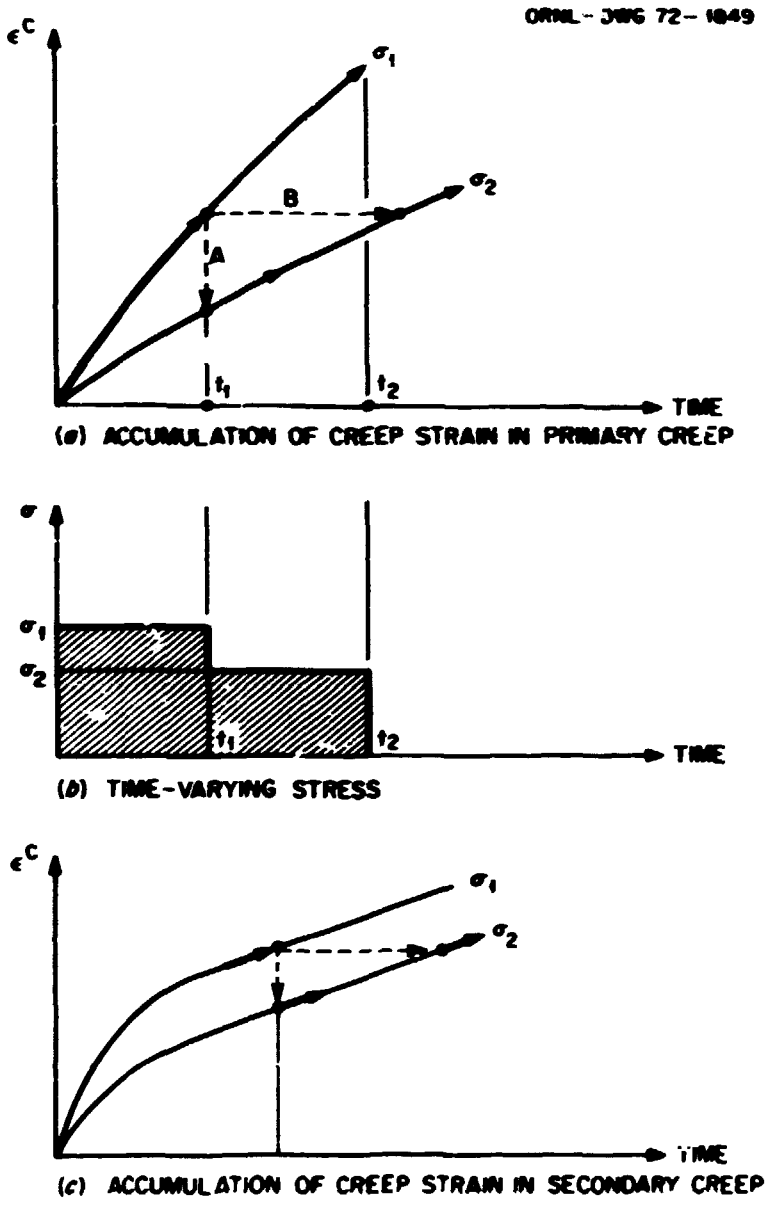

Fig. 17. Comparison of rules for accumulation of creep strain.

6.2.2 Analysis Procedures

In contrast to time-independent plasticity and to linear viscoelastic analysis, only the time incremental–initial strain method has been employed in finite-element creep analysis. This work is summarized in Table 10 and is discussed below.

The earliest application of the finite-element method to creep analysis appears to have been that of the writer, Padlog, and Bijlaard,[13] although it must be noted that the concepts for solution of the established algebraic equations are identical to those presented earlier by Mendelson et al.[33] This approach has been carried through to the present in the references to be discussed.

Chronologically, the next significant contribution to finite-element creep analysis was made by Lansing, Jensen, and Falby.[34] This work, based on the matrix force method and the representation of a stiffened sheet in plane stress subjected to time-varying loads and temperatures by means of axial force members and shear panels, is noteworthy for its contribution of the most significant test data yet reported. The data, which involved strain gage readings and supplementary material property test information, should be useful in the verification of future developments in finite-element creep analysis.

The extension of the subject procedures to axisymmetric solids, together with useful information regarding convergence of the solution process in time, is given by Greenbaum and Rubenstein[35] and by Sutherland.[36] Both papers employ the simplest form of axisymmetric solid element, the triangle with linear displacement fields (joints only at the vertices). (See Sect. 3.2.)

References 35 and 36 identify two limitations on the time-step length in finite-element creep analysis and establish tolerances for each of these within the confines of the problem they study. The first limitation pertains to change of stress within the time interval. First-order creep analysis features the assumption of

Table 10. Applications of time incremental–initial strain method of finite-element creep analysis

Authors and reference	Creep law	Types of element	Problems solved	Remarks
Gallagher et al.[13]	$\dot{\epsilon}^c = B\sigma^n$, strain hardening	Triangle and rectangle in plane stress	Finite-width plate with central hole	
Lansing et al.[34]	$\epsilon^c = at^\gamma(e^{B\sigma}-1)$, strain hardening	Axial member and shear panel	Stiffened rectangular plate	Includes temperature variation with time; careful study of theory vs test
Greenbaum and Rubenstein[35]	$\epsilon^c = B\sigma^{3.61}$, strain hardening and time hardening	Triangle: axisymmetric	Thick-walled cylinder	
Sutherland[36]	$\dot{\epsilon} = B\sigma^{6.35}$, strain hardening	Triangle: axisymmetric, plane strain, plane stress	Axisymmetric cylinder, flow duct, beam	
Swanson and Patterson[38]	$\dot{\epsilon}^c = C_1 e^{C_2/T}(\sinh C_3\sigma/T)^{-C_4}$ C_i = material constants		LMFBR fuel duct	
Chan et al.[44]	Not specified (arbitrary), strain hardening and time hardening	Triangle: plane strain, plane stress	Longitudinal pipe weld	Excludes theory; devoted to description of large-scale analysis; includes time-independent plasticity
Schultz and Van Fossen[45]	$\epsilon^c = at^\gamma\sigma^n$, strain hardening and time hardening	Axisymmetric triangle and quadrilateral	Pressure vessel	Same comment as for Ref. 44

constant stress within the time interval. To keep the error due to this source within acceptable bounds, it would appear appropriate to limit the change in stress to 5%. Sutherland[36] employs both 5 and 10% as the upper limits in the creep analysis of a flow duct using the creep law $\dot{\epsilon}^c = B\sigma^{6.35}$. The differences in the results did not prove significant.

The second limitation applies to the change in creep strain in a given increment in time. It has been found[34] that if the change in creep strain equals the elastic strain, the solutions for succeeding intervals oscillate and diverge. Thus the change in creep strain is limited to some fraction of the elastic strain.[35–38] Alternatively, upon attainment of constancy of the stress field with time, extrapolation may be attempted in order to estimate the displacements for a subsequent time.

A key aspect of any creep analysis program is the automatic selection of time increments, subject to the above criteria. The program described in Ref. 39 asks for an arbitrary initial estimate of the time interval. This is doubled in the next increment if the above criteria are met, or halved (and the increment repeated) if they are not.

Only limited attention has been given to the problem of creep behavior of thin plates or thin shells in flexure. As in the case of the time-independent plasticity analysis of the same structures, retention of the Kirchoff hypothesis demands a numerical integration of behavior across the thickness of the plate.

Hellan[40,41] discusses the finite-element analysis of a rectangular plate in flexure. He bases his development on a simplified rectangular element with quadratic transverse displacement field, the use of "initial stress" concepts in the treatment of accumulated creep strains, and Norton's creep law.[30] Analyses are performed for only a uniformly loaded plate with 21 divisions across the thickness of each element and a 7 X 7 grid of elements. Comparisons are made with only the known stationary solution. Nevertheless, the results presented are the most extensive numerical computations yet to be performed for this class of problem. Similar studies have been performed by Lin[42] and Lin and Ganoung[43]

for circular and rectangular plates, using the finite-difference technique.

6.3 APPLICATIONS EXPERIENCE

This section considers only those published descriptions of analysis that refer to large-scale structures with design significance, as opposed to examples which apply to simple structures, and are intended for the purpose of verifying basic concepts in finite-element creep analysis.

It is fair to say that reported experience in the application of finite-element analysis to large-scale creep problems is quite limited, and for this reason the interested reader is advised to also include the work on finite-difference analysis of rotors as described by Menoelson et al.[33]

Greenbaum and Rubenstein[35] analyzed three types of pressure vessel closures: the spherical, elliptical, and flat head. A total of 520 elements and 301 nodal points were used. An interesting facet of the results was the prediction of change of location of the point of maximum stress intensity as creep deformation progresses

Sutherland[36] examined a hexagonal flow duct; this is an application of triangular elements in plane strain. The creep law employed is of the form

$$\bar{\epsilon}^c = (5.6 \times 10^{-54.05})\, 10^{0.0197T}\, \bar{\sigma}^{6.35} .$$

The solution parameters studied include the changes in elastic stress concentration as time progresses and the amplitude of maximum creep strain as compared with elastic strain.

Chan, Manjoine, and Visser[44] analyzed the stress distribution in the vicinity of a longitudinal pipe weld, including both time-independent plastic deformation (on the basis of deformation theory of plasticity) and creep. A total of 279 plane strain triangular elements comprised the idealization. Schultz and Van Fossen[45] analyzed a pressure vessel for creep and time-independent plasticity using an analytical model composed of 796 quadrilateral and triangular elements.

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7. Dynamic Analysis

Analyses for dynamic phenomena have become increasingly important in the design of nuclear reactor structures in recent years. The prime motivation for interest in this topic stems from the concern of structural integrity in the presence of earth tremors. Other sources of excitation exist, however, including the impacting of nuclear power plants by aircraft and tornados, as well as the mundane mechanical sources of excitation.

It is reasonable to expect that if a finite-element representation is needed for design analyses for the phenomena of principal interest to this report – inelasticity, creep, thermal strain – then the same representation is essential for calculation of the response to dynamic loading. In other words it would appear to be efficient to devise an analytical model that could be used for representation of all behavior mechanisms of interest, that is, for static, dynamic, and thermal phenomena. A practical limit to this objective intervenes, however, due to the expense of analysis in temporal, as well as spatial, coordinates.

This chapter deals only with the relationship of the finite-element method to dynamic analysis. For the overall considerations in dynamic analysis the reader is referred to Biggs,[1] Hurty and Rubenstein,[2] Mierovitch,[3] and Rogers.[4] An excellent development of finite-element dynamic analysis procedures from fundamental principles is given in the latter chapters of the book by Przemieniecki.[5]

This chapter begins with a brief outline of the sources of information regarding dynamic excitation of nuclear reactor structures. The next two sections examine basic questions in the formulation of element relationships for dynamic analysis, without reference to dynamic analysis of the complete system for specific forms of response. These two sections pertain to the inertia and damping representations respectively. Finally, a review is given of relevant considerations in the solution of large-order systems of equations for dynamic phenomena. This aspect divides itself naturally into the topics of frequency determination and analysis for transients.

7.1 DYNAMIC ENVIRONMENT

It has already been noted that in dynamic analyses of nuclear reactor structures the principal source of excitation is earthquakes. The compendium edited by Wiegel[6] gives a good account of the general state of knowledge with respect to earth tremors. An even more recent account of the availability and form of earthquake data for the United States has been presented by Seed.[7] These documents demonstrate that the information regarding earthquakes that have occurred is indeed quite limited.

Measured earthquake data must be arranged in a form that is suitable for analysis input. This undertaking, in itself, constitutes an analysis effort. The generally accepted format for shock-type excitation is the response spectrum, which is the plot of the maximum value of a response parameter (e.g., acceleration) for a single degree-of-freedom system subjected to the relevant excitation.

Thorough expositions of the above transformation and its role in nuclear reactor structural design is given in Refs. 8 through 10; further views of these questions for the general readership are presented by Sharpe.[11] The extent to which available data are deficient for

nuclear power plant design was recently discussed by Housner.[3] However, overriding the considerations of data accumulation are the philosophical and probabilistic questions of a design criteria. A review[12] and a more recent paper[13] by Newmark define progress in this direction.

Tornados, as well as earthquakes, have come to be recognized as sources of environmental hazards to nuclear reactor structural integrity. The present state of knowledge regarding the intensity of tornados is well summarized in Ref. 14.

7.2 INERTIA REPRESENTATIONS

For approximately eight years, since Archer[15] introduced the concept of "consistent" mass matrix representation into finite-element analysis, controversy has existed regarding the relative merit of consistent vs "lumped" representations. The product of the element mass matrix [m] and the vector $\{\ddot{\Delta}\}$ of accelerations of the element degrees of freedom constitutes the inertia force in the element equations of motion (without damping):

$$[m]\{\ddot{\Delta}\} + [k]\{\Delta\} = \{F(t)\}, \tag{1}$$

where $\{F(t)\}$ represents the time-dependent joint forces. For a complete system, upon assembly of the element relationships,

$$[M]\{\ddot{\Delta}\} + [K]\{\Delta\} = \{P(t)\}. \tag{2}$$

During the early period of numerical analysis of multi-degrees-of-freedom systems, the mass matrix (inertia representation) was calculated simply by assigning the mass physically subtended by a particular degree of freedom to that degree of freedom. This is the so-called lumped mass approach, which yields a diagonal mass matrix. The work of Archer,[15] however, and simultaneous efforts by Leckie and Lindberg[16] called attention to the fact that as the stiffness matrix may be derived from strain energy, so also may the inertia term be regarded as being derived from kinetic energy V, where

$$V = \tfrac{1}{2} \int_V (\dot{\Delta})^2 \rho \, dV, \tag{3}$$

and ρ represents the material mass density. Assuming that the usual finite-element procedure of employing shape functions is adopted, as described in Chapter 2, we have

$$\dot{\Delta} = [N]\{\dot{\Delta}\}, \tag{4}$$

where, in analogy to the discussion in Chapter 2, $\{\dot{\Delta}\}$ lists the time derivatives of the degrees of freedom (i.e., $d\Delta_1/dt$, $d\Delta_2/dt$, . . . for t = time), and [N] gives the spatial variation of displacement ($[N] = [N_1(x,y,z), N_2(x,y,z), \text{etc.}]$). Thus, by substitution in Eq. (3),

$$V = \frac{[\dot{\Delta}]}{2}[m]\{\dot{\Delta}\}, \tag{5}$$

with

$$[m] = \rho\left[\int \{N\}[N]\, dV\right], \tag{6}$$

where now [m] is the *consistent* mass matrix, which is so designated because the consistent shape function has been employed in derivation of both the mass and stiffness matrices.

Przemieniecki[5] gives a complete account of explicit consistent mass matrices for the elements he considers. Many papers give both stiffness and consistent mass matrices in presenting new element formulations. With the trend toward numerical integration in element numerical evaluation, however, the requirements for explicit formulations of consistent mass matrices are diminished.

Consistent mass matrices are widely thought to have three advantages: (1) the validity of the pertinent variational principle employed is preserved allowing, for example, the calculation of "upper bound" vibrational frequencies; (2) the solution is expected to be more accurate for a given grid refinement than the solution with lumped or other inconsistent element mass matrices; and (3) the element formulative procedure and the mode of assembly of the complete system equations are unified for both static and dynamic terms.

Advantage (1) is difficult to realize for practical structures. Departures from the conditions for the validity of the pertinent variational principle are difficult to avoid in practical circumstances. These departures are due to such factors as the approximate representation of applied loads and the approximation of geometry, as well as the familiar problem related to lack of interelement continuity of displacement. Gallagher and Lee[17] made a numerical study of the effect of approximations to geometry on the upper-bound character of finite-element vibrational analyses.

Advantages (2) and (3) are more apparent than real. Clough[18] discussed the question of consistent vs

lumped mass matrices in greater depth and found, in certain numerical studies, that the lumped mass model furnished greater solution accuracy at a given level of grid refinement. It is important to note, however, that this conclusion in itself is not general. Consistent mass matrices are more accurate in the classical case of uniform beam vibration and many cases of flat-plate vibration. It should be added that such advantages must be of a scale that would justify the added costs of formulating consistent mass matrices. With respect to item (3), it is, of course, possible to construct element formulation algorithms that develop lumped mass matrices in an automatic manner.

A clear disadvantage of consistent mass matrices stems from their nondiagonal nature. The diagonal form of lumped mass matrices allows major computational efficiencies in the analysis of the complete system.

7.3 DAMPING REPRESENTATIONS

Damping phenomena in real structures are difficult to describe analytically. The forms of representation include viscous, coulomb, and structural damping, but even these often fail to account for the effects that arise from joint slippage, special frictional effects etc. Detailed examinations of damping may be found in the special publication edited by Ruzika[19] and the text by Lazan.[20]

In view of these factors the simplest representation of damping in the viscous form is generally adopted. This choice allows for retention of the linear analysis form, and for the finite-element model we have the governing equation

$$[M]\{\ddot{\Delta}\} + [C]\{\dot{\Delta}\} + [K]\{\Delta\} = \{P(t)\}, \qquad (7)$$

where now [C] is the matrix of viscous damping coefficients.

If Eq. (7) is solved in time by use of numerical integration, wide latitude is available in the representation of the coefficients C_{ij}. (Methods of numerical integration are discussed in the next section.) In practice, however, the direct definition of C_{ij} is generally not feasible since, as noted above, the analytical-based sources of damping are not those that actually predominate.

Perhaps the most realistic approach to the establishment of damping is through testing of the actual structure or from test data of similar structures. The structure is successively excited in the various natural frequencies. For each response frequency a study is made of the decay of amplitude, and from this it is possible to estimate the modal damping coefficient (C_{ij}), that is, the coefficient in a diagonalized representation of the equations of motion. Only a limited number of the lowest modal terms can be established in this manner.

With the above information in hand, two routes open to the analyst; he may either transform the complete analysis representation to modal form and proceed with analysis in time using the modes for which damping data are now available, or he can attempt to "expand" analytically the modal damping data into a full matrix [C]. A procedure for the latter can be devised and is detailed by Clough.[18]

The difficulties and sources of data in establishing damping properties in nuclear reactor structural analysis are documented in Ref. 10 and in the recent paper by Hadjian.[9] Also, Yeh[21] has recently presented one of the few numerical studies of alternatives in the representation and analytical treatment of damping.

7.4 SOLUTION PROCEDURES

7.4.1 Frequency Determination

The basic system calculational problem in structural dynamics is the determination of the frequencies of undamped vibration (eigenvalues) and the associated mode shapes (eigenvectors). This information is of design value per se when the circumstances of design require that certain frequencies not appear in order to avoid resonance. In general, the interest in the frequencies and mode shapes stems from the use of the latter as bases for the transformation of coordinates to uncouple the equations of motion. Thus, in contrast to such eigenvalue problems as elastic instability analysis, the problem of frequency determination involves the calculation of many of the frequencies and mode shapes rather than the single lowest value of frequency.

A definition of the eigenvalue problem, in algebraic terms, is useful to the present discussion. In free, undamped vibration the forcing function {P(t)} of Eq. (2) is zero, and motion is assumed to be described by the function (i.e., harmonic motion)

$$\{\Delta\} = \{\Delta_0\}\cos\omega t, \qquad (8)$$

where $\{\Delta_0\}$ is the mode shape (eigenvector) of the motion and ω is the natural frequency in cycles per second (Hertz). By substitution of Eq. (8) into Eq. (2), we obtain

$$\left[-\omega^2[M] + [K]\right]\{\Delta_0\} = 0, \qquad (9)$$

or

$$[I]\frac{1}{\omega^2}\{\dot{\Delta}_0\} = [K]^{-1}[M]\{\dot{\Delta}_0\}, \qquad (10)$$

where [I] is the unit matrix. A condition for the calculation of ω^2 stems from either the fact that the determinant of $[-\omega^2[M] + [K]]$ must be zero for a nontrivial solution of Eq. (9) to hold or that the premultiplier of $\{\dot{\Delta}_0\}$ on both sides of Eq. (10) must be identical.

There are as many different values (eigenvalues) of ω as there are degrees of freedom in $\{\Delta_0\}$, and each can be determined by application of either of the above conditions. Also, it should be noted that each eigenvector represents the column of a matrix, which, when employed as a transformation-of-coordinates matrix operating on either [K] or [M], will transform these matrices to diagonal form and of an order equal to the number of columns employed in the transformation matrix. The diagonalization operation decouples the degrees of freedom and permits solution of the time history of each degree of freedom from an independent second-order ordinary differential equation.

Many alternative algorithms for the solution of large-order systems of equations are available and well tested (see Sect. 2.3). Despite the great number of contributions to linear equation solving, it is not possible to designate with certainty the most efficient procedure. At best, a limited group of promising alternatives can be identified. The situation is less clear in eigenvalue analysis, possibly because of the late start of interest in this area and the greater complexity and higher cost. It is therefore prudent to view the topic as being in a state of intensive development with the likelihood that important contributions are yet to be made or assimilated into routine practice.

A thorough account of fundamental procedures and concepts in the calculation of eigenvalues of algebraic equations has been given by Wilkinson.[22] Much emphasis is placed upon the "transformation" methods of Jacobi, Givens, and Householder. Forms of these methods are in fact found in widely available algorithms, for example, the NROOT routine of the IBM System Scientific Subroutine Package.[23] Such methods, however, are less attractive in large-scale application than iterative methods, to be discussed in the following, because they are pointed toward simultaneous calculation of all eigenvectors and other factors.

One of the more effective and attractive approaches to eigenvalue determination is the method of conjugate gradients. As noted in Chapter 2, this method permits the development of the system representation without explicit construction of the full set of system equations. This is highly advantageous in finite-element analysis. The method was adapted to eigenvalue analysis by Bradbury and Fletcher[24] and has been cast in the finite-element form by Fox and Kapoor[25] and Prato.[26]

Contributions to finite-element eigenvalue analysis on the "conventional" iterative side have recently been made by Bronlund,[27] Whetstone and Jones,[28] Rosen and Rubenstein,[29] and Dong et al.,[30] among others. As emphasized above, the present early stage of practical application of these alternatives prevents identification of the preferable procedures.

7.4.2 Transients

Two general avenues of approach are adopted in solving Eq. (7) in time; (1) numerical integration of the complete analytical representation and (2) transformation of this representation to modal (diagonalized) form followed by numerical integration. The forcing functions $\{P(t)\}$ imposed in practical structural design analysis situations are nearly always too complicated to permit explicit integration in time. Thus, for both avenues of approach, it is convenient to divide the time history of interest into small, finite intervals.

Methods of numerical integration of the complete system of dynamic equations are summarized in the text by Norris et al.[31] These include the Runge-Kutta methods and Newmark's β-method. The Crank-Nicolson approach, which was originally devised for heat transfer analysis, has been defined for dynamic analysis by Wilson and Clough[32] and by Chan et al.[33] and is among the most popular of the numerical integration schemes. This approach is based upon an assumed variation of acceleration within the chosen time interval and is an implicit scheme. Once the choice is made of the variation of acceleration, it is possible to formulate explicit equations for all solution parameters of interest at the close of a time interval as a function of available data from the start of the interval.

Because of the popularity of the above scheme in dynamic analysis, we elaborate upon it as follows. Assume that the displacement and velocity at time zero, $\{\dot{\Delta}_0\}$ and $\{\Delta_0\}$, are known. On this basis, the acceleration at this point may be directly determined from Eq. (7). Also, assume that the acceleration varies linearly in the time interval $h = t_1 - t_0$:

$$\ddot{\Delta} = \left(1 - \frac{t}{h}\right)\{\ddot{\Delta}_0\} + \frac{t}{h}\{\ddot{\Delta}_1\}. \qquad (11)$$

Then, from kinematic relationships, we have, for the displacement and velocities at the end of the interval,

$$\{\Delta_1\} = \{\Delta_0\} + h\{\dot{\Delta}_0\} + \frac{h^2}{6}\left\{2\{\ddot{\Delta}_0\} + \{\ddot{\Delta}_1\}\right\}. \quad (12)$$

$$\{\dot{\Delta}_1\} = \{\dot{\Delta}_0\} + \frac{h}{2}\left\{\{\ddot{\Delta}_0\} + \{\ddot{\Delta}_1\}\right\}. \quad (13)$$

Also, we have, for the acceleration at the end of the interval [from Eq. (7)],

$$\{\ddot{\Delta}_1\} = [M]^{-1}\left\{\{P_1\} - [K]\{\Delta_1\} - [C]\{\dot{\Delta}_1\}\right\}. \quad (14)$$

Now, Eqs. (12) to (14) can be combined to yield a single equation in which $\{\ddot{\Delta}_1\}$ is expressed in terms of the acceleration, velocity, and displacement at the start of the interval. Once $\{\ddot{\Delta}_1\}$ is solved for, Eqs. (12) and (13) may be employed to give $\{\Delta_1\}$ and $\{\dot{\Delta}_1\}$. The process is repeated for successive time intervals. The accuracy of this process may be enhanced by assuming a higher-order variation of acceleration, but a considerable increase in complexity of the expressions corresponding to Eqs. (12) to (14) is then sustained.

It is of interest to note the development of the concept of "finite elements in time" (Fried[34]). In this scheme one employs a shape function representation of the time variation within the time interval. The n equations of motion are transformed by this means into m X n algebraic equations, where m represents the number of time points spanned in the time period of interest. Because of the associated huge expansion of the number of equations to be solved, this approach has been little used.

With respect to modal methods of transient analysis, it should be observed that the two central questions relate to the number of modes required for adequate representation and the manner of treatment of damping. The number of chosen modes is often limited by the achievable accuracy in the calculation of higher modes of vibration. As already noted, modal damping coefficients may be determined experimentally. When this is not possible and the complete damping matrix has been established by other means, thought must be given to the form of the latter to facilitate the desired uncoupling of the equations of motion. Commonly, a damping representation, which is proportional to either stiffness or mass, or a combination of both, is selected. This permits calculation of the modal damping matrix by simple transformation using the modes of natural vibration. Less restrictive procedures for uncoupling the equations of motion are given by Foss.[35]

The large cost of numerical integration in time, with attendant costs of equation solving or inversion at each step in time, causes the concepts of "condensation" to be highly attractive. Condensation refers to the elimination of degrees of freedom from the basic analytical model prior to performance of the dynamic analysis. In flexural situations, for example, in which the basic analytical model includes both angular and linear displacement components, the analyst may decide to eliminate all angular displacements prior to dynamic analysis. It is important to note that condensation schemes do not limit the choice of degrees of freedom to be eliminated. Since these schemes are necessarily approximate, however, the solution accuracy is dependent upon the number and type of displacement components eliminated.

The simplest approach to condensation is the so-called "static" method (Guyan[36] and Kaufman and Hall[37]), in which the unwanted degrees of freedom are eliminated from the static stiffness matrix. This form of condensation permits the establishment of a matrix that transforms the full set of degrees of freedom to the desired reduced number of degrees of freedom. The transformation is then applied to each component matrix ([M], [C], etc.) of the full set of equations of motion.

A more elegant approach to condensation has been developed by Hurty[38] and refined further by Craig and Bampton.[39] Hurty terms this scheme "component mode synthesis," although it may also be characterized as "dynamic substructuring." In accordance with the latter view, it is assumed that the structure can be divided into major substructures for analysis purposes. Each substructure, excluding the degrees of freedom which attach to adjacent substructures, is subjected to a modal analysis. Combination of the substructures then results in a reduced system of equations of motion in terms of the substructure modal amplitudes and the (conventional) substructure interface joint degrees of freedom.

The limitations of condensation methods with respect to the number of degrees of freedom that may be removed are as yet ill-explored. This factor, together with the complexity of required matrix manipulations, has tended to restrict their application in practice.

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8. Thermal Analysis

The significance of integrated thermal and structural analysis cannot be overemphasized. The most desirable situation occurs when the analytical models for transient thermal analysis and structural analysis are in complete correspondence. Any disparity between these representations is a source of error in an analysis performed for an instant in time. Thermostructural analyses are, in general, time dependent (i.e., transient), and a complete analysis requires solutions at many instants in time. Thus errors in the instantaneous solution are amplified in the integration of the transients in time. Furthermore, uncoordinated thermal and structural analyses are highly inefficient, requiring large costs in transferring data from the thermal to the thermal stress analysis.

Prior to the introduction of digital computers, analytical procedures for thermal gradient conditions in power plant and process industry design were often based upon integrated solutions. Examples are the papers of Fritz[1] and Heisler.[2] Until recently, numerical methods of thermal stress analysis were preoccupied with the various aspects of the structural mechanics side of the problem. It has become recognized, however, that the finite-element method is valid for the solution of a wide variety of problems in mathematical physics, including thermal analysis.

With this acceptance of a broader role for finite-element analysis, rapid strides have been made toward the establishment of integrated thermostructural analysis. The extent of these advances is measured in this chapter. It will be demonstrated that progress in the finite-element analysis of thermal transients has benefited enormously from accomplishments already recorded in finite-element analysis for structural mechanics.

First, a brief review is given on the theoretical basis for finite-element thermal analysis. Attention centers about variational principles, but comments are also made regarding alternative forms of presentation of the basic theory. The extent of available element formulation is described, and methods for solving the time-dependent systems of algebraic equations are reviewed. Finally, an assessment is given of current levels of practice.

8.1 FORMULATIVE BASIS AND SYSTEM EQUATIONS

Approaches to the formulation of element and system equations for finite-element thermal analysis parallel those of structural analysis. Thus a functional in terms of temperature plays the role of "potential energy" and has been termed the "thermal potential."[3] Counterparts of the "mixed" and "complementary" energy principles are also definable in thermal analysis. The complementary functional[4] is based upon heat flux as an independent variable, so that the mixed functional[5] employs both temperature and heat flux parameters.

Because of the expanding nature of the literature related to the subject topic, it is important to note that many new developments associated with "flow problems" (of which transient thermal analysis is one) disregard entirely the concepts of variational principles and functional minimization and concentrate instead upon the "method of weighted residuals" (MWR).[6] The Galerkin form of MWR yields the same element

matrices as the variational approach for those cases where the variational approach is strictly valid. MWR also enables establishment of the governing integrals where variational principles do not strictly apply: such cases are discussed in Refs. 6 and 7.

To date, interest in finite-element heat transfer analysis has centered upon functionals (or integrals produced by MWR) expressed in terms of temperature fields. The subsequent discussions are therefore restricted to this approach, which, as noted above, is the thermal counterpart of the direct stiffness method of structural analysis. The prevalence of thermal potential formulations is due, no doubt, to the popularity of the corresponding principles in structural mechanics. Also, an important advantage is that assumed temperature fields for the various types of elements are readily drawn from the vast body of experience in structural mechanics.

The most commonly cited and utilized variational principle of the above class is due to Gurtin.[8] This functional explicitly accounts for initial conditions; it is an elegant representation that deals with time dependence in terms of convolution integrals. The earliest comprehensive development of finite-element thermal analysis, by Nickell and Wilson,[9] is based upon the functional of Ref. 8, as are numerous subsequent papers.[10–13]

However, alternative forms of the functional for thermal analysis have been employed. Most of these define the functional at an instant of time (i.e., questions relating to the time history of heating are not explicitly considered in the functional), as in the thermal potential of Visser,[3] the MWR-Galerkin development of Zienkiewicz and Parikh,[14] and the divergence-theorem-based formulation of Aguirre-Ramirez and Oden.[15] These alternatives yield identical definitions of the finite-element representations for linear thermal analysis.

With these considerations in hand, it is useful to define the general form of the algebraic equations for finite-element thermal analysis. As indicated, the temperature fields of the respective elements are described by shape functions in terms of node point temperatures. Then, by use of the applicable variational theorem or proper alternative, the element and system algebraic equations are derived in the form:

$$[C]\{\dot{T}\} + [K]\{T\} = \{P\}, \tag{1}$$

where

- $\{P\}$ is the vector of (time-dependent) thermal loads at the element joints,
- $[K]$ is the "conductivity" matrix,
- $[C]$ is the "heat capacitance" matrix,
- $\{T\}$ is the vector of element joint temperatures,
- $\{\dot{T}\}$ represents differentiation of $\{T\}$ with respect to time.

As in the case of structural analysis, the general form of Eq. (1) is valid at both the element and system level. The matrices $[C]$ and $[K]$ are herein collectively termed the element (or system, as the case may be) thermal matrices.

Solution of Eq. (1) for the complete system for the time period of interest yields the prediction of element joint temperature histories. Note that Eq. (1) represents the statement of the structural dynamic analysis problem with the presence of viscous damping, but lacking the inertia terms. Thus a portion of the experience developed for the treatment of problems of dynamic analysis can be adapted to thermal analysis.

A clearer picture of the advantages of the method can now be drawn:

1. Equations governing the heat transfer throughout a complex system can be constructed automatically, as is done in a finite-element structural analysis, based on the concept of element "conductivity" and "capacitance" matrices.
2. The element heat transfer matrices can be calculated as part of the process of formation of element structural analysis matrices. The problem data, as well as certain computational steps, are common to both.
3. Temperature distributions are identically of the form needed for subsequent structural analysis.
4. The scope of the heat transfer analysis capability is widened to account for anisotropic materials, with no significant increase of analytical complexity or computational cost.

8.2 ELEMENT THERMAL MATRICES

Procedures for the selection of shape functions for representation of displacement fields in finite-element structural analysis apply directly to the representation of temperature fields in thermal analysis. The principal distinction is the scalar (single degrees of freedom per

joint) character of first-order representation of temperature. This can be accommodated in existing finite-element analysis programs with little or no modification of logic. The element algebra is, of course, simplified considerably in comparison with structural analysis.

The triangular element with linear temperature field has been used most widely, and explicit formulations are found in Refs. 3, 9, and 16 through 18. Explicit formulation of the first-order triangular ring element is given by Brocci[10] and by Farhoomand.[19] The triangle and the rectangle, for both planar and axisymmetric situations, are described for a large-scale operational computer program in Refs. 20 through 24.

An extension of element relationships into the realm of higher-order representations is presented by Rybicki and Hopper,[25] who describe rectangular elements with 16 and 36 degrees of freedom, where the degrees of freedom in addition to the four joint temperatures are values of the derivatives of the joint temperatures, and by Skjolinstad and Cheung.[26]

Isoparametric formulation concepts, in which the shape functions for description of the temperature field are used as well in the description of the element geometry, are developed by Zienkiewicz and Parikh[14] for both two- and three-dimensional elements. The complexity of such models demands use of numerical integration in establishment of the element thermal matrices. In general, however, the practitioner will find that numerical integration is applied in preference to explicit formulation in large-scale programs, even when the latter option is present. This trend is already apparent in finite-element structural analysis and will be reflected in future developments related to thermal analysis.

8.3 SYSTEM SOLUTION PROCEDURES

The common approach to the solution of the system equations [Eq. (1)] for arbitrary variation of thermal loading is by subdividing the time history into many intervals Δt. Implicit solution procedures, which are unconditionally stable for linear boundary conditions, are directly applicable. However, the simpler explicit schemes may alternatively be employed if a diagonalized capacitance matrix is formed.

An effective implicit approach to the solution of transients is via the Crank-Nicolson method.[27] This approach was presented in the literature of structural dynamics[28,29] and was employed by Nickell and Wilson[9] in their computations. An advantage of the method is the avoidance of the need to iterate within each time increment and latitude in the choice of the variation of the temperature throughout the interval, for example, linear, quadratic, etc.

The range of solution possibilities found in structural dynamics is applicable to finite-element thermal analysis. Thus, Ref. 30 employs a simple step-by-step procedure, while a Runge-Kutta form of numerical integration is utilized by Guymon[12,13] and by Aguirre-Ramirez and Oden.[15] Fujino and Ohsaka[18] reduce the algebraic equations to uncoupled forms by application of the normal mode method. The resulting first-order differential equations are readily solvable by a wide range of methods.

The accuracy and stability of finite-element vs finite-difference numerical analyses of transients has been studied by Lemmon and Heaton[31] and by Emery and Carson.[32] As noted above, finite-element analysis is stable when marching in time since it is in the implicit class of methods in numerical integration. However, explicit methods of numerical integration in time, which do not possess the property of unconditional stability, may be more efficient. For this reason, and because of the costs of assembly of the [C] and [K] matrices as well as a matrix representative of convective heat transfer, Emery and Carson[32] reason that finite-difference solutions are more efficient than finite-element solutions in transient thermal analysis. It must be observed, however, that the finite-element method may be employed in explicit numerical integration schemes simply by approximate diagonalization of the [C] matrix, that is, by "lumping" of the heat capacitance at the nodes. Also, perhaps a small percentage of practical thermal analyses requires assembly of an independent matrix for convective heat transfer; this effect is generally represented by a relatively few individual terms which are additive to [K] and {P}.

A new concept in numerical analysis of transient problems is that of finite elements in time; that is, time is represented functionally and systems of equations are established for each chosen point in time. This concept has already been cited in Chapter 7. If there are n degrees of freedom in the spatial model and r points in time are treated in a single computational cycle, then r X n equations are established directly and must be solved simultaneously. Details of this approach are presented by Fried.[33]

If a problem is sufficiently simple, then classical methods for the solution of first-order differential equations may find application. Visser,[3] for example, explores analytical solution possibilities for small-order

problems, but such schemes do not appear to be workable for large-order systems.

Solution difficulties, which are not cited above, are encountered when the nonlinear boundary conditions are present. The radiation boundary condition represents a severe nonlinearity (the affected joint temperatures are raised to the fourth power), and only recently has this been given attention in finite-element thermal analysis. Farhoomand and Wilson,[19] Richardson and Shum,[30] and Yalamanchili and Chu[34] perform analyses of the problem. References 30 and 34 use the temperatures from the previous time interval in constructing coefficients for the present interval. If the time intervals are too large for the radiation boundary condition or for varying heat flux in the case of convective boundary conditions, instabilities in the computation of the surface temperature are possible. Thus, time increments cannot be too large. To avoid this problem, one may define a variation of temperature within the interval and iterate to convergence to a preassigned tolerance.[11,19]

Another nonlinearity arises when the temperature dependence of conductivity must be taken into account. This problem was studied by workers of Ref. 15, and a numerical solution for a slab with conductivity that is linearly dependent upon temperature was obtained. Solutions for axisymmetric structures are developed in Ref. 19.

8.4 APPLICATIONS EXPERIENCE

Any view of this aspect of finite-element thermal analysis must be highly subjective, since conclusions drawn from the limited amount of published information is apt to be misleading; only a very few public forums for discussion of this aspect have been held to date. Papers describing practical applications experience include Refs. 14, 20 through 24, and 35.

From available information, applications of the finite-element method to nontransient heat conduction have been remarkably widespread. In fact, practitioners whose interests do not extend to development of theoretical concepts or new formulations have performed such analyses. All that is required is the capability for plane stress (or plane strain or 3-D) stiffness analysis and an amenability of the program to cope with redefined constitutive relationships.

Practical applications experience in integrated thermal and stress analysis and in transient thermal analysis are believed to be insignificant in comparison with the overall scale of activity in thermostructural analysis. Published papers deal extensively with transient analysis, but these largely reflect validation of the new concepts in finite-element heat transfer analysis. The transient analysis applications in heat transfer must eventually confront the same aspects already dealt with in dynamic analysis -- computational efficiency via reduction of degrees of freedom through condensation and other schemes. So far as is known, only Ref. 36 has dealt with this question in finite-element heat transfer analysis. In addition, it is important to recognize that the general-purpose finite-element analysis programs are only now reaching an operational level in dynamic (time-dependent) analysis.

Many, and perhaps the large majority of, experimenters whose basic expertise is in heat transfer analysis have been reluctant to employ the finite-element method. A high degree of competence and applications experience has already been established in finite-difference-based heat transfer analysis, and it is widely felt that finite-difference capabilities are more efficient than those evolving from finite-element theory. As indicated previously, the arguments and evidence in support of this view are detailed clearly by Emery and Carson.[32] The writer believes that this debate will be amplified considerably in the future.

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9. Special Problems

The conditions described in previous chapters, which relate to limiting magnitudes of stress or strain under applied loads and temperatures, with due account being taken of inelastic material behavior, continue to be of primary importance in the design analysis cycle. Increasing attention is being given, however, to other specialized modes of failure or behavior mechanisms, in particular, the circumstances of finite displacement behavior and predictions of fracture phenomena. Hence the purpose of this chapter is to identify the sources of information for the relatively voluminous developments in these areas and to ascertain current status.

9.1 FINITE DISPLACEMENTS

9.1.1 Problem Background

Since the term "finite displacement" covers a wide range of seemingly different forms of structural behavior, it is necessary to limit the phenomena considered here. We exclude consideration of both large (or finite) strains, thereby eliminating rubber materials from the following discussion and any treatment of nonconservative (path-dependent) situations. An excellent survey of contributions in these excluded areas is given by Oden.[1]

The subject matter of interest originates in the definition of the strain-displacement equations of a structure to include the first-order nonlinearity.[2] Thus, for axial strain, the extended definition takes the form

$$\epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 , \qquad (1)$$

where now the terms $1/2\,(\partial u/\partial x)^2$, etc., supplement the linear strain-displacement equation $\epsilon_x = \partial u/\partial x$. In general, the phenomenon under consideration is of practical interest in nuclear reactor structural design only for thin shells, where only one of the three indicated quadratic terms (that which involves the transverse displacement w) is of significance; so we have

$$\epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 . \qquad (2)$$

This situation is illustrated in Fig. 18 from which it is clear that for the common case of zero transverse shear deformation, the slope $(\partial w/\partial x)$ equals the rotation of the normal to the neutral surface θ_y. Thus problems in the subject class are frequently referred to as "small strain–large rotation" problems.

The principal significance of finite-displacement (small strain–large rotation) behavior in pressure vessels

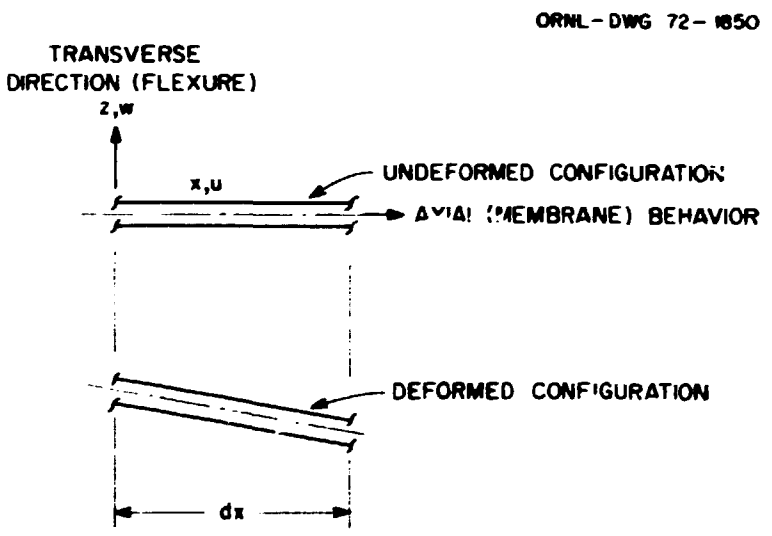

Fig. 18. Sketch of thin-shell behavior.

relates to the buckling phenomenon. Situations are rare in which applied external loads so counterbalance the pressurization effects that regions of compressive membrane stress, which precipitate buckling, are produced. It is well known, however,[3,4] that regions of compression result solely from internal pressurization in torispherical pressure vessel heads and that the risk of buckling failure exists in such cases. Furthermore, situations may arise in which loads are applied in the absence of pressurization.

The "classical" buckling problem involves the determination of the distribution of internal forces based on membrane theory and the calculation of the intensity of such forces to cause instability. The latter constitutes a linear eigenvalue problem. It is increasingly being recognized that the assumption of a membrane, or even linear prebuckling state, may not be correct.[5] Also, designers are showing greater interest in recognition and utilization of postbuckling strength. Additionally, for load intensities less than those required to produce instability, questions have been raised as to the significance in stress analysis of finite-displacement effects. Thus the subject phenomena include prebuckling behavior which may be nonlinear, buckling (or bifurcation), and postbuckling behavior.

An examination of the above with reference to finite-element analysis finds a natural division into the topics of (1) element and system equations, (2) solution procedures for determination of bifurcation, and (3) solution procedures for nonlinear pre- and postbuckling situations. These three topics are outlined in the following; a more elaborate discussion of each may be found in Ref. 6.

9.1.2 Element and System Equations

Published developments of finite-element analysis to encompass finite-displacement behavior are almost exclusively in terms of stiffness procedures. Hence this limitation is adopted in the discussion to follow.

In accordance with concepts detailed by Mallett and Marcal,[7] the element stiffness equations for nonincremental finite-displacement analysis are of the general form

$$[k]\{\Delta\} + [n_1(\Delta)]\{\Delta\} + [n_2(\Delta)]\{\Delta\} = \{F\} \ , \qquad (3)$$

where $\{F\}$ and $\{\Delta\}$ are the joint forces and degrees of freedom, respectively, $[k]$ is the linear (small displacement theory) stiffness matrix, $[n_1(\Delta)]$ is the first-order ("geometric" or "incremental") stiffness matrix, where individual terms are linear functions of the degrees of freedom $\{\Delta\}$, and $[n_2(\Delta)]$ is the second-order (also "geometric" or "incremental") stiffness matrix, with individual terms a quadratic function of the degrees of freedom $\{\Delta\}$.

Upon assembly of the element relationships defined by Eq. (3) to form a representation of the complete structure, the following equations are obtained:

$$[K]\{\Delta\} + [N_1(\Delta)]\{\Delta\} + [N_2(\Delta)]\{\Delta\} = \{P\} \ . \qquad (4)$$

where the definitions of K, N_1, and N_2 for the system representation correspond to those given above for k, n_1, and n_2. The vector $\{P\}$ now represents applied loads.

In linear stability analysis the matrix $[N_2(\Delta)]$ is disregarded, and certain simplifications are introduced into $[N_1(\Delta)]$ to eliminate dependence on the full displacement vector $\{\Delta\}$. Such simplifications are generally associated with a membrane (no flexure) prebuckling analysis. We designate the modified $[N_1(\Delta)]$ matrix as $[N]$, so that Eq. (4) reduces to

$$[K]\{\Delta\} + [N]\{\Delta\} = \{P\} \ . \qquad (5)$$

Furthermore, for stability analysis, one sets the load vector to zero and multiplies the matrix $[N]$ by a parameter λ, which is the ratio of the load intensity for buckling to the load intensity used in the calculation of $[N]$. Thus we have

$$[K]\{\Delta\} + \lambda[N]\{\Delta\} = 0 \ . \qquad (6)$$

We will examine the solution of this relationship for λ in the next section.

It was noted above that Eq. (3) applies to nonincremental solution procedures. Incremental solution procedures are popular, however, and for these a somewhat different form of the element and system equations is needed. Such relationships are developed in Refs. 7 and 8. It should also be noted in conjunction with Eq. (3) that matrix representation of the terms of N_1 and N_2 is extremely inconvenient and inefficient. This was observed by Wissman[9] and has more recently been examined by Vos[10] and by Morin,[11] who recommend indicial notation.

Only limited progress has been made in the establishment of N_1 and N_2 components of element stiffness relationships. A survey of this topic has been presented by Gallagher.[12] On the other hand, computer programs based upon application of numerical integration algorithms represent a capability for the calculation of the

N_1 and N_2 matrices for all elements contained in such programs. The extent of shell-element formulations for buckling analysis is discussed further below.

9.1.3 Solution Procedures – Buckling

As noted previously, in connection with Dynamic Analysis (Chap. 7), a thorough account of procedures for eigenvalue calculation, as of 1965, is given by Wilkinson.[13] More recent developments were also cited in Chapter 7. Here we discuss only the development of finite-element buckling procedures as they pertain to elements for thin shells.

A previous chapter examined thin-shell elements but without reference to instability analysis. Again, the division into axisymmetric and general thin-shell elements appears appropriate. With respect to the former, Navaratna, Pian, and Witmer[14] formulated a meridionally curved element and employed this in extensive numerical computations. In conjunction with general thin-shell elements, Gallagher and Yang[15] dealt specifically with the doubly-curved thin-shell instability problem. Curved-shell analysis developments by Brebbia and Connor[16] and by Bogner et al.[17] deal with geometric nonlinearities, of which stability has already been noted to be a special case.

The use of flat elements in representation of curved shells is noteworthy because of the relatively limited experience in finite-element thin-shell instability analysis. Such applications have been described by Gallagher et al.[18]

Clearly, additional numerical evidence of the finite-element method in linear stability analysis of thin shells would be desirable. In particular, no finite-element analysis has been reported of the stability analysis of pressurized torispherical shells.

9.1.4 Solution Procedures – Nonlinear Pre- and Postbuckling

Methods of geometrically nonlinear finite-element analysis fall in two general categories, iterative and step-by-step methods. These are analogous to the initial strain and tangent stiffness methods, respectively, of time-independent inelastic analysis.

It is also useful to distinguish between prebuckling and postbuckling analysis. In prebuckling analysis, one operates directly on the solution of Eq. (4), and the suitability and efficiency of a given method depends in part upon the severity of the nonlinearity. Methods in the prebuckling analysis regime are reviewed in detail by Haisler, Stricklin, and Stebbins.[19]

The simplest approach is direct iteration, in which all nonlinear terms in Eq. (4) are transferred to the left-hand side and the resulting "initial forces" are successively improved; that is,

$$[K]\{\Delta\} = \{P\} - [N_1(\Delta)]\{\Delta\} - [N_2(\Delta)]\{\Delta\} = \{P\} - \{P^i\}, \quad (7)$$

where $\{P^i\}$ represents the pseudo-initial forces. This method is widely regarded as inefficient, since it possesses only "first-order" convergence characteristics. Also, it is usually necessary to proceed step by step in load increments, even though the method is not a step-by-step procedure, since convergence difficulties are encountered when the changes in the nonlinear terms depart too greatly from the starting point.

On the other hand, recent experience[20] in direct iterative analysis indicates that the character of convergence is not seriously disadvantageous compared with methods possessing second-order convergence. Furthermore, direct iteration permits a solution of the prebuckling "fundamental path" behavior beyond the point of buckling. In this way interpolation may be employed to define the buckling point, which is otherwise difficult to define in the presence of a nonlinear prebuckling behavior due to the singularity of the stiffness equations at that point.

Methods of step-by-step geometrically nonlinear analysis are reviewed by Martin.[21] The majority of these methods are based upon the tangent stiffness concept (Ref. 7), although particular note should be taken of the work of Murray and Wilson,[22] which employs the usual linear stiffness at each step but updates the geometry to account for changes associated with the displaced state and for such changes as they affect the equilibrium conditions.

The updating of geometry after one or more increments of load is essential only if the displacements sustained are truly large. The updating operation may, on the other hand, be desirable from the standpoint of computational efficiency, since it should cause a reduction of the number of iterations in a given increment of load. The method of Ref. 22 is totally dependent upon updating; that is, it supplants rather than supplements the incremental-iterative process.

Before outlining postbuckling analysis, some aspects of thin-shell buckling should be discussed. It is well known that thin-shell structures often collapse at load levels which are less than those predicted by the linear stability theory described in the preceding section. This

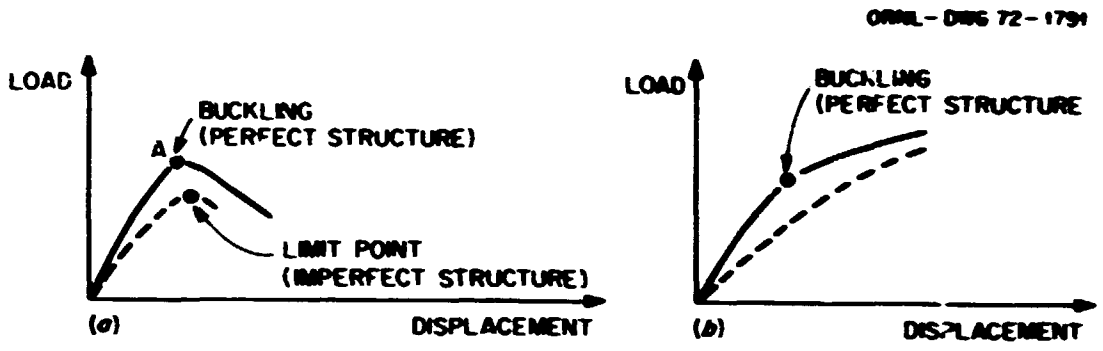

Fig. 19. Buckling behavior of thin-shell structure.

is due to the role played by initial imperfections and geometric nonlinearity.

Figure 19a is a representative load-displacement plot for such a situation. The solid line, which applies to a "perfect" structure, indicates buckling at point A followed by a descending postbuckling path. The dotted line shows the behavior of the real structure, one possessing fabricational inaccuracies. In this case the maximum, or "limit," load is substantially less than the buckling load. The difference between the buckling load and the limit load depends upon the magnitude and distribution of imperfections.

No logical means has yet been found to determine the magnitude and distribution of imperfections in structures not yet fabricated (i.e., being designed), although some progress is being made through statistical concepts.[23] The best that can be hoped for is a method of determining the slope of the initial postbuckling behavior, that is, whether the slope is negative (Fig. 19a), in which case the actual strength is less than the buckling load, or positive (Fig. 19b), where postcritical strength exceeds the buckling load.

A complete theory for answering the above question has been developed by Koiter[24,25] and is the basis for the largest share of current theoretical work in shell postbuckling analysis. Additional theoretical contributions have been made by Budiansky and Hutchinson,[26] Sewell,[27] and Thompson.[28] These developments are based on "perturbation" concepts, in which the postbuckling behavior is described by series expansion about the bifurcation point. Initial imperfections may also be taken into account, in which case the limit point behavior (dotted line, Fig. 19a) is approximated.

Extensions of Koiter's procedure to the format of finite-element analysis have recently appeared. This work is reviewed by Mau and the writer in Ref. 20. The most general of these approaches appear to be the development of Ref. 20 and the more approximate "equivalent structure" scheme of Haftka, Mallet, and Nachbar.[29]

It is important to note that perturbation schemes are valid only in the vicinity of bifurcation. Thus, if one seeks a numerical solution at points removed from buckling, as in the determination of displacements at higher load in the behavior represented by Fig. 19b, or if a recovery to a higher load level occurs in the extension of the dotted line of Fig. 19a, then perturbation schemes cannot be employed. Such determinations can be accomplished in finite-element analysis using procedures described in Refs. 20 and 30.

9.2 FRACTURE MECHANICS ANALYSIS

9.2.1 Basic Considerations

The importance of fracture mechanics analysis procedures to the verification of the structural integrity of nuclear pressure vessels has grown in parallel with the development of the fracture mechanics technology itself. This growth extends back little more than a decade. More recently, it has become recognized that the finite-element method is ideally suited to the calculation of stress intensity factors, the basic components of any fracture mechanics study, and many papers have appeared in a short period of time. Each of these advocates a somewhat different approach, and it is by no means apparent which is the most reliable or efficient. Indeed, the appropriate method of interpreting the results is not clearly defined.

The finite-element analysis methods for linear fracture mechanics can be grouped as (1) direct methods, (2) energy-based procedures, (3) superposition schemes, and (4) singularity function formulations. Each of these methods is reviewed below; a more detailed exposition is given by the writer in Ref. 30.

In the following we restrict attention to linear fracture mechanics, which works with the results of

elastic analysis. Progress is being made toward a more realistic, inelastic fracture mechanics theory, but this is in the early stages. For additional information consult Ref. 31.

9.2.2 Direct Methods

Direct methods involve the performance of a straightforward finite-element analysis with a high degree of grid refinement in the region of the crack tip and the determination of the stress intensity factor by direct interpretation of the calculated stresses or displacements.

Kobayashi et al.[32] employ the "crack-opening-displacement" (COD) procedure, wherein the displacement at a point near the crack tip is correlated with the analytical form of the general solution for displacement. An extremely large number of degrees of freedom are required to achieve an acceptable level of accuracy.

In the approach of Chan, Tuba, and Wilson[33] and Cruse and Vanburen,[34] the stress intensity factor is determined by study of many solution points via extrapolation along a radius centered at the crack tip. This method requires considerably fewer degrees of freedom to attain a given level of accuracy than the COD method and is therefore to be preferred. A choice exists in the extrapolation method between the interpretation of the solved-for displacements or stresses. The former is found to be superior in displacement-based finite-element analysis. Results from the direct methods have generally been approximately 5% in error.

9.2.3 Energy-Based Procedures[34–41]

Three subsidiary approaches are grouped under this heading: the total energy method, the local energy method, and the line integral method.

The total energy method[35–39] exploits the relationships between the rate of change of strain energy with crack opening and the stress intensity factor, using finite-element idealization of the total structure in calculation of the former. The strain energy is occasionally represented in terms of flexibility coefficients (the compliance form of the method) or in terms of stiffness and solved-for displacements. In each case the rate of change of strain energy has been computed from data for two or more analyses at different crack lengths. Thus a number of large-scale analyses are required. All results reported for the total energy method demonstrate satisfactory accuracy for solution times that are much smaller than those for the direct methods.

In the local energy method, a classical relationship[40] between the stress intensity factor and the strain energy contained within a region surrounding the crack tip forms the basis for computation of the former. The difficulties with this method are that a high degree of element grid refinement is needed in the vicinity of the crack, the conventional geometric form of the individual element does not conform to the curved (circular) boundary of the region of interest, and the solution data for the circular region of interest must be isolated and extracted from the total analysis. An advantage of the method is that only one analysis is required for determination of the stress intensity factor.

The line integral method depends upon a relationship due to Rice[41] between the stress intensity factor and a line integral of a function of strain energy and edge tractions on an arbitrarily defined region containing the crack. Thus the method is similar to the local energy method except that it is less restrictive in geometric characterization of the region of interest.

9.2.4 Superposition Schemes[42,43]

Approaches in this class are based upon an appropriate combination of a classical solution and the finite-element analysis results. Thus one defines a classical solution that most closely represents the problem of interest and which, of course, applies only to a regular structure in the region close to the crack. Since the solution will not satisfy all conditions of the actual problem, the disparities can be interpreted as body forces and edge loadings. The latter are applied to a finite-element analysis (with relatively coarse gridwork) in reverse direction, and the classical and finite-element solutions are superimposed.

The method, however, is awkward to apply because it requires extensive hand computation or a highly specialized computer program but is perhaps the most efficient approach from the standpoint of computer operating costs.

9.2.5 Singularity Function Formulations[44–47]

One of the most appealing approaches, from both a theoretical and computational view, is to formulate a special element containing a singularity and which may be used in the region of the crack tip. Regions remote from the crack tip are idealized by means of conventional elements.

The most sophisticated singularity function element formulation published to date is due to Byskov,[44] who constructs the basic element relationships with use of the complex stress functions of Muskhelishvili.[45] Using stress parameters, displacements, and the hybrid approach[46] to element formulation, Walsh[47] and also Rao, Krishnamurthy, and Raju[48] derived stiffness matrices for a variety of stress concentration conditions. Levy et al.[49] established a four-sided sector element in polar coordinates. The degenerate case of this element, in which two points coalesce, is applied in representation of the region adjacent to the crack tip; it should be noted that the singularity represented here is more descriptive of plastic behavior at the crack tip than the presumed elastic behavior of linear fracture mechanics.

An important new development was recently published by Pian et al.[50] This approach, based upon the hybrid method of element formulation, treats the stress intensity factor as an independent solution parameter. The computational costs appear to be minimal, approximately two orders of magnitude less than those for the direct approach first advanced in Ref. 32.

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10. Closure

This section draws upon the detailed perspectives of the prior sections to form an overall view of the subject. The major advantages and disadvantages are reasserted and specific directions of future work suggested. For convenience this presentation follows the divisions of subject matter represented by the report sections.

In conjunction with methods of linear analysis of the complete system, cornerstones of any form of nonlinear analysis, it is apparent that a comprehensive and reliable approach to virtually all practical problems is provided by currently available general-purpose computer programs. These programs are based exclusively on the displacement-based element stiffness formulations. In many areas, however, major advantages in computational efficiency will result from procedures based on the assumption of stress fields or mixed stress–displacement fields. Existing programs are generally inflexible in the manipulation of large blocks of data, and this too must be modified to gain economically feasible analyses. We have in mind here such operations as substructuring, condensation, the use of harmonic representations of response for geometrically axisymmetric structures, and the imposition of constraint conditions.

Solid element formulations are of demonstrated reliability. A dilemma arises in the choice of element representation, that is, tetrahedronal vs hexahedronal, regular vs isoparametric coordinates, and conventional vs higher-order degrees of freedom. No single approach is optimum; the choice depends principally upon the application at hand. Caution must be exercised in the use of higher-order representations due to potential difficulties in numerical conditioning.

Shell-element formulations have only recently emerged in forms that satisfy all basic requirements in finite-element analysis, and no single formulation has received broad acceptance. These sophisticated formulations, being of types that are phrased in stiffness matrix format, can be incorporated in available finite-element programs of reasonable generality. Their degree of sophistication, however, coupled with the complexities attendant upon analysis for inelastic deformation, reflects unfavorably upon the economics of the solution process. Shell-element formulations based on both stress and displacement fields are therefore highly promising in this regard.

The analysis of time-independent inelastic effects has reached a plateau in both theoretical development and practical application. The correlation of alternative approaches (initial strain vs tangent stiffness) has been accomplished, and nearly all techniques of classical plasticity analysis have been exercised. If attention is subsequently directed toward minimization of computational costs, for example, through substructuring to isolate the region of plastic deformation and via multimode schemes wherein initial strain and tangent stiffness operations are judiciously combined, wider latitude will be available for needed examination of cyclic plasticity. There is a body of opinion which holds that the conventional representations of constitutive laws are not the most promising bases for modeling of observed inelastic phenomena. Thus it may be necessary to construct such representations with use of finite-element modeling concepts themselves.

In creep analysis, the finite-element solutions have again duplicated nearly all existing classical solutions and available test data. In the opinion of the writer, finite-element analysis represents the only feasible means of solution for the practical, complex structure,

time history, and constitutive law. The relatively few practical circumstances that are intrinsically simple in these respects are, of course, exempted. It follows then that comparison results are not available, and the validity of finite-element solutions to creep analysis problems must be measured on the basis of test data but more often against the criterion of "reasonableness." Once again we find that engineering judgment is more significant in computer analyses than is generally supposed.

Dynamic analysis represents an increase in cost of linear static analysis and inevitable uncertainties in the characterization of damping. The concepts of condensation to effect reduction in the size of the system prior to marching in time are well established here and are supplemented by advantages gained with use of approximate inertia force representations. Proper attention must be given to the choice of time intervals in transient analyses.

Thermal analysis via the finite-element method has progressed rapidly in terms of acceptance and codification and is completely reliable with respect to steady-state conditions. Transient analysis, however, also is confronted with all the uncertainty of numerical procedures for initial value (transient dynamic analysis) problems. The choice of time interval size, as in dynamic analysis, is critical to computational efficiency and, if an explicit solution approach is adopted, to the stability of solution. Implicit methods also present difficulties in solution reliability as a function of interval size. Practitioners must maintain an awareness of numerical methods in the analysis of transients as they appear in the literature.

Of the two problems described here as "special problems," those in the category of geometric nonlinearities are the widest in scope. If the term "geometric nonlinearities" is construed to apply to linear elastic instability analysis, a fully reliable solution approach now exists in finite-element analysis. Unfortunately, linear stability predictions apply only to a restricted class of situations for thin shells. Nonlinear stability analyses have been demonstrated to be feasible, but insufficient experience has been accumulated in practical applications. In addition, there are open questions with respect to basic theoretical representation in the range of severe nonlinearity.

The special problem of linear fracture mechanics furnishes a case study of the history of application of finite-element concepts to a "new" technological area. The space of three years saw the progress in this respect from a straightforward utilization, which was immediately identified as economically infeasible, to the establishment of a novel formulation of finite-element analysis that was specifically addressed to the efficient solution of the problem at hand. It also demonstrated that progress of this type is possible only if computer program capabilities are adaptable to other than displacement-based stiffness formulations.

In conclusion, it should be noted that widely available general-purpose finite-element structural analysis computer programs have achieved distribution and operational status within design offices only during the past five years, although the concept of the general-purpose program gained acceptance many years earlier and certain organizations had long possessed in-house capabilities of this form. As a group, the currently available programs often do not yet meet originally envisioned objectives nor account for phenomena that had not been recognized in the initial stages of their development. Adaptation of the general-purpose program is feasible, however, if the associated developmental team continues to be active with the program.