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Computer search in projective planes for the sizes of complete arcs

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Abstract. The spectrum of possible sizes k of complete k-arcs in finite projective planes $PG(2, q)$ is investigated by computer search. Backtracking algorithms that try to construct complete arcs joining the orbits of some subgroup of collineation group $P\Gamma L(3, q)$ and randomized greedy algorithms are applied. New upper bounds on the smallest size of a complete arc are given for $q = 41, 43, 47, 49, 53, 59, 64, 71 \leq q \leq 809,\ q \neq 529, 625, 729$, and $q = 821$. New lower bounds on the second largest size of a complete arc are given for $q = 31, 41, 43, 47, 53, 125$. Also, many new sizes of complete arcs are obtained for $31 \leq q \leq 167$.

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1. Introduction

Let $PG(2, q)$ be the projective plane over the Galois field $GF(q)$. A k-arc in $PG(2, q)$ is a set of k points, no three of which are collinear. A k-arc in $PG(2, q)$ is called complete if it is not contained in a $(k + 1)$-arc of $PG(2, q)$. For an introduction to these geometric objects, see [8]. A complete arc in a plane $PG(2, q)$, points of which are treated as 3-dimensional q-ary columns, defines a parity check matrix of a q-ary linear code with codimension 3, Hamming distance 4, and covering radius 2 [9, Section 1.3]. For information on the covering radius of a code, see [2].

It can be shown that arcs and linear maximum distance separable codes (MDS codes) are equivalent objects, see [9]. Indeed, it was the coding theory problem which provided the initial motivation for our study on the spectrum of values of k for which a complete arc exists.

We use the following notation in $PG(2, q)$: $m_2(2, q)$ is the size of the largest complete arc, $m'_2(2, q)$ is the size of the second largest complete arc, and $t_2(2, q)$ is the size of the smallest complete arc. The corresponding *best known* values are denoted by $\overline{m}'_2(2, q)$ and $\bar{t}_2(2, q)$.

In this work a number of new values of $\overline{m}'_2(2, q)$ and $\bar{t}_2(2, q)$ are obtained by computer search. Many new sizes k for which a complete k-arc in $PG(2, q)$ exists are also obtained. In particular, new upper bounds on the smallest size $t_2(2, q)$ of a complete arc are obtained for $q = 41, 43, 47, 49, 53, 59, 64, 71 \le q \le 809, q \ne 529, 625, 729$, and $q = 821$. These new upper bounds give

$$t_2(2, q) < 4\sqrt{q} \text{ for } 3 \le q \le 809 \text{ and } q = 821. \tag{1}$$

New lower bounds on the second largest size $m_2'(2, q)$ of a complete arc are obtained for $q = 31, 41, 43, 47, 53, 125$. These new lower bounds are as follows:

$$m_2'(2, 31) \ge 22,\ m_2'(2, 41) \ge 30,\ m_2'(2, 43) \ge 28,\ m_2'(2, 47) \ge 32,$$
$$m_2'(2, 53) \ge 42,\ m_2'(2, 125) \ge 66. \tag{2}$$

For previously known results concerning the values of $m_2(2, q)$, $m_2'(2, q)$, and $t_2(2, q)$, see [4], [5], [9], [14], [15], [18] and the references therein. As examples of such references we mention the works [12], [13].

In Section 2 we outline the computer search approach used. In Section 3 we give the sizes of the known complete arcs with $25 \le q \le 167$ and in Section 4 the minimal known sizes of the complete arcs with $3 \le q \le 809$ and $q = 821$ are mentioned. Here, for $q \le 128$, several new small complete k-arcs with $k = \bar{t}_2(2, q)$ are obtained; the new complete arcs are listed in the Appendix.

2. Approaches to computer search

Computer search for complete arcs and caps is considered in many works; see, for example, from [12] to [17]. In this paper for computer search we used two distinct approaches.

One of the ways is based on randomized greedy algorithms that are convenient for relatively large q and for obtaining examples of different sizes of complete caps. At every step an algorithm minimizes or maximizes an objective function f but some steps are executed in a random manner. The number of these steps and their ordinal numbers have been taken intuitively. Also, if the same extremum of f can be obtained in distinct ways, one way is chosen randomly.

We begin to construct a complete arc by using a starting set of points S_0. At every step one point is added to the set. As the value of the objective function f we consider the number of points in the projective space that lie on bisecants of the set obtained. As S_0 we can use a subset of points of an arc obtained in previous stages of the search. A generator of random numbers is used for a random choice. To get arcs with distinct sizes, starting conditions of the generator are changed for the same set S_0. In this way the algorithm works in a convenient limited region of the search space to obtain examples improving the size of the arc from which the fixed points have been taken.

In [16] another approach has been used, constructing complete arcs with the constraint of being stabilized by some particular group. In this paper a similar approach has been used trying to construct complete arcs joining the orbits of some subgroup of $P\Gamma L(3, q)$. This algorithm has been implemented using MAGMA, a system for symbolic computation developed at the University of Sydney. The algorithm starts by fixing a subgroup S of $P\Gamma L(3, q)$ and calculating the set O consisting of the orbits of S that are arcs. Then it performs a backtracking process joining the orbits contained in O. Utilizing this procedure the search is more effective as backtracking considers a whole orbit of points at each step; besides, it allows us to find interesting arcs with non-trivial stabilizers.

3. On the spectrum of sizes of complete arcs in $PG(2, q)$

We repeat here Table 2.1 from [4] as the upper part of Table 1.

k	Conditions on q	Reference
$(q+3)/2$, $(q+5)/2$	$q \equiv 1$ or 3 (mod 4)	[4]
$(q+7)/2$	$q = 4t-1,\ t \neq 2^r$, or $q = 2p-1$, p an odd prime	[4]
$(q+6)/2$	$q = 2^h,\ h \geq 4$	[4]
$(q+8)/3$	$q = 2^{2i},\ i \geq 3$	[4]
$q - \sqrt{q} + 1$	$q = p^{2i} > 4$	[4]
$4(\sqrt{q} - 1)$	$q = p^2$, p is odd, $q \leq 1681$ or $q = 2401$	[6]

Table 1. Infinite sequences of non-oval complete k-arcs in $PG(2, q)$

The values $m_2(2, q)$ are achieved by ovals or hyperovals. Bounds on $t_2(2, q)$ and $m'_2(2, q)$ are taken from [4, Tables 2.3, 2.4], [5, Table 2], and [9, Table 2.3]. The following bound comes from [4], [5], and [19]:

$$t_2(2, q) > \sqrt{2q} + 1. \tag{3}$$

In $PG(2, q)$ if K is a complete k-arc, then

$$k(k-1)(q+1)/2 - (k-2)(2k-3) \geq |PG(2, q)|; \tag{4}$$

$$t_2(2, q) > \sqrt{3q} + 1/2 \text{ if } q = p^i,\ i = 1, 2, 3,\ p \text{ is prime}. \tag{5}$$

Table 2 gives sizes of the known complete arcs in $PG(2, q)$. Values of the cardinality of complete arcs come from [1], [4, Table 2.4], [5, Tables 2,6], [13], and Table 1. New sizes of arcs are obtained by computer. Note that some sizes are obtained by using both approaches at the same time, as described in Section 2; for example, for $k = 34$ in $PG(2, 53)$ and $k = 28$ in $PG(2, 47)$.

Table 2 gives new lower bounds for the second largest size $m'_2(2, q)$ of a complete arc for $q = 31, 41, 43, 47, 53, 125$. These new lower bounds are $m'_2(2, 31) \geq 22$, $m'_2(2, 41) \geq 30$, $m'_2(2, 43) \geq 28$, $m'_2(2, 47) \geq 32$, $m'_2(2, 53) \geq 42$, and $m'_2(2, 125) \geq 66$.

q	$t_2(2, q)$	Sizes k of the known complete arcs with $t_2(2, q) \leq k \leq m'_2(2, q)$	$m'_2(2, q)$	$m_2(2, q)$	References
25	12	$12 \leq k \leq 18 = (q + 11)/2$, $k = 21$	21	26	[4], [5], [14]
27	12	$12 \leq k \leq 19 = (q + 11)/2$, $k = 22$	22	28	[4], [5], [15]
29	13	$13 \leq k \leq 20 = (q + 11)/2$, $k = 24$	24	30	[4], [15]
31	≥ 11	$14 \leq k \leq 22 = (q + 13)/2$	≤ 30	32	[4], [13]
32	≥ 10	$14 \leq k \leq 24 = (q + 16)/2$	≤ 26	34	[3], [4]
37	≥ 12	$15 \leq k \leq 23 = (q + 9)/2$	≤ 36	38	[3], [4], [11]
41	≥ 12	$16 \leq k \leq 26$, $k = 28$, $k = 30 = (q + 19)/2$	≤ 40	42	[4], [13]
43	≥ 12	$16 \leq k \leq 25$, $k = 28 = (q + 13)/2$	≤ 42	44	[4], [5], [13]
47	≥ 13	$18 \leq k \leq 28$, $k = 32 = (q + 17)/2$	≤ 46	48	[1], [4], [13]
49	≥ 13	$18 \leq k \leq 30$, $k = 36$, $k = 43 = (q + 37)/2$	≤ 48	50	[4], [6], [7], [13]
53	≥ 14	$k = 18$, $20 \leq k \leq 30$, $k = 34, 42 = (q + 31)/2$	≤ 51	54	[4], [13]
59	≥ 14	$20 \leq k \leq 33 = (q + 7)/2$	≤ 57	60	[3], [4], [13]
61	≥ 15	$20 \leq k \leq 34 = (q + 7)/2$	≤ 59	62	[3], [4], [11]
64	≥ 13	$22 \leq k \leq 35 = (q + 6)/2$, $k = 57$	57	66	[1], [4]

Table 2. The sizes of the known complete k-arcs in $PG(2, q)$

q	$t_2(2,q)$	Sizes k of the known complete arcs with $t_2(2,q) \le k \le m_2'(2,q)$	$m_2'(2,q)$	$m_2(2,q)$	References
67	≥ 15	$23 \le k \le 37 = (q+7)/2$	≤ 65	68	[3], [4], [17]
71	≥ 16	$22 \le k \le 39 = (q+7)/2$	≤ 69	72	[3], [4], [11]
73	≥ 16	$24 \le k \le 40 = (q+7)/2$	≤ 71	74	[3], [4]
79	≥ 16	$26 \le k \le 43 = (q+7)/2$	≤ 77	80	[3], [4]
81	≥ 14	$26 \le k \le 44 = (q+7)/2$, $k = 73$	≤ 79	82	[3], [4], [6]
83	≥ 17	$27 \le k \le 45 = (q+7)/2$	≤ 81	84	[3], [4]
89	≥ 17	$28 \le k \le 47 = (q+5)/2$	≤ 87	90	[3], [4]
97	≥ 18	$30 \le k \le 51 = (q+5)/2$	≤ 94	98	[3], [4]
101	≥ 18	$30 \le k \le 53 = (q+5)/2$	≤ 98	102	[3], [4]
103	≥ 19	$31 \le k \le 55 = (q+7)/2$	≤ 100	104	[3], [4]
107	≥ 19	$32 \le k \le 57 = (q+7)/2$	≤ 104	108	[3], [4]
109	≥ 19	$32 \le k \le 57 = (q+5)/2$	≤ 106	110	[3], [4]
113	≥ 19	$33 \le k \le 59 = (q+5)/2$	≤ 110	114	[3], [4]
121	≥ 20	$34 \le k \le 64 = (q+7)/2$, $k = 111$	≤ 119	122	[1], [3], [4], [6],[7]
125	≥ 20	$35 \le k \le 66 = (q+7)/2$	≤ 124	126	[3], [4]
127	≥ 21	$35 \le k \le 66 = (q+5)/2$	≤ 124	128	[3], [4]
128	≥ 18	$36 \le k \le 67 = (q+6)/2$	≤ 114	130	[3], [4]
131	≥ 21	$36 \le k \le 69 = (q+7)/2$	≤ 128	132	[3], [4]
137	≥ 21	$37 \le k \le 71 = (q+5)/2$	≤ 134	138	[3], [4]
139	≥ 21	$37 \le k \le 73 = (q+7)/2$	≤ 135	140	[3], [4]
149	≥ 22	$39 \le k \le 77 = (q+5)/2$	≤ 145	150	[3], [4]
151	≥ 22	$39 \le k \le 79 = (q+7)/2$	≤ 147	152	[3], [4]
157	≥ 23	$40 \le k \le 82 = (q+7)/2$	≤ 153	158	[3], [4]
163	≥ 23	$41 \le k \le 85 = (q+7)/2$ and $k \ne 80, 82, 83$	≤ 160	164	[3], [4]
167	≥ 23	$42 \le k \le 87 = (q+7)/2$ and $k \ne 83, 84$	≤ 164	168	[3],[4]

Table 2 (continued). The sizes of the known complete k-arcs in $PG(2,q)$

4. Small complete arcs in $PG(2,q)$

The smallest *known* sizes $\bar{t}_2(2,q)$ of complete arcs for $3 \le q \le 809$ and $q = 821$ in planes $PG(2,q)$ are given in Table 3 where $A_q = \lfloor 4\sqrt{q} - \bar{t}_2(2,q) \rfloor$. The values of $\bar{t}_2(2,q)$ for

$q \le 167$ are taken from Table 2 and [4, Table 2.2]; see also references in Table 3. A dot after the entry indicates that $\bar{t}_2(2, q) = t_2(2, q)$. For $169 \le q \le 809$ we obtained $\bar{t}_2(2, q)$ by computer.

Since $t_2(2, q) \le \bar{t}_2(2, q)$, Table 3 gives new upper bounds on the smallest size $t_2(2, q)$ of a complete arc for $q = 41, 43, 47, 49, 53, 59, 64, 71 \le q \le 809, q \ne 529, 625, 729$, and $q = 821$.

THEOREM. *In* $PG(2, q)$,

(i) $t_2(2, q) < 4\sqrt{q}$ *for* $3 \le q \le 809$ *and* $q = 821$;
(ii) $t_2(2, q) \le 4\sqrt{q} - 8$ *for* $23 \le q \le 257$,
$t_2(2, q) \le 4\sqrt{q} - 7$ *for* $19 \le q \le 317$,
$t_2(2, q) \le 4\sqrt{q} - 6$ *for* $9 \le q \le 383$,
$t_2(2, q) \le 4\sqrt{q} - 5$ *for* $8 \le q \le 443$,
$t_2(2, q) \le 4\sqrt{q} - 4$ *for* $7 \le q \le 512$,
$t_2(2, q) \le 4\sqrt{q} - 2$ *for* $3 \le q \le 601$.

From Table 3,

$$4\sqrt{q} - 12 < \bar{t}_2(2, q) \text{ for } 3 \le q \le 809 \text{ and } q = 821,$$

but

$$4\sqrt{53} - 11 > \bar{t}_2(2, 53).$$

Hence

$$4\sqrt{q} - B_q \le t_2(2, q), \quad \lceil B_q \rceil \ge 12. \tag{6}$$

q	$\bar{t}_2(2, q)$	$4\sqrt{q}$	A_q	References	q	$\bar{t}_2(2, q)$	$4\sqrt{q}$	A_q	References
3	4.	6.9	2	[8]	8	6.	11.3	5	[8]
4	6.	8	2	[8]	9	6.	12	6	[8]
5	6.	8.9	2	[8]	11	7.	13.3	6	[8]
7	6.	10.4	4	[8]	13	8.	14.4	6	[18]

Table 3. The minimal known sizes $\bar{t}_2(2, q)$ of complete arcs in planes $PG(2, q)$, $A_q = \lfloor 4\sqrt{q} - \bar{t}_2(2, q) \rfloor$

q	$\bar{t}_2(2,q)$	$4\sqrt{q}$	A_q	References	q	$\bar{t}_2(2,q)$	$4\sqrt{q}$	A_q	References
16	9	16	7	[18]	107	32	41.4	9	[3]
17	10	16.5	6	[18]	109	32	41.8	9	[3]
19	10	17.4	7	[18]	113	33	42.5	9	[3]
23	10	19.2	9	[18]	121	34	44	10	[1]
25	12	20	8	[5], [14], [18]	125	35	44.7	9	[3]
27	12	20.8	8	[5], [15], [18]	127	35	45.1	10	[3]
29	13	21.5	8	[15], [18]	128	36	45.3	9	[3]
31	14	22.3	8	[18]	131	36	45.8	9	[3]
32	14	22.6	8	[3], [18]	137	37	46.8	9	[3]
37	15	24.3	9	[11]	139	37	47.2	10	[3]
41	16	25.6	9	[13]	149	39	48.8	9	[3]
43	16	26.2	10	[13]	151	39	49.2	10	[3]
47	18	27.4	9	[1], [13]	157	40	50.1	10	[3]
49	18	28	10	[13]	163	41	51.1	10	[3]
53	18	29.1	11	[13]	167	42	51.7	9	[3]
59	20	30.7	10	[13]	169	43	52	9	[3]
61	20	31.2	11	[11]	173	44	52.6	8	[3]
64	22	32	10	[1]	179	44	53.5	9	[3]
67	23	32.7	9	[17]	181	45	53.8	8	[3]
71	22	33.7	11	[11]	191	46	55.3	9	[3]
73	24	34.2	10	[3]	193	47	55.6	8	[3]
79	26	35.6	9	[3]	197	47	56.1	9	[3]
81	26	36	10	[3]	199	47	56.4	9	[3]
83	27	36.4	9	[3]	211	49	58.1	9	[3]
89	28	37.7	9	[3]	223	51	59.7	8	[3]
97	30	39.4	9	[3]	227	51	60.3	9	[3]
101	30	40.2	10	[3]	229	52	60.5	8	[3]
103	31	40.6	9	[3]	233	52	61.1	9	[3]

Table 3 (continued) The minimal known sizes $\bar{t}_2(2,q)$ of complete arcs in planes $PG(2,q)$, $A_q = \lfloor 4\sqrt{q} - \bar{t}_2(2,q) \rfloor$

q	$\bar{t}_2(2,q)$	$4\sqrt{q}$	A_q	References	q	$\bar{t}_2(2,q)$	$4\sqrt{q}$	A_q	References
239	53	61.8	8	[3]	379	71	77.9	6	[3]
241	53	62.1	9	[3]	383	71	78.3	7	[3]
243	54	62.4	8	[3]	389	73	78.9	5	[3]
251	55	63.4	8	[3]	397	73	79.7	6	[3]
256	56	64	8	[3]	401	74	80.1	6	[3]
257	56	64.1	8	[3]	409	75	80.9	5	[3]
263	57	64.9	7	[3]	419	76	81.9	5	[3]
269	57	65.6	8	[3]	421	77	82.1	5	[3]
271	58	65.8	7	[3]	431	77	83.04	6	[3]
277	59	66.6	7	[3]	433	78	83.2	5	[3]
281	59	67.1	8	[3]	439	78	83.8	5	[3]
283	60	67.3	7	[3]	443	79	84.2	5	[3]
289	60	68	8	[3]	449	80	84.8	4	[3]
293	61	68.5	7	[3]	457	81	85.5	4	[3]
307	63	70.1	7	[3]	461	81	85.9	4	[3]
311	63	70.5	7	[3]	463	82	86.1	4	[3]
313	63	70.8	7	[3]	467	82	86.4	4	[3]
317	64	71.2	7	[3]	479	83	87.5	4	[3]
331	66	72.8	6	[3]	487	84	88.3	4	[3]
337	66	73.4	7	[3]	491	84	88.6	4	[3]
343	67	74.1	7	[3]	499	85	89.4	4	[3]
347	67	74.5	7	[3]	503	85	89.7	4	[3]
349	68	74.7	6	[3]	509	85	90.2	5	[3]
353	68	75.2	7	[3]	512	86	90.5	4	[3]
359	69	75.8	6	[3]	521	88	91.3	3	[3]
361	69	76	7	[3]	523	88	91.5	3	[3]
367	70	76.6	6	[3]	529	88	92	4	[3], [6], [7]
373	71	77.3	6	[3]	541	89	93.04	4	[3]

Table 3 (continued). The minimal known sizes $\bar{t}_2(2,q)$ of complete arcs in planes $PG(2,q)$, $A_q = \lfloor 4\sqrt{q} - \bar{t}_2(2,q) \rfloor$

q	$\bar{t}_2(2,q)$	$4\sqrt{q}$	A_q	References	q	$\bar{t}_2(2,q)$	$4\sqrt{q}$	A_q	References
547	90	93.6	3	[3]	673	103	103.8	0	[3]
557	91	94.4	3	[3]	677	103	104.1	1	[3]
563	92	94.9	2	[3]	683	103	104.5	1	[3]
569	93	95.4	2	[3]	691	104	105.1	1	[3]
571	93	95.5	2	[3]	701	105	105.9	0	[3]
577	93	96.1	3	[3]	709	105	106.5	1	[3]
587	94	96.9	2	[3]	719	106	107.2	1	[3]
593	95	97.4	2	[3]	727	106	107.8	1	[3]
599	95	97.9	2	[3]	729	104	108	4	[6]
601	96	98.1	2	[3]	733	107	108.3	1	[3]
607	98	98.5	0	[3]	739	108	108.7	0	[3]
613	97	99.04	2	[3]	743	109	109.03	0	[3]
617	97	99.4	2	[3]	751	109	109.6	0	[3]
619	98	99.5	1	[3]	757	109	110.05	1	[3]
625	96	100	4	[6]	761	109	110.3	1	[3]
631	99	100.5	1	[3]	769	110	110.9	0	[3]
641	100	101.3	1	[3]	773	111	111.2	0	[3]
643	100	101.4	1	[3]	787	112	112.2	0	[3]
647	99	101.7	2	[3]	797	112	112.9	0	[3]
653	101	102.2	1	[3]	809	113	113.8	0	[3]
659	100	102.7	2	[3]	821	114	114.6	0	[3]
661	101	102.8	1	[3]					

Table 3 (continued). The minimal known sizes $\bar{t}_2(2,q)$ of complete arcs in planes $PG(2,q)$, $A_q = \lfloor 4\sqrt{q} - \bar{t}_2(2,q) \rfloor$

Appendix

For $q \leq 128$ we give a list of new (in comparison with [4] and [5]) small complete k-arcs with $k = \bar{t}_2(2,q)$. Similarly to [17], we represent elements of a Galois field $GF(q)$ as follows:

$\{0, 1, \ldots, q-1\}$ if q is prime and we operate on these modulo q;

$\{0, 1 = \alpha^0, 2 = \alpha^1, \ldots, q-1 = \alpha^{q-2}\}$, where α is a primitive element, if $q = p^n$, p prime.

For calculation in $GF(q)$ when q is not a prime, a primitive polynomial and hence a primitive element with its powers are used, as in [10],[17]. Here, the primitive polynomials are x^5+x^3+1 for $q = 32$, x^2+x+3 for $q = 49$, $x^6+x^4+x^3+1$ for $q = 64$, x^4+x+2 for $q = 81$, $x^2 + 4x + 2$ for $q = 121$, $x^3 + 3x + 2$ for $q = 125$, and $x^7 + x + 1$ for $q = 128$, [10]. An arc is written as a set of points.

$\bar{t}_2(2, 41) = 16$:
(1, 27, 2), (1, 0, 0), (1, 33, 22), (1, 33, 11), (1, 28, 21), (1, 7, 2), (1, 6, 39), (1, 7, 1), (1, 22, 8), (1, 28, 31), (1, 22, 16), (1, 9, 19), (1, 1, 15), (1, 12, 1), (1, 25, 17), (1, 9, 30)

$\bar{t}_2(2, 43) = 16$:
(1, 19, 32), (1, 4, 13), (0, 0, 1), (1, 33, 2), (1, 30, 19), (1, 14, 0), (0, 1, 14), (1, 22, 29), (1, 38, 4), (1, 35, 13), (1, 40, 21), (1, 36, 24), (1, 18, 19), (1, 13, 31), (1, 2, 34), (1, 24, 1)

$\bar{t}_2(2, 47) = 18$:
(1,34, 23), (1, 7, 46), (1, 5, 24), (1, 34, 20), (1, 17, 28), (1, 43, 41), (1, 28, 43), (1, 14, 32), (1, 0, 41), (1, 25, 17), (1, 46, 46), (1, 13, 19), (1, 25, 36), (0, 1, 44), (1, 26, 11), (1, 31, 10), (1, 26, 24), (1, 10, 29)

$\bar{t}_2(2, 49) = 18$:
(1, 0, 0), (1, 15, 17), (0, 0, 1), (1, 45, 24), (1, 20, 45), (1, 44, 17), (1, 5, 22), (1, 8, 34), (1, 46, 42), (1, 29, 0), (1, 43, 29), (1, 4, 32), (1, 13, 6), (1, 40, 32), (1, 28, 37), (1, 2, 14), (1, 23, 14), (1, 17, 37)

$\bar{t}_2(2, 53) = 18$:
(1, 13, 29), (1, 11, 13), (1, 50, 52), (1, 50, 48), (1, 9, 43), (1, 5, 11), (1, 16, 47), (1, 0, 44), (1, 1, 36), (1, 42, 36), (1, 5, 25), (1, 51, 16), (1, 39, 32), (1, 2, 3), (1, 13, 12), (1, 44, 16), (1, 19, 10), (1, 4, 11)

$\bar{t}_2(2, 59) = 20$:
(1, 16, 37), (1, 39, 18), (1, 13, 43), (1, 37, 32), (1, 21, 40), (1, 11, 13), (1, 53, 58), (1, 57, 50), (1, 15, 13), (1, 3, 38), (1, 24, 16), (1, 56, 44), (1, 18, 25), (1, 47, 18), (1, 20, 3), (1, 34, 32), (1, 56, 6), (1, 24, 41), (1, 47, 37), (1, 30, 33)

$\bar{t}_2(2, 73) = 25$:
(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 2, 55), (1, 45, 5), (1, 23, 57), (1, 47, 33), (1, 5, 70), (1, 18, 26), (1, 71, 10), (1, 72, 27), (1, 61, 14), (1, 37, 30), (1, 52, 38), (1, 69, 34), (1, 17, 29), (1, 38, 46), (1, 28, 35), (1, 49, 24), (1, 39, 18), (1, 13, 13), (1, 62, 64), (1, 34, 22), (1, 64, 54), (1, 48, 41)

$\bar{t}_2(2, 79) = 26$:
(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 2, 13), (1, 48, 72), (1, 31, 66), (1, 64, 7), (1, 18, 21), (1, 35, 17), (1, 33, 48), (1, 45, 18), (1, 38, 51), (1, 72, 54), (1, 12, 75), (1, 22, 20), (1, 20, 38), (1, 77, 6), (1, 54, 46), (1, 69, 16), (1, 65, 68), (1, 11, 43), (1, 9, 78), (1, 28, 60), (1, 46, 5), (1, 63, 39), (1, 19, 10)

$\bar{t}_2(2, 81) = 26$:
(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 75), (1, 41, 25), (1, 31, 65), (1, 51, 55), (1, 71, 5), (1, 61, 35), (1, 21, 45), (1, 11, 6), (1, 49, 28), (1, 70, 63), (1, 39, 26), (1, 48, 57), (1, 40, 11), (1, 18, 62), (1, 63, 1), (1, 16, 14), (1, 27, 15), (1, 17, 37), (1, 64, 79), (1, 73, 18), (1, 24, 13), (1, 5, 51), (1, 58, 69)

$\bar{t}_2(2, 83) = 27$:
(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 46), (1, 8, 67), (1, 61, 43), (1, 56, 58), (1, 53, 44), (1, 35, 60), (1, 20, 82), (1, 54, 69), (1, 10, 33), (1, 41, 61), (1, 72, 42), (1, 71, 47), (1, 9, 55), (1, 28, 2), (1, 16, 36), (1, 67, 7), (1, 76, 65), (1, 69, 35), (1, 81, 54), (1, 22, 76), (1, 24, 73), (1, 39, 66), (1, 17, 28), (1, 13, 37)

$\bar{t}_2(2, 89) = 28$:
(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 2, 52), (1, 13, 7), (1, 79, 49), (1, 84, 77), (1, 31, 30), (1, 75, 25), (1, 77, 75), (1, 82, 60), (1, 54, 46), (1, 55, 69), (1, 59, 57), (1, 57, 31), (1, 83, 45), (1, 32, 3), (1, 70, 37), (1, 67, 40), (1, 18, 36), (1, 64, 26), (1, 20, 63), (1, 88, 64), (1, 41, 11), (1, 85, 13), (1, 23, 22), (1, 40, 58), (1, 39, 18)

$\bar{t}_2(2, 97) = 30$:
(1, 0, 0), (0, 1, 44), (1, 47, 32), (1, 82, 8), (1, 14, 45), (1, 19, 94), (1, 19, 25), (1, 68, 86), (1, 71, 19), (1, 39, 2), (1, 84, 28), (1, 32, 30), (1, 27, 38), (1, 86, 59), (1, 33, 69), (1, 68, 4), (1, 17, 50), (1, 1, 46), (1, 6, 58), (1, 5, 39), (1, 47, 27), (1, 14, 11), (1, 84, 43), (1, 94, 11), (1, 29, 85), (1, 92, 33), (1, 59, 17), (1, 31, 30), (0, 1, 53), (1, 62, 42)

$\bar{t}_2(2, 101) = 30$:
(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 2, 17), (1, 38, 10), (1, 75, 20), (1, 4, 23), (1, 76, 30), (1, 57, 74), (1, 9, 27), (1, 74, 98), (1, 61, 40), (1, 24, 100), (1, 73, 62), (1, 10, 33), (1, 67, 2), (1, 13, 24), (1, 92, 82), (1, 90, 19), (1, 85, 42), (1, 91, 18), (1, 18, 16), (1, 97, 6), (1, 12, 67), (1, 69, 72), (1, 25, 65), (1, 17, 96), (1, 3, 87), (1, 60, 89), (1, 23, 93)

$\bar{t}_2(2, 103) = 31$:
(1, 0, 0), (0, 1, 44), (1, 44, 49), (1, 77, 10), (1, 13, 49), (1, 26, 0), (0, 1, 88), (1, 2, 65), (1, 77, 43), (1, 7, 1), (1, 43, 3), (1, 43, 55), (1, 44, 102), (1, 19, 31), (1, 21, 55), (1, 38, 73), (1, 86, 29), (1, 49, 57), (1, 90, 69), (1, 13, 71), (1, 56, 74), (1, 100, 43), (1, 52, 91), (1, 98, 29), (1, 68, 89), (1, 48, 98), (1, 8, 76), (1, 68, 18), (1, 65, 97), (1, 79, 85), (1, 48, 50)

$\bar{t}_2(2, 107) = 32$:
(1, 0, 0), (0, 1, 86), (1, 47, 16), (1, 22, 87), (1, 28, 0), (1, 13, 17), (1, 52, 100), (1, 80, 93), (1, 13, 81), (1, 102, 36), (1, 96, 29), (1, 90, 65), (1, 90, 30), (1, 69, 42), (1, 88, 63), (1, 79, 4), (1, 15, 15), (1, 59, 10), (1, 96, 49), (1, 36, 97), (1, 99, 48), (1, 0, 42), (1, 61, 8), (1, 50, 104), (1, 82, 72), (1, 33, 17), (1, 89, 12), (1, 82, 41), (1, 104, 70), (1, 21, 37), (1, 57, 70), (1, 57, 47)

$\bar{t}_2(2, 109) = 32$:
(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 55), (1, 2, 23), (1, 47, 1), (1, 100, 70), (1, 37, 60), (1, 95, 50), (1, 54, 87), (1, 70, 102), (1, 52, 84), (1, 25, 8), (1, 74, 83), (1, 65, 62), (1, 46, 58), (1, 57, 24), (1, 4, 86), (1, 29, 17), (1, 36, 26), (1, 33, 21), (1, 93, 89), (1, 99, 40), (1, 64, 45), (1, 17, 44), (1, 21, 10), (1, 49, 13), (1, 55, 64), (1, 19, 95), (1, 86, 54), (1, 32, 82), (1, 12, 65)

$\bar{t}_2(2, 113) = 33$:
(1, 0, 0), (0, 1, 53), (1, 35, 85), (1, 35, 102), (1, 74, 70), (1, 0, 93), (1, 30, 43), (1, 100, 74), (1, 60, 56), (1, 109, 20), (1, 1, 47), (1, 78, 29), (1, 58, 40), (1, 55, 111), (1, 91, 55), (1, 13, 58), (1, 44, 37), (1, 110, 96), (1, 70, 62), (1, 6, 79), (1, 1, 58), (1, 77, 26), (1, 14, 95), (1, 78, 48), (1, 104, 91), (1, 28, 51), (1, 55, 100), (1, 15, 24), (1, 34, 90), (1, 100, 19), (1, 14, 108), (1, 50, 105), (1, 66, 20)

$\bar{t}_2(2, 121) = 34$:
(1, 0, 0), (0, 1, 40), (1, 16, 42), (1, 17, 113), (1, 15, 84), (1, 80, 60), (1, 16, 48), (1, 6, 26), (1, 58, 58), (1, 70, 24), (1, 108, 104), (1, 54, 88), (1, 79, 56), (1, 7, 35), (1, 47, 31), (1, 23, 111), (1, 95, 82), (1, 117, 33), (1, 117, 51), (1, 50, 116), (1, 22, 99), (1, 8, 1), (1, 106, 91), (1, 12, 104), (1, 25, 19), (1, 49, 9), (1, 43, 119), (1, 61, 13), (1, 26, 41), (1, 105, 84), (1, 36, 26), (1, 95, 19), (1, 101, 10), (1, 4, 29)

$\bar{t}_2(2, 125) = 35$:
(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 72), (1, 6, 24), (1, 71, 77), (1, 96, 93), (1, 108, 40), (1, 48, 114), (1, 28, 28), (1, 11, 33), (1, 24, 88), (1, 110, 119), (1, 21, 42), (1, 79, 29), (1, 36, 81), (1, 105, 63), (1, 38, 19), (1, 55, 97), (1, 97, 32), (1, 115, 34), (1, 15, 87), (1, 4, 59), (1, 13, 39), (1, 22, 123), (1, 59, 12), (1, 73, 100), (1, 32, 111), (1, 19, 48), (1, 64, 98), (1, 81, 38), (1, 29, 37), (1, 93, 18), (1, 10, 26), (1, 91, 75)

$\bar{t}_2(2, 127) = 36$:
(1, 0, 0), (0, 1, 50), (1, 11, 49), (1, 19, 102), (1, 2, 19), (1, 43, 54), (1, 106, 36), (1, 120, 55), (1, 12, 14), (1, 68, 63), (1, 28, 50), (1, 91, 111), (1, 63, 6), (1, 37, 98), (1, 21, 1), (1, 35, 116), (1, 8, 40), (1, 25, 88), (1, 20, 107), (1, 117, 47), (1, 80, 15), (1, 94, 35), (1, 112, 68), (1, 110, 81), (1, 90, 38), (1, 54, 115), (1, 0, 86), (1, 70, 24), (1, 24, 25), (1, 67, 9), (1, 14, 14), (1, 105, 27), (1, 50, 11), (1, 41, 61), (1, 92, 72), (1, 69, 71)

$\bar{t}_2(2, 128) = 36$:
(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 88), (1, 10, 9), (1, 119, 127), (1, 115, 97), (1, 118, 107), (1, 11, 117), (1, 14, 110), (1, 31, 64), (1, 80, 74), (1, 98, 34), (1, 49, 122), (1, 40, 120), (1, 89, 81), (1, 35, 55), (1, 94, 21), (1, 77, 106), (1, 6, 80), (1, 29, 56), (1, 28, 67), (1, 126, 33), (1, 70, 58), (1, 2, 5), (1, 30, 13), (1, 68, 17), (1, 79, 125), (1, 76, 78), (1, 3, 52), (1, 18, 123), (1, 93, 112), (1, 112, 25), (1, 13, 84), (1, 120, 101), (1, 19, 60)

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