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COMPUTATIONAL MODELING OF TURBULENT TRANSPORT[1]

by

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Abstract

A rational closure technique is presented for the first and second moment-equations in a stratified, contaminated turbulent flow. Following the application of high Reynolds/Peclet number approximations, remaining third moments are expanded about the isotropic, homogeneous state. The stratified, uncontaminated case reduces to seventeen equations in seventeen unknowns. Other authors have suggested some of the terms generated, but some have been using the wrong terms, or the right terms for the wrong reasons. The approximation is kinetic-theoretic (turbulence/mean motion scales assumed small, and turbulence nearly in equilibrium) and results in a relaxation time, and in generalized gradient transport forms; however, gradients of one quantity can produce fluxes of another. The model relates the time scale for return to isotropy to the Lagrangian integral time scale (reducing to K-theory in a homogeneous parallel flow

with orthogonal temperature gradient). Some coefficients are estimated, and preliminary computations are presented of the unstratified 2-D turbulent wake; only component energies near the centerline are not well reproduced, probably due to the omission of a term with which temporary computational difficulties were being experienced. Stratified, contaminated 3-D calculations appear to be practical.

Introduction

Inexpensive semi-quantitative numerical simulation of turbulent transport would have many applications. In particular, when applied to pollution dispersal in an urban atmospheric environment, or in an estuary, it would permit rational decision making by the government bodies involved. There are, in addition, problems in oceanography and meteorology (such as thermocline formation) on which a reasonably realistic, though semi-empirical, computational technique could shed light.

Before attempting to construct such a method, we should consider two basic, related questions: is the method within our grasp conceptually and computationally? That is, do we understand turbulence well enough to model it with acceptable accuracy, and can the model provide simulation at acceptable cost, with available computational facilities? These are very real questions for, although we understand a great deal about turbulence, there is even more that we do not know; also, even techniques that do not exceed the capacity of the most recent generation of computers can easily exceed the ability (or willingness) of reasonable men to pay, if computer time must be accounted for.

A great deal of success has been had with direct numerical simulation of turbulence (Orszag and Patterson, 1972). This involves no dynamical modeling. However, Fox and Lilly (1972) have shown that such an approach rapidly receeds beyond economic reach as the Reynolds number increases. The next most successful approach is direct simulation with so-called sub-grid scale modeling (Deardorff, 1973, for example). This recognizes that it is economically impossible to carry in the computation the smallest scales at high Reynolds numbers; the grid scale is made as small as is economically feasible, and the motion on scales below the grid scale is modeled, making use of the well-supported property of turbulence dynamics (Tennekes & Lumley, 1972) that the precise nature of the dissipative mechanism does not influence the large scales of the motion, if the Reynolds number is high enough.

The success of this method depends not only on having a sufficiently high Reynolds number to have an inertial subrange (so that energy-containing and dissipative scales are, to a first approximation, dynamically related only by the value of the spectral energy flux) but also on being able to use a grid scale that is small enough to lie in the inertial subrange. It suffers from the disadvantage, that one calculation is not sufficient: each calculation is a realization in an ensemble, and a sufficient number of independent runs must be made to obtain stable statistics; for example, of the order of 200 runs to obtain 10% accuracy in second order quantities. To date, this technique has been used primarily for flows having a homogeneous direction; in such flows, the number of runs necessary to obtain stable statistics can be substantially reduced by spacial averaging in the homogeneous direction.

If one is attempting to model the flow in a fully three-dimensional region such as an urban environment, two facts rapidly become clear: the number of points required to make even a crude model of the region precludes the use of a grid scale lying in the inertial subrange, and a single calculation is so expensive as to preclude the possibility of doing statistics on an ensemble of them (since there is no homogeneous direction for averaging). We must then use a grid scale lying in the energy containing range, and compute only the statistical properties of the turbulence. The so-called sub-grid scale motions now contain virtually all the turbulence, and the dynamical modeling becomes much more critical. In fact, since the entire influence of the turbulence is being computed through the moments, it is no longer correct to think of turbulence quantities as being sub-grid scale; for example, the scale of the turbulence is not now related to the grid scale, but must be obtained from dynamical considerations.

No good direct model of second order turbulence quantities exists. The only practical model is the so-called eddy diffusivity, or "K-theory" model, which has been used with some success in simple situations (Tennekes & Lumley, 1972) to predict first order quantities. This model is known to fail, however, in situations which are rapidly changing in space or time. A number of authors, realizing that a good model is desirable but that good second-order models are not available, have decided to carry the equations for second moments exactly, and model third order terms (Donaldson, 1972; Mellor, 1973; Daly and Harlow, 1970; Jones and Launder, 1972; Ng and Spalding, 1972). There is some justification for this approach (which we will follow); as we shall show later, while it is not possible to construct a rational model at second order, it is at third order. However, the model constructed still rests on a fallacy: kinetic theory concepts are embodied, implying that length and time scales of the transporting mechanism (the turbulence) are small relative to length and time scales of the mean motion. This is, of course, known not to be the case for turbulence. There is thus an article of faith involved: if a crude assumption for second moments predicts first moments adequately, perhaps a crude assumption for third moments will predict second moments adequately.

Some of these third order closure schemes are incomplete in the sense that they provide no prediction for one of the scales (which may be taken to be equivalent to a length scale) (Donaldson, 1972; Mellor, 1973). Others (Daly and Harlow, 1970; Ng and Spalding, 1972; Jones and Launder, 1972) do provide a supplementary equation equivalent to one for a length scale. All of them, however, suffer from a basic flaw: they do not present any method for generating the models used for the third order terms. Since the models are constructed on an *ad-hoc* basis, usually being required to have only the same general tensor character as the terms modeled, models are occasionally constructed that behave incorrectly with Reynolds number (Corrsin, 1972), or as we shall see, the right term is included for the wrong reason, or important terms are omitted.

We will present here two related techniques which make it possible to generate, in a consistent and straightforward manner, models of all orders of the third moments, and of all order in Reynolds number. The technique is equally applicable to stratification, to pollution dispersal, to chemical reactions, etc. Many of the terms generated are essentially those suggested by other authors on an ad-hoc basis. However, in the case of the third-order transport terms, we will find that it is inconsistent within the model not to allow the flux of one second order quantity to be produced by gradients of *all* the others, much as a molecular flux of salt can be produced in a liquid by a temperature gradient, and vice versa. This opens the possibility of up-gradient diffusion, an important process in atmospheric modeling. Unlike the situation in kinetic theory, where the cross-diffusion coefficients are ordinarily small, the turbulent cross-diffusion coefficients may be substantial. However, in the (artificial) situation of constant eddy viscosity and constant structure, the forms obtained reduce to the classical forms assumed by other authors on an *ad-hoc* basis.

It is possible that we will conclude ultimately that these third order closures, though within our reach computationally, are, for some purposes, inadequate models of the second moments (although preliminary results, as we shall see later, appear quite favorable). It should not be necessary to point out, however, that we cannot reach a rational conclusion on this question unless we are sure that the closure used is based on a small number of explicitly stated, readily grasped principles, and that all terms, and only those terms, generated by these principles are used. Otherwise, we will not know if an unsatisfactory result can be attributed to the omission of a vital term, the inclusion of an extraneous one, or the use of an incorrect basic principle. The model that we will present, in common with the other third order closures, contains many undetermined constants. It has long been part of the folk wisdom in turbulence that a model can be made to fit a flow, given sufficiently many constants[2]. While there is some justice to this, it is not quite true. A model may be incapable of reproducing a certain qualitative behavior, regardless of the values assigned to the constants. Models with many constants can still be intellectually satisfactory, so long as the constants are not optimized for each flow, or group of flows, and so long as the physical interpretation of the constant is clear. It is important that the values of the constants governing each physically distinct effect be determined by computation in a situation in which that effect is not influenced by others. Otherwise, one is in danger of adjusting the wrong constant for the right reason. As an aside, if this principle is applied conscientiously, it quickly becomes clear that despite the wealth of experimental data collected over the past few decades, there is a remarkable dearth of well-documented *elementary* turbulent flows, in which one effect at a time is carefully studied.

The High Reynolds Number Approximation and the Dissipation Equations

In Tennekes and Lumley (1972), it is shown how orders of magnitude may be assigned to various correlations appearing in the dynamical equations. Roughly, instantaneous quantities appearing in the correlations are of two types, belonging either to the energy containing range of eddies, or to the dissipation range. The former has characteristic frequency u'/ℓ (where $\overline{\varepsilon} = u'^3/\ell$, $3u'^2$ is twice the mean fluctuating energy q^2, and $\overline{\varepsilon}$ is the mean dissipation of energy per unit mass) while the latter has characteristic frequency u'/λ, where $\lambda \sim 4\ell R_\ell^{-1/2}$, $R_\ell = u'\ell/\nu$. The correlation coefficient between two quantities from the same range may usually be taken as unity, but the coefficient between two quantities, each from a different range, is of the order of the time scale ratio, $\lambda/\ell \sim 4R_\ell^{-1/2}$.

In addition, of course, we may make use of the more familiar fact that derivatives which are external to correlations correspond to scales in the energy containing range, while derivatives within the correlation correspond to dissipation scales. We wish to apply this sort of reasoning to every term appearing in the equations, but particularly to the equations for the dissipation of energy (and of temperature or concentration variance); applied to these equations, it is particularly productive, because the dynamics of these quantities is dominated by the small scales, and interacts only weakly with the energy containing eddies. Proceeding in this way, the equation for the mean dissipation of energy may be reduced (as is done with the (equivalent) vorticity equation in Tennekes & Lumley, 1972) to the form

$$\dot{\bar{\varepsilon}} + \bar{\varepsilon}_{,j}U_j + (\overline{\varepsilon u_j})_{,j} = -2\nu\, \overline{u_{i,\kappa}u_{i,j}u_{j,\kappa}} - 2\nu^2\, \overline{u_{i,\kappa j}u_{i,\kappa j}} \qquad (1)$$

As is discussed in Tennekes & Lumley (1972), the two terms on the right are of order one but differ by order $R_\ell^{-1/2}$. The remaining terms are of order $R_\ell^{-1/2}$. Other terms (many of which appear in the full equations) are of higher order. The first term on the right represents the production of velocity gradients by stretching by fluctuating strain rate, while the second represents the destruction of these gradients by viscosity.

Several authors (Daly and Harlow, 1970; Jones and Launder, 1972, Reynolds, 1970, Ng and Spalding, 1972) have retained the terms on the left-hand side

$$2\, u_{i,\kappa}u_{i,j}\, U_{j,\kappa} \qquad (2)$$

and another term of similar form, correctly feeling that there must be some source of dissipation. However, these terms are of (relative order R_ℓ^{-1}, since (as is shown in Lumley, 1970)

$$\nu \overline{u_{i,\kappa} u_{i,j}} = \varepsilon(\delta_{\kappa j}/3 + O(S_{\kappa j}\ \lambda/u')) \qquad (3)$$

where $S_{\kappa j}$ is the mean strain rate; since U_i is incompressible, only the second term contributes. Since a term like (3), namely $\nu\ \overline{u_{i,\kappa} u_{j,\kappa}}$ appears in the Reynolds stress equation, we should mention here that this term also has the same behavior as (3) (c.f. Corrsin, 1972). It has been modeled by some authors (Daly and Harlow, 1970; Donaldson, 1972) as proportional to $\overline{u_i u_j}$, whereas the ratio of off-diagonal to diagonal terms must vanish as $R_\ell^{-1/2}$, as shown by (3).

The proper source of the production of dissipation is in the first term in the right hand side of (1). In the following section, we will apply a formal procedure to obtain an unambiguous expression for the right-hand side of (1), and we will find that we obtain a term of form similar to that retained by the authors mentioned in connection with (2), so that in a sense they have been using the right term for the wrong reason. Before going to the formal procedure, however, it will be instructive to carry out a physical analysis of the right-hand side of (1), to see how a production term can be retained at infinite Reynolds number.

The right hand side of (1) represents a balance between stretching and dissipation, and it must be possible for the (relatively small) mismatch to be of either sign. That is, consider the stretching of a single vortex to equilibrium: if the stretching is suddenly increased, momentarily the first term will dominate the second, setting up more vorticity until equilibrium is again attained, at a higher level. If the stretching is reduced, the process is reversed. Put in statistical terms, if the spectral energy flux increases, the first term should dominate the second until the dissipation has been increased to match the flux, and vice versa. There is, in addition an unsteady effect; in a fluctuating turbulent velocity field, equilibrium is never attained, since the strain rate changes before it can be achieved. Hence, there is always a fluctuating mismatch; although to first order, we would expect this to average to zero, we would expect non-linear effects to produce a small net loss.

The response to these effects should depend on the time scale ratio. The time scale of the dissipative eddies is $(\nu/\bar{\varepsilon})^{1/2}$; if the turbulence were isotropic and decaying, a time scale descriptive of the unsteady stretching would be $q^2/2\bar{\varepsilon}$. If there is an input to the spectral flux, there will be another time scale associated with this input. In a homogeneous flow, the input is characterized by P, the production, and the time scale will be $q^2/2P$. In an inhomogeneous situation, it is more difficult to find a simple way of characterizing the input, since part of P is transported. In the next section we will obtain by formal means an appropriate expression in the inhomogeneous situation; here we will retain P, which may be thought of as an approximation for small inhomogeneity. This is effectively what was done by Daly and Harlow (1970), Jones and Launder (1972) and Ng and Spalding (1972). The inverse of the time scale thus may be written as $(2\overline{\varepsilon}/q^2)F(P/\overline{\varepsilon})$, where F is an unknown function. If the production (and hence the anisotropy) is small, we may expand to obtain $(2\overline{\varepsilon}/q^2)(1-aP/\overline{\varepsilon})$. It is, of course, not legitimate to use such an expression for values of $P/\overline{\varepsilon} \sim O(1)$; the expression will serve at least to predict qualitative behavior, however, since it reverses sign as we have reasoned it must.

The difference on the right hand side of (1) might consequently be modeled as

$$- \varepsilon\sqrt{\overline{\varepsilon}/\nu}\ \{0 + b\,\frac{2\overline{\varepsilon}}{q^2}\,(1 - a\,\frac{P}{\overline{\varepsilon}})\,\sqrt{\frac{\nu}{\overline{\varepsilon}}} + O(1/R_\ell)\} \qquad (4)$$

where the 0 symbolizes the equality of the terms at infinite Reynolds number. To second order in time, the direct response to the mean flow distortions will appear through S_{ij}, etc.; these cannot appear to first order because they are of the wrong tensor rank; a scalar term is needed, and can be made from S_{ij} only by a quadratic form. Note that the terms are the same order as the others retained in the equations because the time scale $(\nu/\overline{\varepsilon})^{1/2}$ cancels one of the factors in the magnitude $(\overline{\varepsilon}/\nu)^{1/2}$;

i.e. - as the Reynolds number increases, the magnitude of each term grows, but the difference shrinks at the same rate.

In Lumley (1970), a similar analysis was carried out, though less physical and more formal; consequently, although the order of the term obtained there was correct, the fact that it should be reversible was missed. The general conclusion obtained there, regarding the continual growth of the length scale in a homogeneous flow is correct, however, so long as $a \neq 1$, as may be easily verified using (4) in the analysis there.

As in Lumley (1970), one of the coefficients may be identified by reference to homogeneous decay; we find that

$$b = 2 \quad . \tag{5}$$

Thus, (1) becomes

$$\dot{\bar{\varepsilon}} + \bar{\varepsilon}_{,j} U_j + (\overline{\varepsilon u_j})_{,j} = - 4 \frac{\bar{\varepsilon}^2}{q^2} + 4a \frac{\bar{\varepsilon} P}{q^2} \quad . \tag{6}$$

This is essentially the form used by the author referred to above.

The temperature (or contaminant) dissipation equation may be attacked in exactly the same way. Presuming that the Prandtl number is of order one (a very large or very small value can lead to the retention or discard of different terms) we obtain

$$\dot{\bar{\varepsilon}}_\theta + \bar{\varepsilon}_{\theta,j} U_j + (\overline{\varepsilon_\theta u_j})_{,j} = - 2\kappa \overline{\theta_{,j} \theta_{,i} u_{i,j}} - 2\kappa^2 \overline{\theta_{,ij} \theta_{,ij}} \tag{7}$$

The interpretation of the terms is exactly the same, and the dynamical reasoning is the same. The time scale characterizing the small scales (again presuming the Prandtl number to be of order unity) is $(\nu/\overline{\varepsilon})^{1/2}$; the time scale characterizing the fluctuating stretching is $q^2/2\overline{\varepsilon}$, and that characterizing the input to the spectral flux is $q^2/2P$ (again, we will obtain a better expression than P for the input to the spectral flux later). Hence, the form is very similar to (4), and we obtain

$$- \varepsilon_\theta \sqrt{\overline{\varepsilon}/\nu}\ \{0 + c \frac{2\overline{\varepsilon}}{q^2} (1 - d \frac{P}{\overline{\varepsilon}}) \sqrt{\frac{\nu}{\overline{\varepsilon}}} + 0(1/R_\ell)\} \qquad (8)$$

giving

$$\dot{\overline{\varepsilon}}_\theta + \overline{\varepsilon}_{\theta,j} U_j + (\overline{\varepsilon_\theta u_j})_{,j} = -5 \frac{\overline{\varepsilon\varepsilon_\theta}}{q^2} + 5d \frac{\overline{\varepsilon}_\theta}{q^2} P \qquad (9)$$

Again, the relation between the coefficients is obtained by reference to homogeneous decay data, specifically Gibson and Schwarz (1963). In the following section, we will obtain by formal methods improved approximations to (6) and (9).

Modeling the Third Moments .

If we consistently apply the Reynolds/Peclet number order of magnitude analysis given here, we obtain the set of equations given below (in addition to (1) and (7), we are considering here only velocity and temperature; the treatment of a passive contaminant, or of an active contaminant other than temperature, can be handled in exactly the same way. We write the equations in the Boussinesq approximation - Phillips, 1966).

$$\dot{U}_i + U_{i,j}U_j + (\overline{u_iu_j})_{,j} = -P_{,i}/\rho_o + \delta_{3i}g\Theta/T_o, \quad U_{i,i} = 0 \tag{10}$$

$$\dot{\Theta} + \Theta_{,i}U_i + (\overline{\theta u_i})_{,i} = 0 \tag{11}$$

$$\dot{\overline{u_iu_\kappa}} + U_{i,j}\overline{u_ju_\kappa} + U_{\kappa,j}\overline{u_ju_i} + (\overline{u_iu_\kappa})_{,j}U_j + (\overline{u_iu_\kappa u_j})_{,j} =$$

$$-(\overline{u_\kappa p_{,i}} + \overline{u_ip_{,\kappa}})/\rho_o + (\overline{u_\kappa\theta}\,\delta_{3i} + \overline{u_i\theta}\,\delta_{3\kappa})g/T_o + - 2\overline{\varepsilon}\delta_{i\kappa}/3 \tag{12}$$

$$\dot{\overline{\theta^2}} + 2\Theta_{,j}\overline{u_j\theta} + \overline{\theta^2}_{,j}U_j + (\overline{\theta^2u_j})_{,j} = -2\varepsilon_\theta \tag{13}$$

$$\dot{\overline{\theta u_i}} + U_{i,j}\overline{\theta u_j} + \Theta_{,j}\overline{u_iu_j} + (\overline{\theta u_i})_{,j}U_j + (\overline{\theta u_iu_j})_{,j} =$$

$$-\overline{\theta p_{,i}}/\rho_o + \delta_{3i}\overline{\theta^2}g/T_o \tag{14}$$

where Θ and θ are, respectively, the mean and fluctuating parts of the temperature. Equations (10)-(14), plus (1) and (7) constitute the set we will consider.

The averages are to be understood as ensemble averages, so that the equations can accomodate evolution of the turbulent field, or changing mean or boundary conditions. In addition, such phenomena as internal waves can be accomodated so long as the period is long compared to any characteristic time of the turbulence, so that there is no direct coupling. We have neglected $\kappa\Theta_{,jj}$, $\nu U_{i,jj}$, $\nu(\overline{u_i u_\kappa})_{,jj}$, $\kappa\overline{\theta^2}_{,jj}$, $\nu(\overline{\theta u_{i,j}})_{,j}$ and $\kappa(\overline{u_i\theta_{,j}})_{,j}$ which are of order R_ℓ^{-1} in their respective equations, and $\nu\overline{\theta_{,j}u_{i,j}}$ and $\kappa\overline{u_{i,j}\theta_{,j}}$ which are of order $R_\ell^{-1/2}$. In the atmosphere, typically $R_\ell^{1/2} \sim 10^3$, so that the neglect of these terms may be expected to be an excellent approximation. We are also neglecting off-diagonal components of the last term in (12), in accord with (3).

Let us consider first the term

$$- (\overline{u_\kappa p_{,i}} + \overline{u_i p_{,\kappa}})/\rho_o \tag{15}$$

in equation (12). Part of this term is a transport term; let us subtract the trace, and consider

$$- (\overline{u_\kappa p_{,i}} + \overline{u_i p_{,\kappa}})/\rho_o + 2(\overline{u_j p})_{,j}\delta_{i\kappa}/3\rho_o = A_{i\kappa}, \text{ say} \quad . \tag{16}$$

Now, from equations (10), (11) and (12), if we knew the distributions of $\overline{u_i u_j}$, $\overline{\theta u_i}$ and the value of g/T_o for all time and space, we would know U_i and Θ; having U_i and Θ, together with $\overline{u_i u_j}$, $\overline{u_i \theta} g/T_o$ and $\overline{\varepsilon}$, equation (12) would give us the transport term plus $A_{i\kappa}$. Thus it must be possible to write the sum of $A_{i\kappa}$ and the transport term as a functional of $\overline{u_i u_j}$, $\overline{\theta u_i}$, $\overline{\varepsilon}$, g/T_o (the latter occurring with and without $\overline{\theta u_i}$). The trace of the transport term itself may be so written (by applying the same reasoning to the trace of equation (12), since $A_{ii} = 0$). It is surely a small step to assume that the *whole* transport term may be so written, and hence that $A_{i\kappa}$ may be so written.

$$A_{i\kappa} = F_{i\kappa}\{\overline{u_i u_\kappa}, \overline{\theta u_i}, \overline{\varepsilon}, g/T_o\} \quad . \tag{17}$$

where the functional extends over all space, and all earlier time. We have not included the direction of the gravity vector, since this is part of the structure of the equations; information about the direction of gravity must be forced to appear in $\overline{u_i u_j}$ and θu_i.

This far, no approximation is involved (except the assumption that either of the third order terms in (12) may be written as (17) if both may be). Now, we wish to introduce the approximation of *weak anisotropy* (which implies *weak inhomogeneity*) and quasi-steadiness. Both these concepts are kinetic theory concepts: that time and length scales of inhomogeneity are large relative to time and length scales of the turbulence, and that the turbulence is nearly in equilibrium (isotropic). These are known to be poor descriptors of turbulence, equivalent to gradient transport concepts. However, we are applying them here to third-order quantities, handling the second-order ones exactly; it is hoped that the predictions will be less sensitive to assumptions made at this level. The assumption at least provides an exact model of a physically realizable process (something like a very rarified gas), obtainable from turbulence in a conceptually (though not physically) possible way (by letting the turbulent length and time scales become short), so that one may hope that the predictions will not be unphysical (i.e. - producing negative energy, etc.) and should bear some qualitative resemblance to turbulence.

To implement the approximation of weak anisotropy, we will expand the right hand side of (17) in a functional power series (assuming fading memory and limited awareness, as in Lumley, 1967); in keeping with quasi-steadiness, we will neglect time derivatives. Carrying out the expansion requires the following steps: express $\overline{u_i u_j}$ as $\overline{u_i u_j} - q^2\delta_{ij}/3 = a_{ij}$, say, and q^2, writing $\overline{\theta u_i} = b_i$. The functional (17) is now a function of a symmetric tensor a_{ij}, a vector b_i, and three scalars, $\overline{\varepsilon}$, q^2 and g/T_0. In the isotropic limit a_{ij}, b_i and g/T_0 all vanish, as do all gradients. Thus $a_{ij,k}$ would be a second order term, and $q^2{}_{,i}$ a first order term. First, form the functional Taylor series in the gradients of the arguments. Second, the tensor coefficients are now functions of the local values of a_{ij}, b_i, etc.; express them in invariant form, arranging them according to order (note that the invariants of a_{ij} are of second and third order, etc.).

Expand in powers of g/T_0; finally, apply dimensional analysis to the coefficients.

The net result consists of terms of two types: those that would be present in a homogeneous flow, and corrections for inhomogeneity. Through third order, the homogeneous terms are

$$A_{ij} = - \{(1 + B_1 II/q)a_{ij} + B_2(a_{ij}^2 - II\delta_{ij}/3)/q^2\}/T \qquad (18)$$

where $a_{ij}^2 = a_{i\kappa}a_{\kappa j}$, and $II = a_{ij}a_{ji}$, the second invariant. $T = cq^2/\bar{\varepsilon}$, where c may be evaluated from the initial rate of return to isotropy (see, for example, Tucker and Reynolds, 1968). Later, we will identify T as the Lagrangian integral time scale, which permits approximate evaluation of $c \sim 1/8$.

The lowest order term, a_{ij}/T, may be identified as that suggested by Rotta (1951). The form (18) is consistent with the observations of Champagne, et. al. (1970) that the principal axes of $A_{i\kappa}$ and $a_{i\kappa}$ were the same, in a homogeneous flow. Initially, it had been hoped that first order terms would provide an adequate model; however, the second and third order terms prove to be necessary, at least in the wake. The third order term speeds the return to isotropy for large anisotropy, while the second provides some redistribution in the presence of shear. In the wake, the peak of $\overline{w^2}$ is directly attributable to the second order term only, while the peak in $\overline{u^2}$ cannot be reduced to reasonable proportions without the third order term.

Through third order, the terms involving first derivatives are buoyancy corrections:

$$\beta_1(g/T_o)(b_i\delta_{j\kappa} + b_\kappa\delta_{ij} - b_j\delta_{i\kappa}2/3)\; q^2_{,j}/q^2$$
$$\beta_2(g/T_o)(b_i\delta_{j\kappa} + b_\kappa\delta_{ij} - b_j\delta_{i\kappa}2/3)\; \overline{\varepsilon}_{,j}/\overline{\varepsilon} \qquad (19)$$

where use has been made of the fact that $A_{i\kappa} = A_{\kappa i}$, $A_{ii} = 0$. There are also tensorially appropriate terms in $q^2_{,ij}$, $\overline{\varepsilon}_{,ij}$, $q^2_{,i}\overline{\varepsilon}_{,j}$, $q^2_{,i}q^2_{,j}$, $\overline{\varepsilon}_{,i}\overline{\varepsilon}_{,j}$ and $gb_{i,j}/T_o$, the coefficients of the first five being first order functions of $\overline{\varepsilon},q^2$ and a_{ij}; the sixth term, being a third order buoyancy correction, has a coefficient which is a function only of $\overline{\varepsilon},q^2$. Finally, there are third order terms in $a_{i\kappa,j\ell}$, $a_{i\kappa,j}q^2_{,\ell}$ and $a_{i\kappa,j}\overline{\varepsilon}_{,\ell}$, with coefficients in $\overline{\varepsilon},q^2$. It is not expected that all these terms will be equally important. In the (isothermal) wake, we found that only the terms in $q^2_{,ij}$, $\overline{\varepsilon}_{,ij}$ and $a_{i\kappa,j\ell}$ were dynamically important.

Note that the type of second order terms suggested by Rotta (1951) and Reynolds (1970) and others do not appear; our formalism excludes *explicit* dependence on the mean velocity profile, and forces implicit dependence through second order quantities.

Now, we may apply the same reasoning to the term in (14). Examining the equations, we find that

$$- \overline{\theta p_{,i}}/\rho_o = F_i\{\overline{\theta u_j}, \overline{u_i u_j}, g\overline{\theta^2}/T_o, g/T_o, \overline{\varepsilon}\} \tag{20}$$

The dissipation $\overline{\varepsilon}$ does not appear directly, since (assuming high Reynolds and Peclet numbers) this equation has no dissipation terms. However, in the homogeneous case, $\overline{\theta u_i}$ and $\overline{u_i u_j}$ (and g/T_o) are not sufficient to determine U_i and Θ; some other quantity must be included to fix a length scale for the turbulence. This is reflected in the group in (19) being dimensionally incomplete without $\overline{\varepsilon}$ (in the homogeneous case); that is, a non-trivial form can be found only at an order higher than the first. We will include $\overline{\varepsilon}$, noting that including redundant quantities can do no harm.

Applying our expansion procedure, we find for the homogeneous part (to third order)

$$\begin{aligned} - \overline{\theta p_{,i}}/\rho_o = - \{&(17/16 + a_1 II/q^4 + a_2 T^2 g^2 \overline{\theta^2}/q^2 T_o^{\,2})\delta_{ij} \\ &+ a_3 a_{ij}/q^2 + a_4 a_{ij}^2/q^4\} \, b_j/T \end{aligned} \tag{21}$$

where T is the same as in (18), and the coefficient has been evaluated by reference to homogeneous decay data again. That is, the criterion was applied that, in the limit of small anisotropy, *i.e.* - late in the decay, $\overline{\theta u_i} \sqrt{q^2/\overline{\theta^2}}$ should decay at the same rate as a_{ij}. The lowest order part of (21), b_i/T, has been suggested on an *ad-hoc* basis by Donaldson (1972).

The inhomogeneous terms may easily be generated; through second order they are (with numerical coefficients)

$$\overline{g\theta^2}_{,i}/T_o; \quad (qT\overline{g\theta^2}/T_o)q^2_{,i}; \quad (T^2/q)(\overline{g\theta^2}/T_o)\overline{\varepsilon}_{,i} \tag{22}$$

These are all buoyancy corrections; third order terms bring in

$$b_{i,jk}; \quad b_{i,j}\overline{\varepsilon}_{,k}; \quad b_{i,j}q^2_{,k}; \quad \overline{g\theta^2}_{,i}\varepsilon_{,j}/T_o; \quad \overline{g\theta^2}_{,i}q^2_{,j}/T_o \tag{23}$$

plus anisotropy corrections to the first and second order terms. It will be interesting to see by calculation whether it is necessary to retain terms of this order.

The Transport Terms

The various transport terms may be attacked in the same way. Adopting the forms (18) and (20), we find

$$\overline{\varepsilon_\theta u_i} = F_i\{\overline{u_i u_j}, \overline{\theta u_i}, \overline{\varepsilon}, \overline{\varepsilon_\theta}, g/T_o\} \tag{24}$$

Following exactly the same procedure, we obtain to first order

$$\overline{\varepsilon_\theta u_i} = -A_{41}\overline{\varepsilon}_\theta T(q^2/2)_{,i} - A_{42}\overline{\varepsilon}_\theta T^2\overline{\varepsilon}_{,i} - A_{44}(q^2/3)T\overline{\varepsilon}_{\theta,i} + A_{45}(\overline{\varepsilon}_\theta/T)^{1/2}\,\overline{\theta u_i} \tag{25}$$

so that gradients of several quantities, and another flux, can produce a flux of $\overline{\varepsilon}_\theta$.

The same reasoning will produce

$$\overline{\theta^2 u_i} = F_i\{\overline{u_i u_j}, \overline{\theta u_i}, \overline{\theta^2}, \overline{\varepsilon}_\theta, g/T_o\} \tag{26}$$

so that

$$\overline{\theta^2 u_i} = -A_{31}\overline{\theta^2}T_\theta(q^2/2)_{,i} - A_{33}(q^2/3)T_\theta \overline{\theta^2}_{,i} - A_{34}T_\theta^{\,2}(q\,/3)\overline{\varepsilon}_{\theta,i} + A_{35}\overline{\theta^2}^{\,1/2}\,\overline{\theta u_i} \tag{27}$$

where

$$T_\theta = \overline{\theta^2}/8\overline{\varepsilon}_\theta \quad .$$

The number of additional constants introduced in (25) and (27) is somewhat startling. It is encouraging that we find a simpler result when we examine $\overline{\theta u_i u_j}$. Thus

$$\overline{\theta u_i u_j} = F_{ij}\{\overline{\theta u_i}, \overline{u_i u_j}, \overline{\theta^2} g/T_o, \overline{\varepsilon}, g/T_o\} \tag{28}$$

which leads (to first order) to

$$\overline{\theta u_i u_j} = A_{53} Tq\overline{\theta^2}\delta_{ij} g/T_o \tag{29}$$

Whether or not it is necessary to carry second order terms in any particular case is a point that can be settled in general only by computation. We would normally expect to need only first order terms in the fluxes, since taking the divergence raises the order by one. One situation, however, provides a clear need for second order terms: if the first order terms vanish identically in a particular physical problem. In (28), if gravity is not important, the transport of thermal flux vanishes. Since this is surely not true, second order terms are necessary to provide an adequate model when stratification is weak:

$$\overline{\theta u_i u_j} = \delta_{ij}(A_{53}\mathcal{T}q\overline{\theta^2}g/T_o + A_{54}q^2\mathcal{T}(\overline{\theta u_p})_{,p})$$
$$+ A_{55}\mathcal{T}q(3\overline{u_i u_j}/q^2-\delta_{ij})\theta^2 g/T_o + A_{56}q^2\mathcal{T}[(\overline{\theta u_i})_{,j} + (\overline{\theta u_j})_{,i}] \qquad (30)$$

Only the terms in A_{54} and A_{56} will remain if gravitational effects are relatively weak. The term in A_{55} simply corrects the term in A_{53} for anisotropy. We may give a simple physical interpretation to the terms in A_{53} and A_{55}: $\theta g/T_o$ is the buoyant acceleration, and $\mathcal{T}q$ the turbulent length scale; hence, $\mathcal{T}q\theta g/T_o$ is that part of the turbulent energy that is correlated with the temperature fluctuation. The terms in A_{54} and A_{56} are simply gradient transport.

The purely mechanical transport terms are also relatively simple. For the mechanical dissipation we have

$$\overline{\varepsilon u_i} = F_i\{\overline{u_i u_j}, \overline{\theta u_i}, \overline{\varepsilon}, g/T_o\} \qquad (31)$$

leading to

$$\overline{\varepsilon u_i} = - A_{21}(q^2/3)(q^2/2)_{,i} - A_{22}T(q^2/3)\overline{\varepsilon}_{,i} \tag{32}$$

In an exactly similar way, we find

$$\overline{(u_i u_\kappa + 2\delta_{i\kappa}p/3\rho_o)u_j} = F_{i\kappa j}\{\overline{u_i u_j}, \overline{u_i\vartheta}, \overline{\varepsilon}, g/T_o\} \tag{33}$$

which gives (to first order)

$$\overline{(u_i u_\kappa + 2\delta_{i\kappa}p/3\rho_o)u_j} = - A_{11}T(q^2/3)[\delta_{i\kappa}\delta_{j\ell} + a(\delta_{ij}\delta_{\kappa\ell}+\delta_{i\ell}\delta_{\kappa j})]q^2_{,\ell}$$

$$- A_{12}T^2(q^2/3)[\delta_{i\kappa}\delta_{j\ell} + b(\delta_{ij}\delta_{\kappa\ell} + \delta_{i\ell}\delta_{\kappa j})]\overline{\varepsilon}_{,\ell} \tag{34}$$

where a and b are constants. In each case we have tried to choose signs and numerical coefficients which will result in the constants A_{pq} being positive and of order one.

Just as we found for the transport of thermal flux, there is a situation in which the first order transport of Reynolds stress vanishes, and second order terms are consequently necessary. Specifically, in a simple shear $U_1(x_2)$, in which only two-gradients are non-zero, $\overline{u_1 u_2 u_2} = 0$ from (33). Since this is a dynamically important quantity, we will need second order terms (in the wake calculations to be described later, the omission of this transport results in undesirable, and unrealistic, hyperbolic behavior, with steepening fronts, etc.). The second order terms are

$$\begin{aligned} & T(q^2/3)[\alpha_1 a_{i\kappa,j} + \alpha_2(a_{ij,\kappa} + a_{\kappa j,i}) \\ & \quad + \alpha_3\delta_{i\kappa}a_{jp,p} + \alpha_4(\delta_{ij}a_{\kappa p,p} + \delta_{\kappa j}a_{ip,p})] \\ & + T[\beta_1\delta_{i\kappa}a_{j\ell} + \beta_2 a_{i\kappa}\delta_{j\ell} + \beta_3(\delta_{ij}a_{\kappa\ell} + \delta_{\kappa j}a_{i\ell}) \\ & \quad + \beta_4(\delta_{i\ell}a_{\kappa j} + \delta_{\kappa\ell}a_{ij})]q^2_{,\ell} \\ & + T^2[\gamma_1\delta_{i\kappa}a_{j\ell} + \gamma_2 a_{i\kappa}\delta_{j\ell} + \gamma_3(\delta_{ij}a_{\kappa\ell} + \delta_{\kappa j}a_{i\ell}) \\ & \quad + \gamma_4(\delta_{i\ell}a_{\kappa j} + \delta_{\kappa\ell}a_{ij})]\varepsilon_{,\ell} \end{aligned} \tag{35}$$

The Dissipation Equations

Finally, we may apply the same procedure to obtain the forms for the right hand sides of (1) and (7). The right hand side of (1) must be a scalar functional of the same variables as (31). Complete to terms of third order, the right hand side of (1) is

$$\mathrm{RHS}(1) = -4\overline{\varepsilon}^2/q^2 + \overline{\varepsilon}\{c_1 II/q^4 + c_2 III/q^6\}/T \tag{36}$$

and for (7) we obtain (with the same dependency as (24)

$$\text{RHS}(7) = -5\bar{\varepsilon}\bar{\varepsilon}_\theta/q^2 + \bar{\varepsilon}_\theta\{d_1 II/q^4 + d_2 III/q^6 + d_3 b_i b_i/\bar{\varepsilon}_\theta T q^2$$

$$+ d_4 a_{ij} b_i b_j/\bar{\varepsilon}_\theta T q^4 + d_5 g\bar{\varepsilon}_\theta^{1/2} T^{3/2}/T_o q + d_6 g\bar{\varepsilon}_\theta^{1/2} II T^{3/2}/T_o q^5 \qquad (37)$$

$$+ d_7 b_i b_i T^{1/2} g/\bar{\varepsilon}_\theta^{1/2} q^3 T_o\}$$

We have accepted the coefficients determined from the homogeneous decay data. III is the third invariant of a_{ij}, $III = a_{ij}a_{jk}a_{ki}$.

If we look at (36); keeping only second order terms, we see that what we sould have used in (6) and (9) in place of P is $II = a_{ij}a_{ji}$ (with a suitable coefficient to make the dimensions correct). It is reasonable to associate II with the spectral flux, since the inequality of components indicates that straining of the turbulence is taking place. II is something like P; if we made the simplistic K-theory approximation that $a_{ij} \propto S_{ij}$ where S_{ij} is the mean strain rate, then $P \propto II$. In reality, however, $II \neq 0$ in regions of most flows where P vanishes. Hence, we will still have production of $\bar{\varepsilon}$ there. The same reasoning applies to (37), where it is also evident that, even to second order, our physical reasoning resulted in the neglect of a number of important terms.

Evaluation of Coefficients

The evaluation (or elimination) of most of these coefficients will have to await detailed calculation of flows in which many measurements exist. A few statements can be made, however, by considering simple flows.

First, consider a steady, homogeneous, isotropic turbulence with a linear temperature gradient, and no gravity or mean velocity. Then (14) reduces to (using (21))

$$\Theta_{,j}\,\overline{u_i u_j}\,16T/17 = -\overline{\theta u_i} \qquad (38)$$

This is just the Lagrangian transport form for a passive scalar. Hence, evidently $16T/17$ may be identified with the Lagrangian integral time scale. This scale has been estimated from first principles (by a very crude technique) as $\ell/3u'$, and from wake decay data as $\ell/2.8u'$ (see Tennekes and Lumley, 1972, p. 229), where $\bar{\varepsilon} = u'^3/\ell$, and $3u'^2 = q^2$. This gives roughly $T = q^2/8\bar{\varepsilon}$.

If we consider a steady, homogeneous flow with $U_i = (U_{1,3}x_3, 0, 0)$, $U_{1,3}$ = const., and $\Theta = \Theta_{,3}x_3$, and no gravity, we do obtain "K-theory" forms

$$\overline{\theta u_3} \propto -T\overline{u_3^2}\,\Theta_{,3}\,16/17; \quad \overline{u_1 u_3} \propto -T\overline{u_3^2}\,U_{1,3} \qquad (39)$$

To first order, the coefficients are unity. However, if we keep our higher order terms, the coefficients are complicated functions of $U'T$

and the anisotropy. Thus, although the ratio K_M/K_H begins at 17/16 for very weak shear, it rapidly changes as the shear becomes more intense (measured in terms of $U'T$).

If the first order model is applied to the constant stress layer of a neutral turbulent boundary layer, we obtain (using $T = q^2/8\bar{\varepsilon}$)

$$\overline{u_1^2} = q^2/2, \ \overline{u_2^2} = \overline{u_3^2} = q^2/4, \ -\overline{u_1u_2}/(\overline{u_1^2}\ \overline{u_2^2})^{1/2} = 1/2$$

$$\overline{u_2^2}/u_*^2 = \overline{u_3^2}/u_*^2 = \sqrt{2}\ , \ \overline{u_1^2}/u_*^2 = 2\sqrt{2} \tag{40}$$

Including higher order terms precludes algebraic evaluation. In addition, one may obtain a relation between A_{22} and the c_1 of equation (36). In a similar way, if a constant heat flux is added to the neutral constant stress layer, a relation may be obtained among the constants A_{42}, A_{44} and A_{45}.

Further evaluation of the constants will have to await further computations. One general guideline has suggested itself, however, which provides at least an estimate of the magnitudes of some of the constants, and eliminates others: in a region of constant eddy viscosity, and constant structure (which exists only conceptually), the transport terms should reduce to the K-theory forms[3]. That is (considering the purely mechanical case), in such a region, writing $Q_{ij} = \overline{u_iu_j}$,

$$\varepsilon \propto q^4, \quad Q_{ij}/q^2 = \text{const.} \tag{41}$$

which leads to

$$2q^2_{,i}/q^2 = \varepsilon_{,i}/\varepsilon, \quad Q_{ij,\kappa} = (Q_{ij}/q^2)q^2_{,\kappa} \tag{42}$$

If, for example, we are considering $q^2_{,\kappa}$, then equally good expressions are

$$q^2_{,\kappa}; \quad (q^2/2\varepsilon)\varepsilon_{,\kappa}; \quad q^2 Q^{ji} Q_{ij,\kappa}/3 \tag{43}$$

where Q^{ji} is the inverse of Q_{ij}, $Q^{pi}Q_{ij} = \delta^p_j$. We should thus expect the transport of q^2 to be a form like

$$TQ_{ij}\{Aq^2_{,j} + B(q^2/2\varepsilon)\varepsilon_{,j} + Cq^2 Q^{ji} Q_{ij,\kappa}/3\} \tag{44}$$

where $A + B + C \sim 1$. This expression may be expanded, keeping only second order terms, to eliminate many of the unknown constants.

Computation in the Two-Dimensional Isothermal Wake

For a first computation, we should begin with an isothermal, mechanical flow; when the various constants have been evaluated, if the results are satisfactory, we can proceed to a flow with temperature fluctuations, but without stratification, and finally to flows with stratification.

We have selected for our first computation the two-dimensional wake[4]. We feel that a flow without boundaries is more sensitive to the values of the constants, and our calculations have borne out that feeling. In a flow with boundary conditions, interior points can never depart very far from the boundary values. In a flow without boundaries, however, the variables must develop on their own. Other reasons for selecting the wake are the existence of up gradient energy transport near the center line, which provides a demanding test of the model, the existence of a similarity solution, and the fact that it is one-dimensional.

The equations programmed neglect streamwise transport; they are thus the equations describing the lateral development with *time* of a linear wake of constant cross-section, created instantaneously. Making use of symmetry, only one-half the wake was programmed.

A modified leap-frog method was used: all terms other than transport terms were evaluated at level κ, and the time difference was centered at level κ. Centered space differences were used, evaluated at level κ-1. Since this produces a second order difference equation in time (modeling a first order differential equation), an extra initial condition must be supplied; hence, the first two time steps are set equal. The second solution is oscillatory, and corresponds to amplification of the error (from the true solution) made in this assignment of the value at the second time step. This manifests itself by a gradual separation of the values of the solution at alternate time steps. This is a well understood phenomenon, and we used the classical cure: the solutions at κ and $\kappa+1$ were averaged and the process restarted every 10 time steps.

This differencing scheme is only conditionally stable, so that a limitation must be placed on the size of the time step. The classical limitation is $\Delta\tau \leq 1/4\ \Delta y^2$, in dimensionless variables in which the diffusion coefficients are unity. In our variables, normalized by the velocity defect on the centerline and the standard deviation of the defect profile, the diffusion coefficient is 12.5 (roughly) so that $\Delta t \leq 3.1\ \Delta y^2$ is the appropriate restriction. We used $\Delta t = 3\ \Delta y^2$, and $\Delta y = 0.1$.

Because the wake width grows continually, and the centerline defect shrinks, development is slower and slower, and larger and larger time and space steps can be used as time progresses, both from the viewpoint of truncation error and from the viewpoint of stability. In addition, the size of the computational mesh required constantly increases. Consequently, as the computation progressed, $\overline{v^2}$ was continually monitored at the last mesh point; when it had grown to a value about 1% of the peak, a new, empty mesh point was added. When the number of meshpoints had doubled, the computation was stopped, every other mesh point was discarded, the values renormalized by the new velocity and length scales, and the computation restarted, effectively increasing Δy by 2 and Δt by 4. Each of these doublings corresponded to a dimensionless time lapse of about 5 (defined by $\int dt/T$, where $T = q^2/8\varepsilon$ on the centerline). About three doublings are required to get within 10% of the final values.

If the initial wake width is taken as θ, the momentum thickness, then three doublings corresponds to x/d = 500 (c.f. Tennekes and Lumley, 1970). Each successive doubling corresponds to a four-fold increase in streamwise distance. To get within 1% of the final values, roughly two more doublings are required, that is, an x/d of roughly 8000 (see Figure 1).

Coefficients were approximated by the following techniques: in the wake, outboard of the zero of the advection (c.f. Tennekes and Lumley, 1972) there is a region in which $-\overline{uv} \sim q^2 \sim \overline{v^2}$, production ~ dissipation, advection ~ transport, and hence $\overline{v^2}T \sim$ const. In this region, the equations may be solved exactly; compatability among the equations requires that the coefficient in the net q^2 transport be unity, that in the ε transport be 1/2 and that in the $-\overline{uv}$ transport be v^2/q^2. Using this, plus the concepts described in equations (41)-(44), and in addition neglecting pressure transport (*i.e.* - requiring three-fold symmetry of (33)) reduced the transport coefficients to five, two of known magnitude.

We found generally that the values of the various transport coefficients are not critical, so long as the overall magnitude of the transport is correct. The result is most sensitive to the coefficients in the pressure gradient-velocity correlation, and in the production of dissipation. The latter controls the overall energy level. As already mentioned, the peak in $\overline{w^2}$ is directly attributable to the second order term in the pressure gradient-velocity correlation, and the proportion of the peak in $\overline{u^2}$ to the third order term.

As can be seen from Figures 2-8 (where the calculations are compared with the data of Townsend, 1956), the axial values of $\overline{u^2}$ and $\overline{w^2}$ are higher, while those of $\overline{v^2}$ are lower, than Townsend's measurements. Use of the second order inhomogeneity correction to the pressure gradient-velocity correlation, proportional to $q^2_{,ii}$, and $\bar{\varepsilon}_{,ii}$ has the effect of increasing the relative intensity of the component normal to an energy or dissipation trough (which would help fill-in the trough). This term would transfer energy from $\overline{u^2}$ and $\overline{w^2}$ to $\overline{v^2}$ on the axis, and from $\overline{v^2}$ to $\overline{u^2}$ and $\overline{w^2}$ near the energy peak. This would push the ratios in the right direction on the axis. At the peak, the anisotropy is already so great that little more would be produced due to the counter balancing third order terms.

Unfortunately, we were not able to use significant values of this term due to a second order non-linear computational instability, triggered by the addition of points at the edge of the mesh; even very small values of this coefficient required drastic decreases in the size of the time step. This is not a fundamental difficulty, however; we must simply find a differencing scheme for these terms which is more stable, or find a way of adding points to the mesh which will not excite the instability.

The peak in the curve of $\bar{\varepsilon}$ will also be essentially removed by the use of this term; at present, the flow is too isotropic near the center line, and consequently there is not enough $\bar{\varepsilon}$ production there. Increasing the anisotropy there will make the $\bar{\varepsilon}$ production more uniform and permit a reduction in the overall level of $\bar{\varepsilon}$.

It is worth mentioning that in the range of values of x/d in which Townsend's measurements were made (500-1000), our values changed only a total of 4%, which is about the variation observed in Townsend's data; it thus appears perfectly possible that Townsend's measurements are 6%-10% from the true self-preserving values (but would appear to be self-preserving). This is borne out by the fact that his measured Reynolds stress is not self-preserving (*i.e.* - is not proportional to yU, where U is the mean velocity defect, an exact result of self-preservation) by about this percentage. The direction of the error and trend (if it exists) in each component depends on the initial values, and would not necessarily be the same in our computation and in the real experiment.

Although the number of terms involved boggles the imagination, the cost of such a calculation is not prohibitive. One run of the wake calculation, to six doublings, programmed on the IBM 370/165, costs about $15.00. It must be remembered that this is an experimental calculation, and the program was written so that the various constants could be adjusted at will; no attempt was made to minimize cost. In the final form, the cost should be substantially lower perhaps of the order of $10.00. If we envision a three-dimensional calculation of a stratified flow with boundaries (which converges some five times as fast as a boundary-free flow) involving some 10^4 points and 17 equations instead of six, we would have an estimated cost of perhaps $3000. This is not a particularly accurate way to make such an estimate, because the larger calculation cannot be done in core, and requires reading out to disc storage. In addition, the pressure must be computed in a three-dimensional calculation. A detailed, direct estimate of the number of computations involved, and storage requirements, suggests a cost of the order of $1500. The difference between the estimates probably represents the different proportion of I/O, CPU and auxiliary storage access time, and reflects the inefficiency of our present program. It is never-the-less clear that such a calculation is within our grasp for a few thousand dollars.

Acknowledgments

We have profited from many discussions with J. W. Deardorff, R. Owen, W. C. Reynolds, H. Tennekes and J. Wyngaard. A remark of Deardorff led to the understanding of the dissipation production; a remark of Owen suggested the importance of the second and third order terms in the pressure gradient-velocity correlation; Reynolds carried out many calculations in an early stage of the investigation, and contributed much analysis and comment which led to a better understanding of the structure of the equations, and an appreciation of the importance of the second order transport terms; Tennekes of course shares responsibility for the Reynolds number order-of-magnitude analysis, and has made extensive contributions by applying the model to simple situations, bringing flaws to light; Wyngaard has made independent calculations of the stratified atmospheric boundary layer, and his findings influenced the development of the model.

We are particularly grateful to our assembly language programmer, M. Hurvitz, who has been absolutely indispensable.

Generally, the development of the theory has been the responsibility of JLL and the execution of the calculations the responsibility of BK-N, although there has been intensive and extensive interactions between the two, so that they cannot be separated so simply. These calculations will form part of the doctoral dissertation of BK-N.

We wish to acknowledge the assistance of the Applied Research Laboratory in the preparation of the final figures.

FIGURE CAPTIONS

Figure 1. Convergence of the normalized mean square total velocity (twice the energy). The squares indicate points at which the wake was renormalized. Thus, between such points, the scale is linear, but increases by a factor of four at each point (hence the change in slope). Each point indicates a doubling of the width. The abscissa values hence are multiples of the initial wake width. The corresponding values of x/d can be obtained as roughly eight times the square of the abscissa. Hence, the third doubling corresponds roughly to x/d = 512.

Figure 2. In all the figures, the experimental points are those of Townsend (1956) scaled from the published graphs, renormalized and machine plotted. The solid line is machine plotted from the calculated points, approximately 80 per figure, with linear interpolation. The mean velocity-defect profile.

Figure 3. The Reynolds stress. In the self-preserving state, this can be obtained directly from the first moment of the mean velocity defect profile. Townsend's values are not quite self-preserving. The constant of proportionality determines the growth; our wake is growing somewhat slower than Townsend's. All turbulent quantities normalized by the mean velocity defect at the origin.

Figure 4. The normalized mean square value of the streamwise fluctuating velocity.

Figure 5. The normalized mean square value of the fluctuating velocity normal to the plane of the wake.

Figure 6. The normalized mean square value of the cross-stream fluctuating velocity in the plane of the wake.

Figure 7. The normalized mean square value of the total fluctuating velocity (twice the energy). The experimental points are the sum of the three component intensities read from Townsend's faired curves.

Figure 8. The normalized dissipation. Townsend's values of the abscissa are clearly incorrect, since they result in a displacement of the peak of the production, which can be obtained without assumption from the mean velocity curve. The abscissa has been renormalized so that the peak of the measured production occurs in the same place as that obtained from the mean velocity profile.

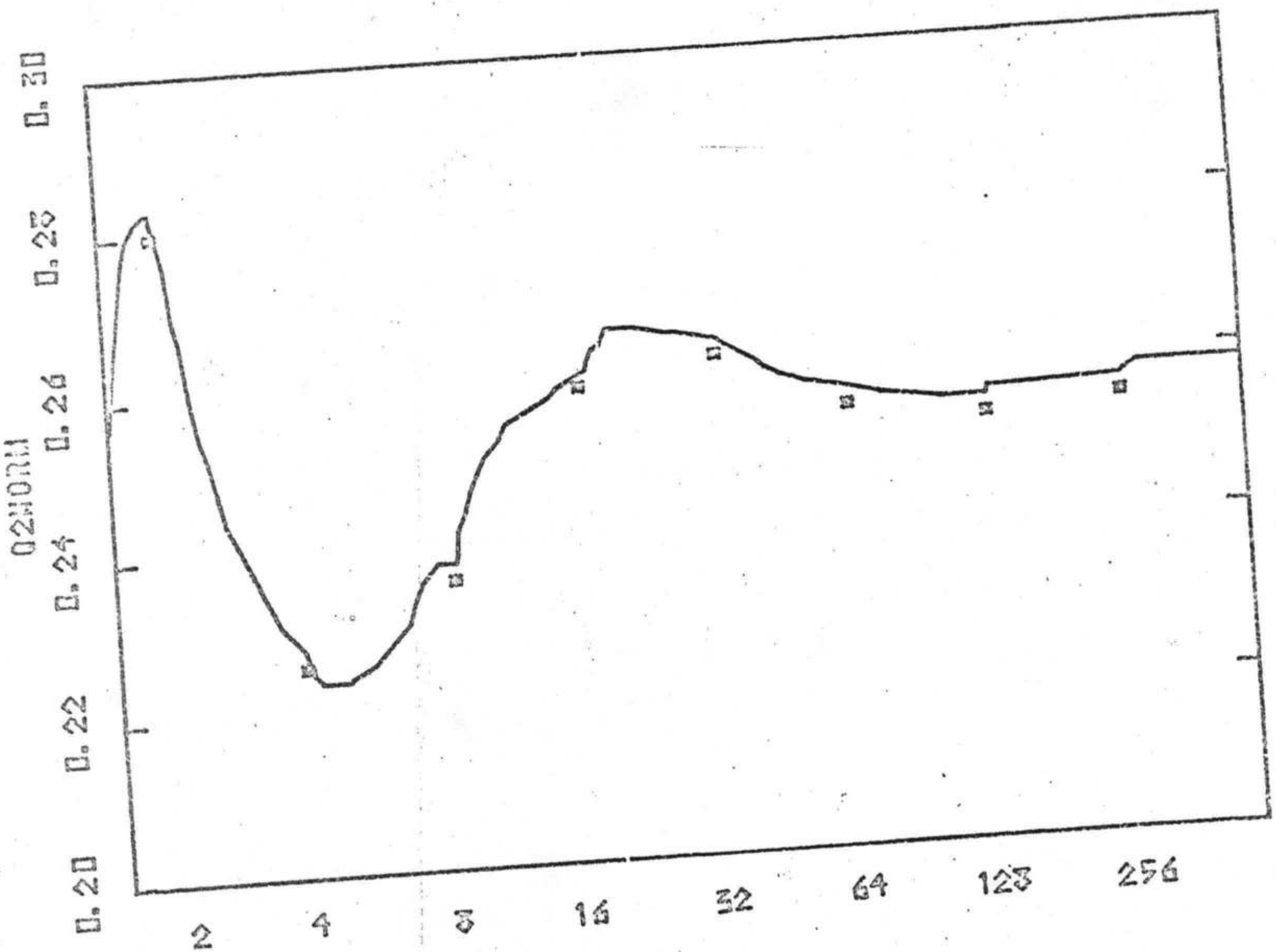

FIGURE 1

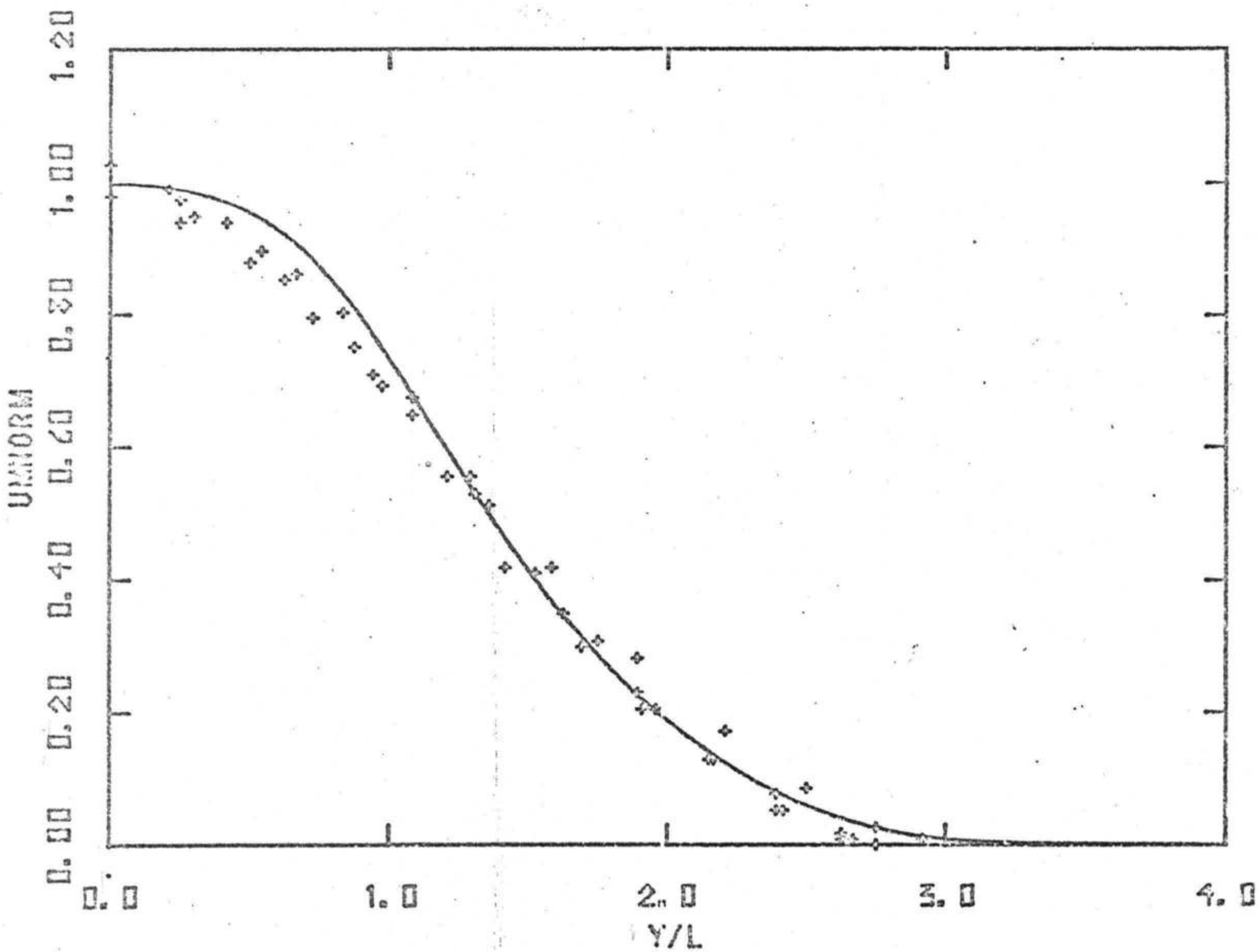

FIGURE 2

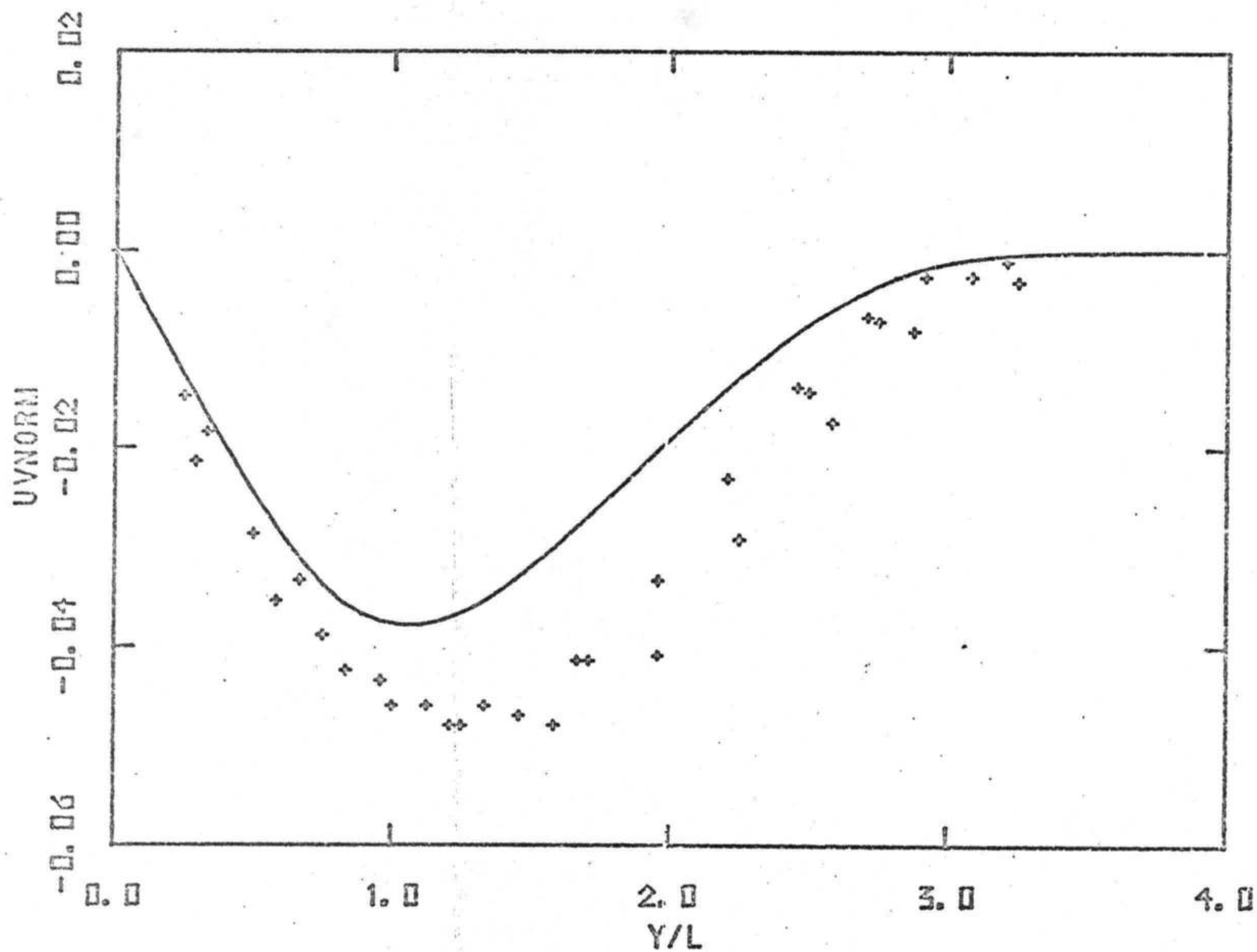

FIGURE 3

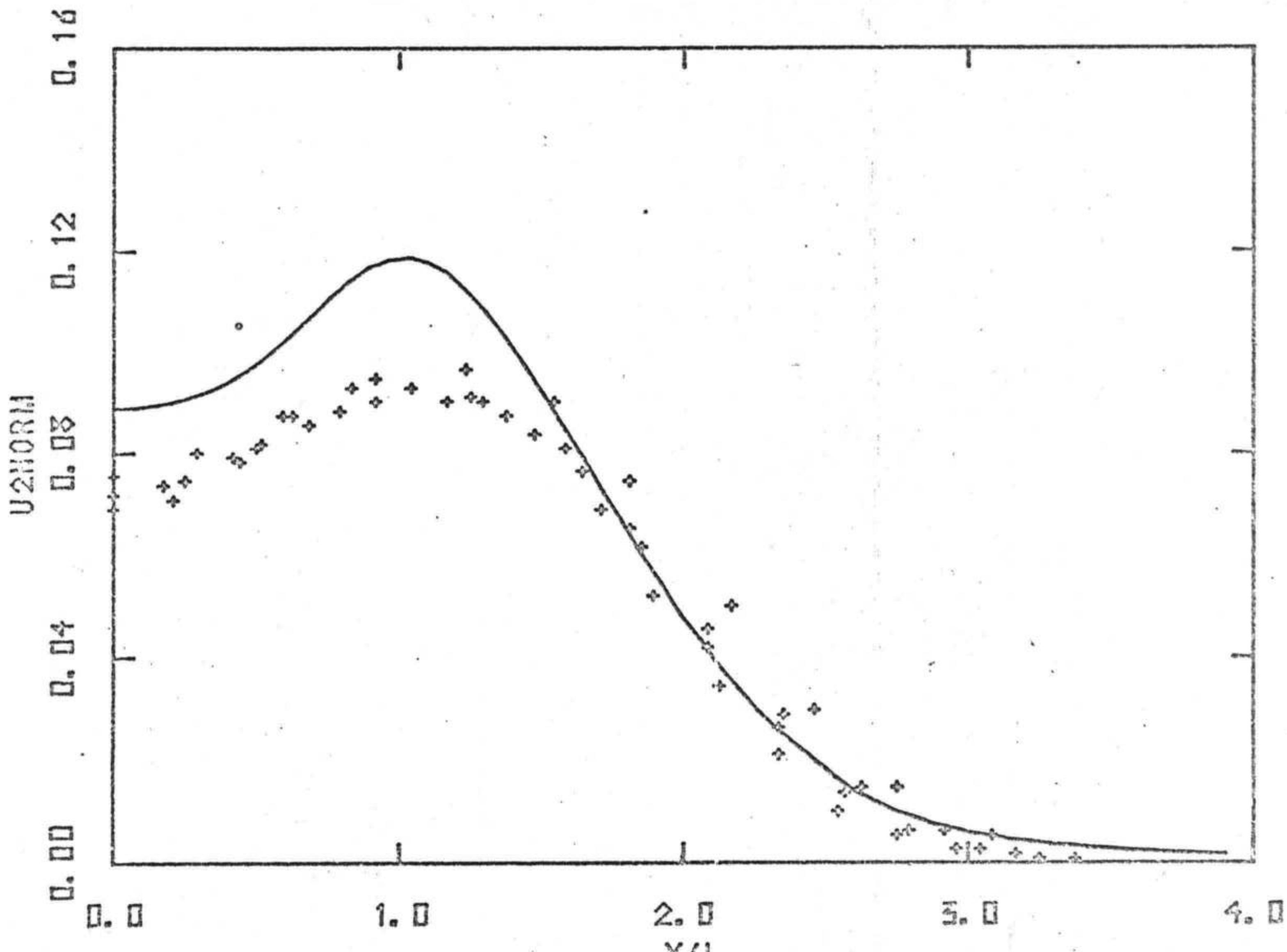

FIGURE 4

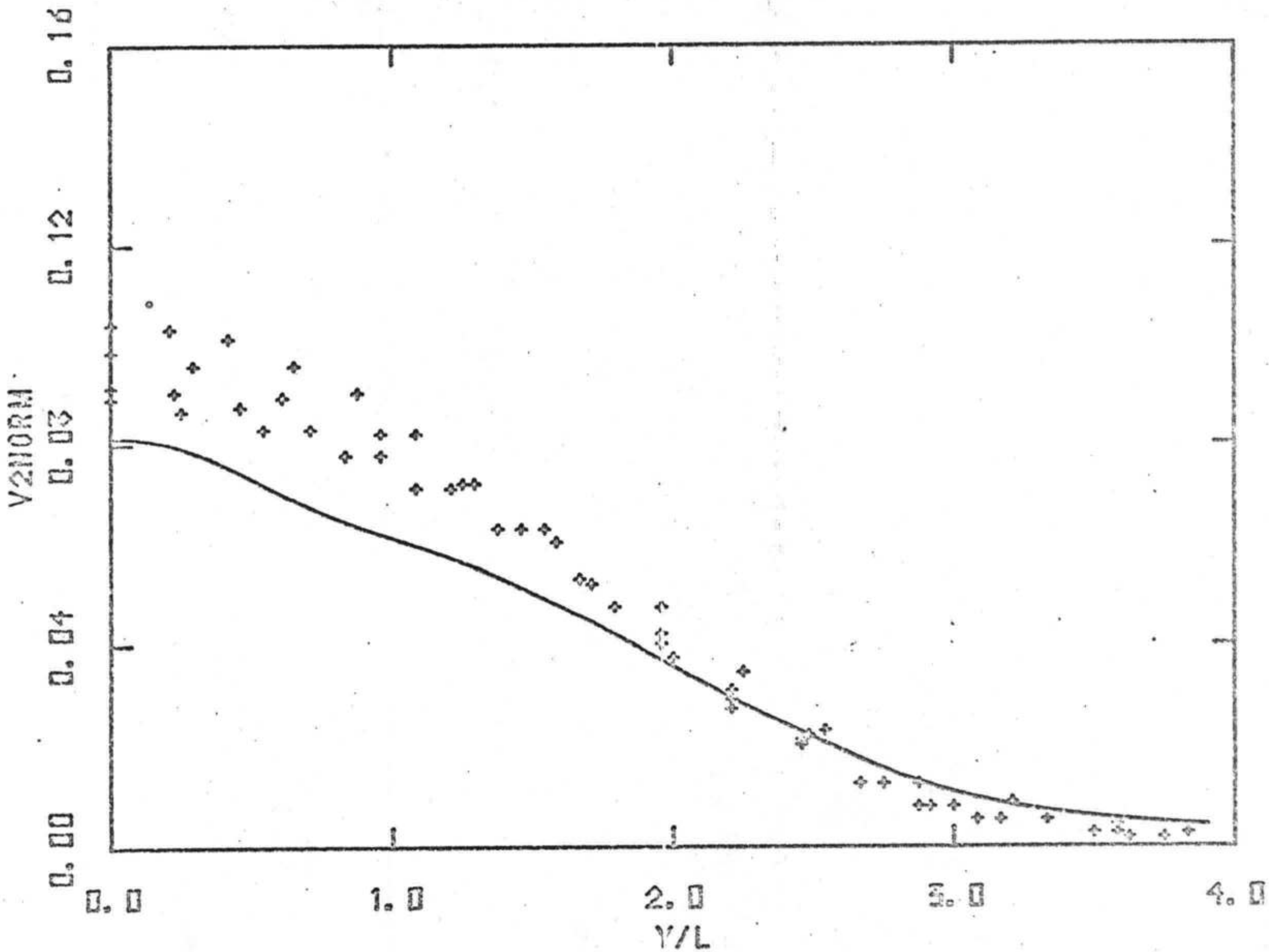

FIGURE 5

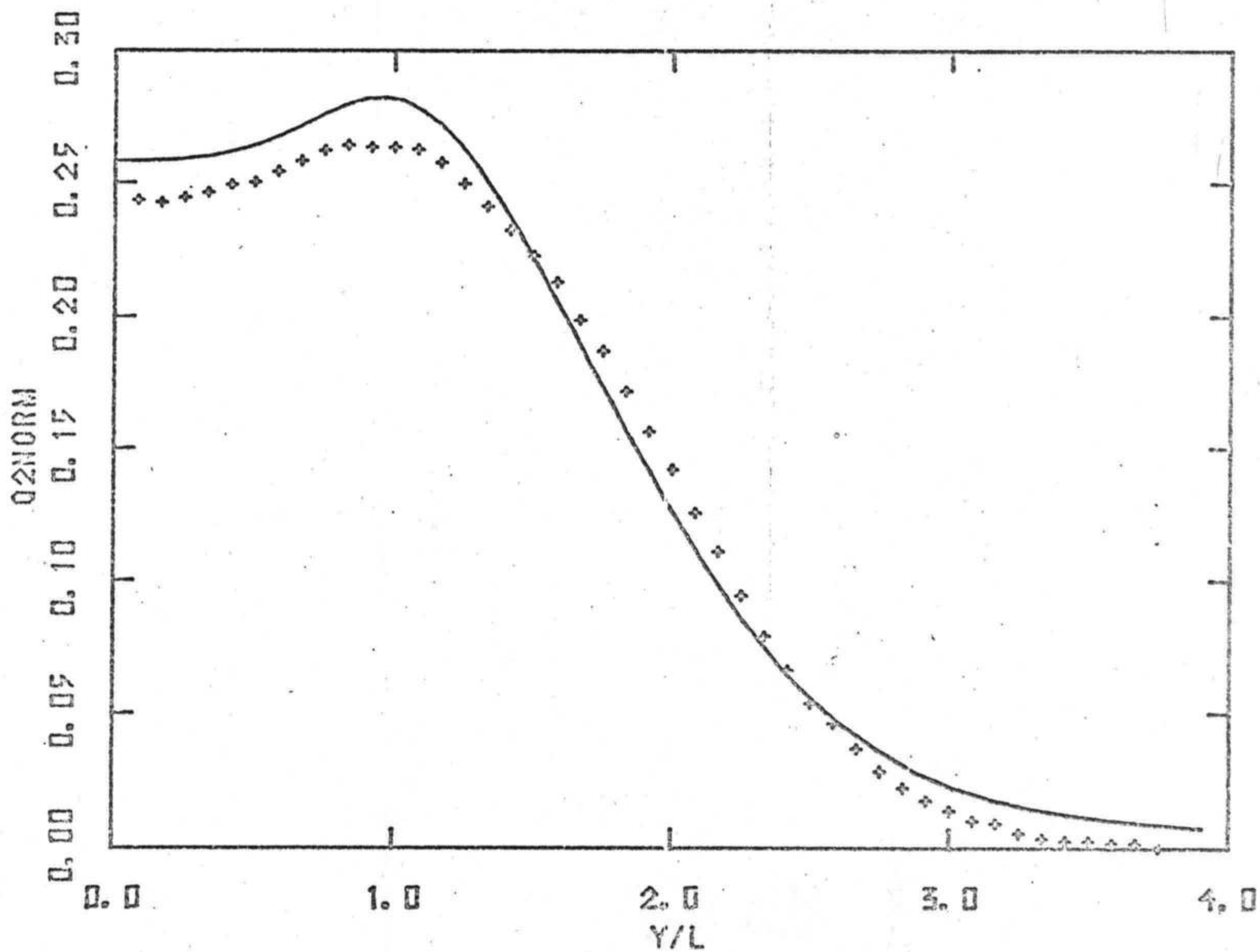

FIGURE 7

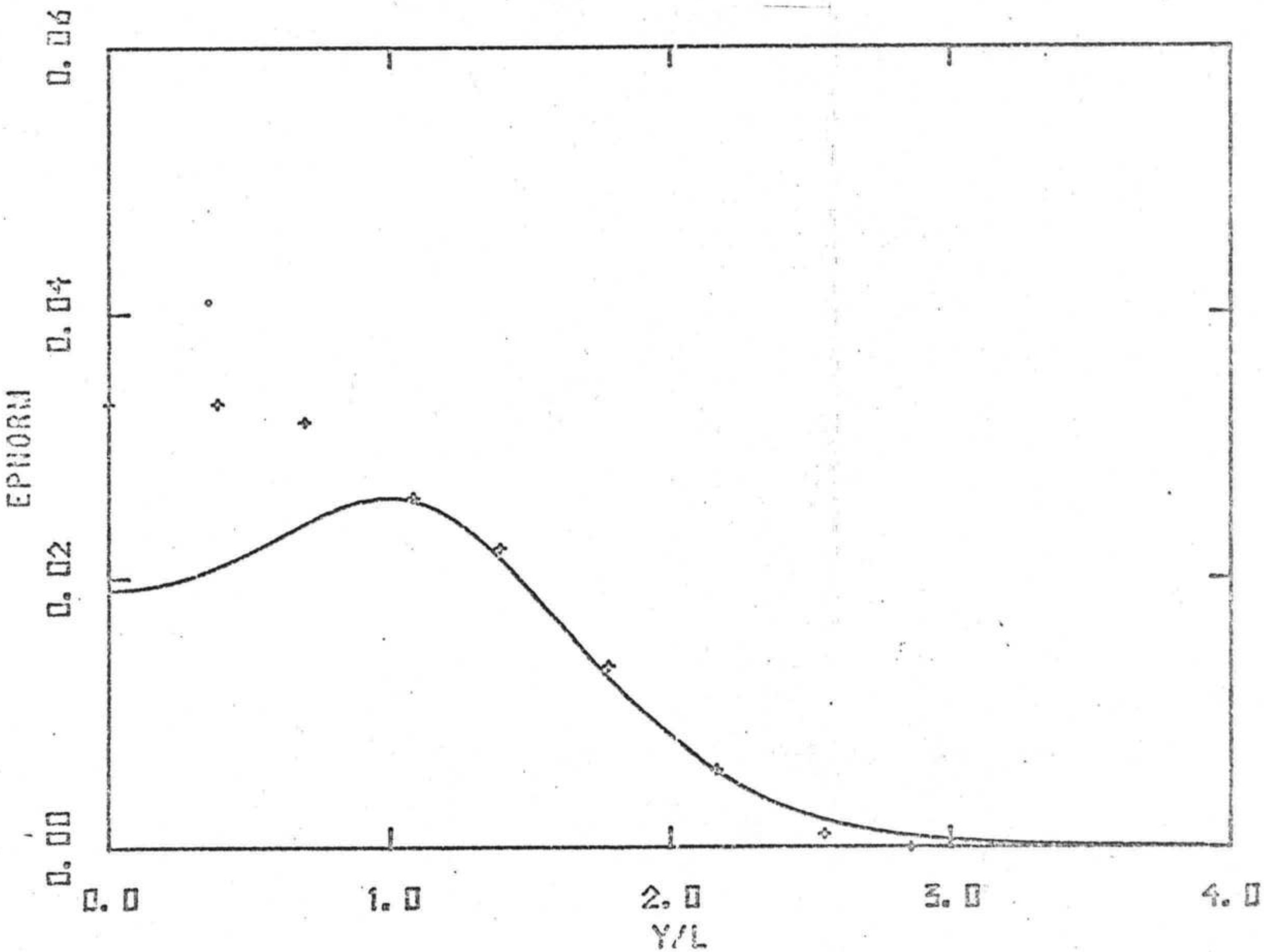

FIGURE 8

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FOOTNOTES

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[2] Bradshaw is credited with the remark, at the Stanford Conference on Computation of Turbulent Boundary Layers, that with six constants he could create an elephant.

[3] Because such a region is characterized by a single length and velocity scale at each point, and only in such a region would K-theory forms be expected to hold (c.f. Tennekes and Lumley (1972)).

[4] Properly speaking, we should begin with a homogeneous flow, like that of Champagne, et. al. (1970), to evaluate the coefficients in the homogeneous forms. When we began the computation, however, we did not realize that second- and third-order terms would be necessary, and hence were under the impression that the only such first order coefficient (c in T) had been evaluated in Lumley (1970). It will now, of course, be necessary to redo this calculation.