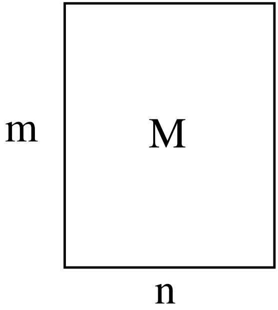
Computing a Nonnegative Matrix Factorization – Provably

Ankur Moitra, IAS

joint work with Sanjeev Arora, Rong Ge and Ravi Kannan

June 20, 2012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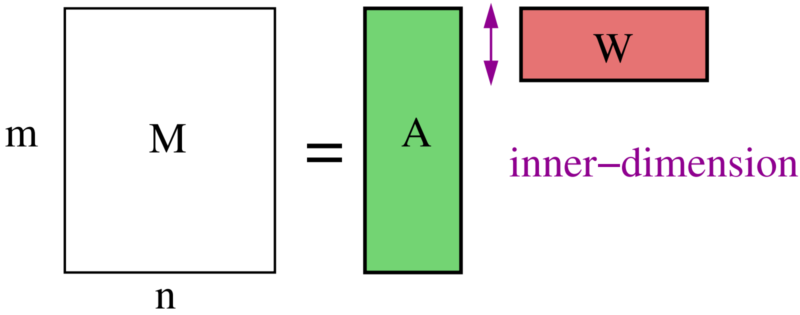
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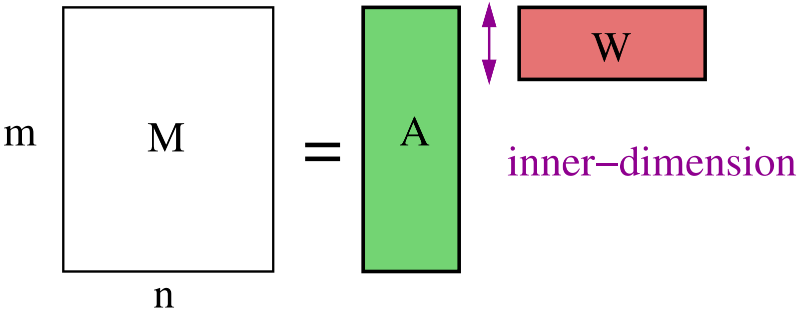
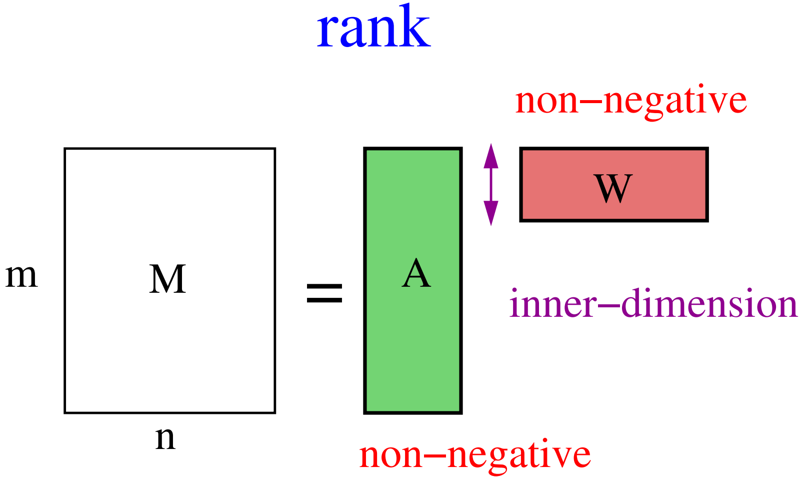
inner–dimension

rank

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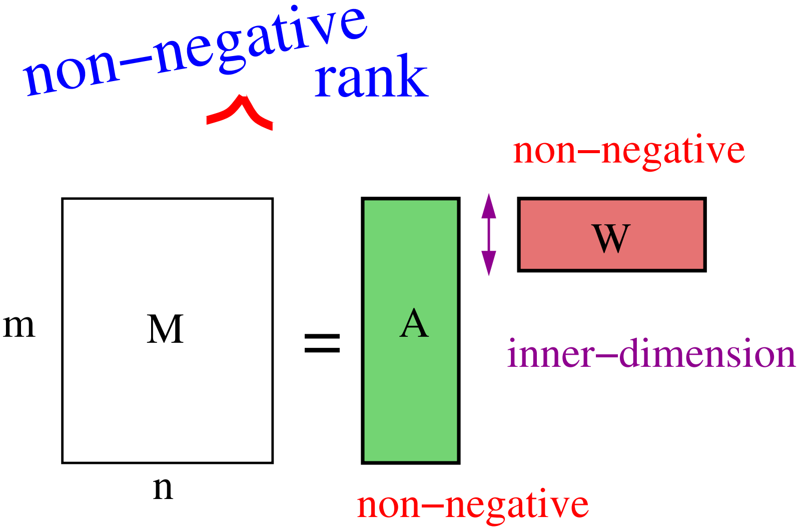
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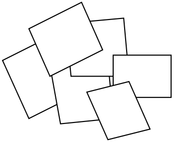
Information Retrieval

documents:



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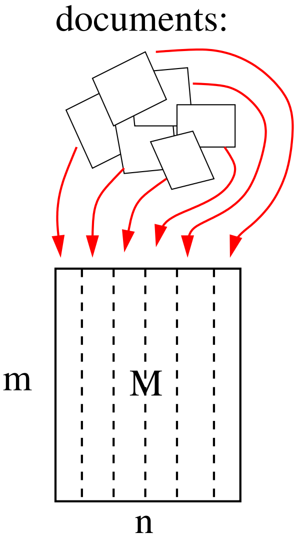

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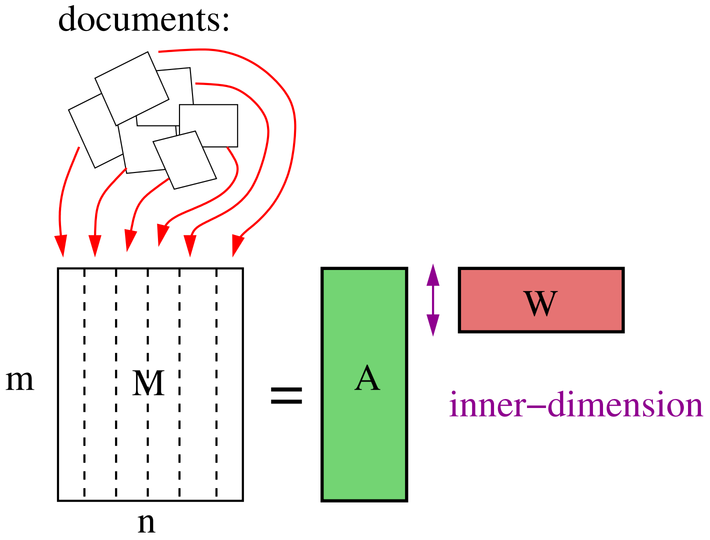

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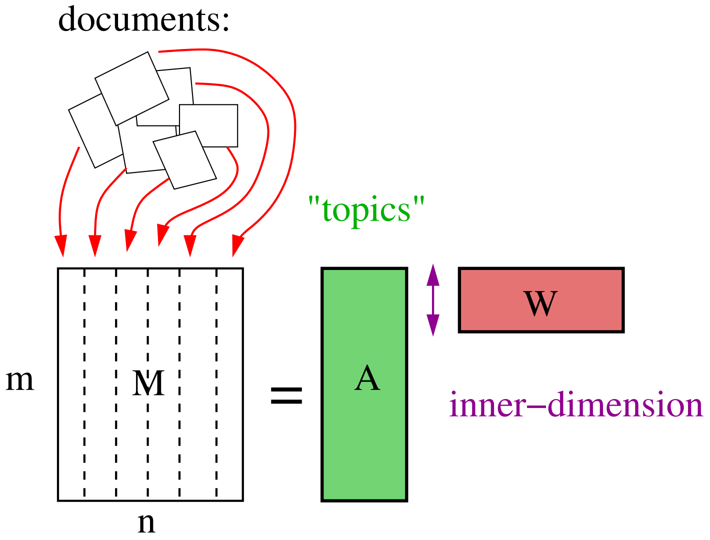

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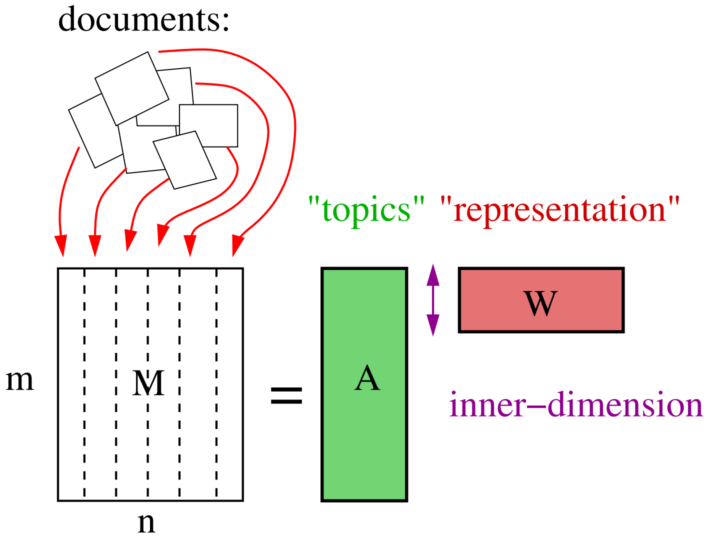

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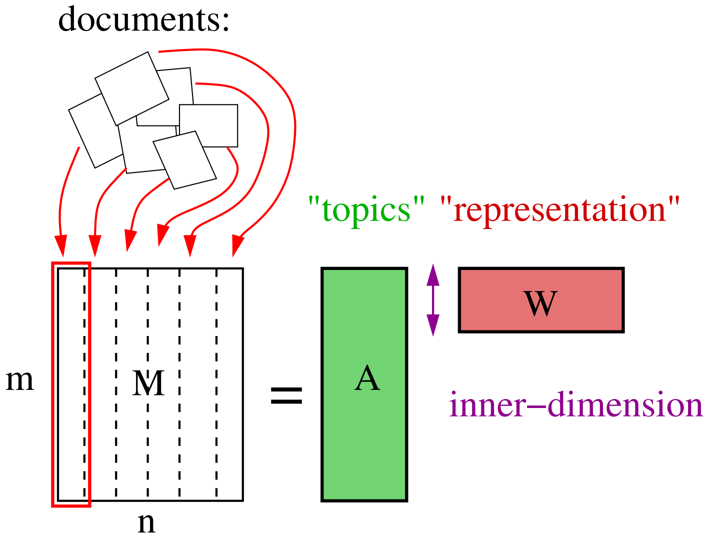

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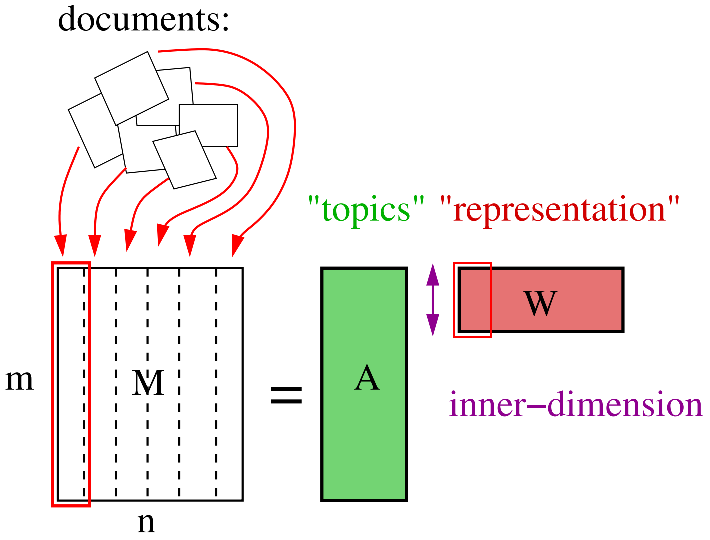

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- Statistics and Machine Learning:
  - extract latent relationships in data
  - image segmentation, text classification, information retrieval, collaborative filtering, ...

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- Physical Modeling:
  - interaction of components is additive
  - visual recognition, environmetrics

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Hardness of NMF

Theorem (Vavasis)

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How quickly can we solve NMF if r is small?

The Worst-Case Complexity of NMF

Theorem (Arora, Ge, Moitra, Kannan)

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Separability was introduced to understand when NMF is unique – Is it enough to make NMF easy?

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Theorem (Arora, Ge, Kannan, Moitra)

There is a polynomial time exact algorithm for NMF when the topic matrix A is separable

What if documents do not contain many words compared to the dictionary? (e.g. we are given samples from M)

In fact, the above algorithm can be made robust to noise:

Theorem (Arora, Ge, Moitra)

There is a polynomial time algorithm for learning a separable topic matrix A in various probabilistic models - e.g. LDA, CTM

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Semi-algebraic sets: s polynomials, k variables, Boolean function B

$$S = \{x_1, x_2 ... x_k | B(\text{sgn}(f_1), \text{sgn}(f_2), ... \text{sgn}(f_s)) = \text{"true"} \}$$

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In fact, best known algorithms (e.g. [Renegar]) for finding a point in S run in $(ds)^{O(k)}$ time

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What is the smallest formulation, measured in the number of variables? Can we use only $f(r)$ variables?

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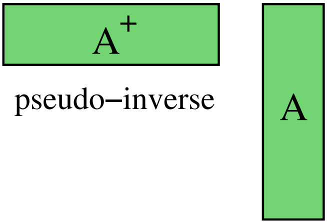

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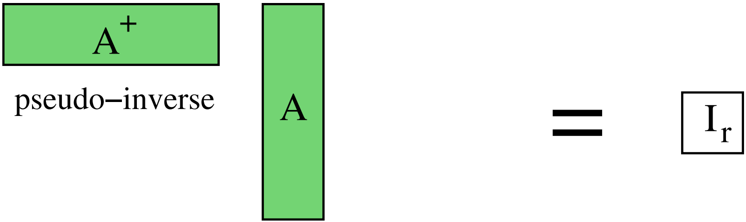

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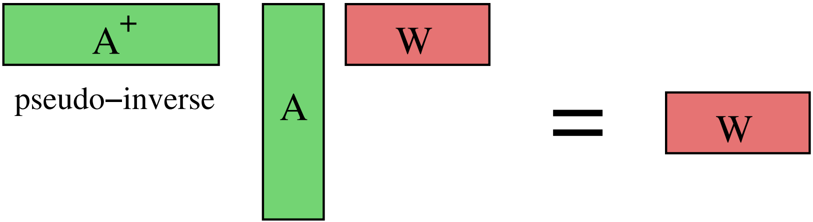

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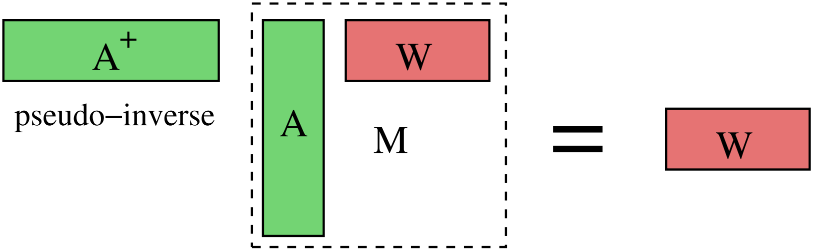

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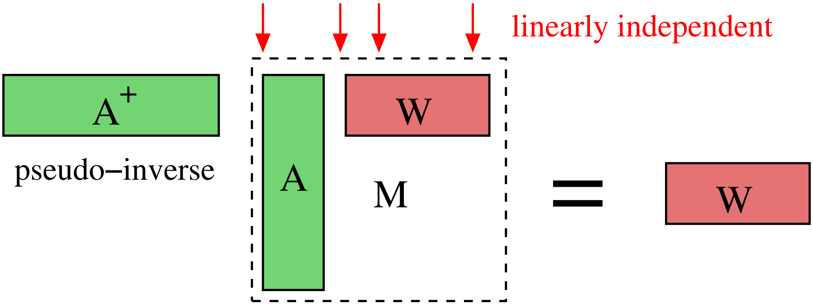

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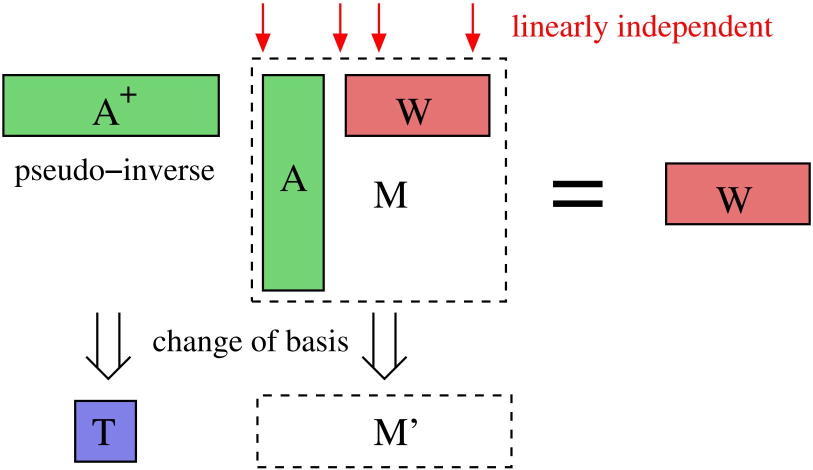

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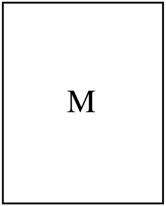

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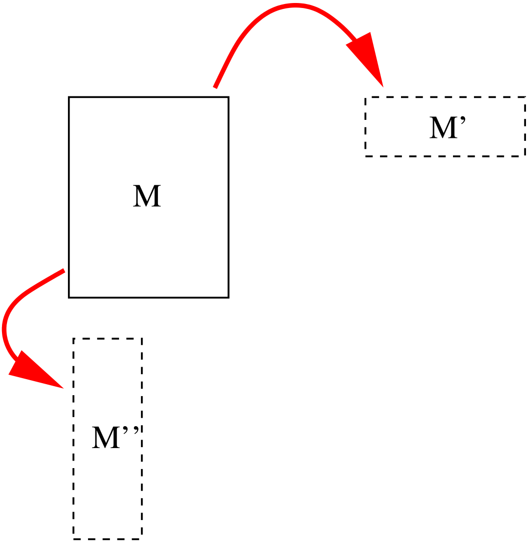

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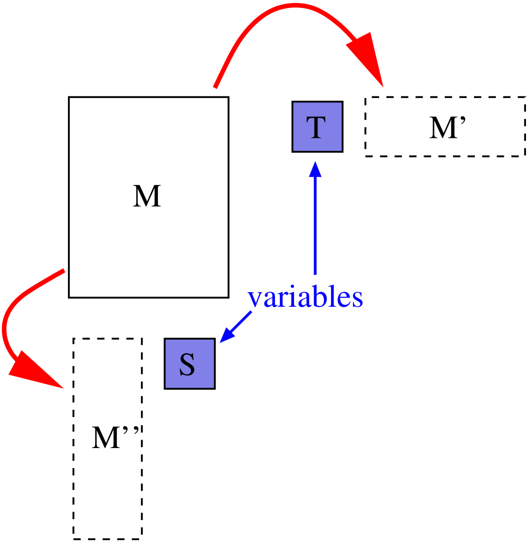

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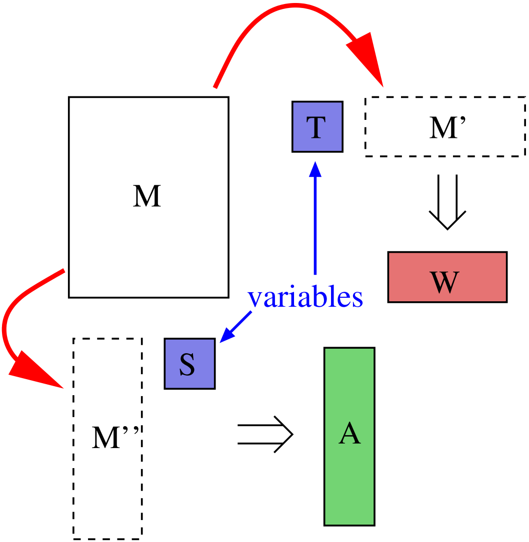

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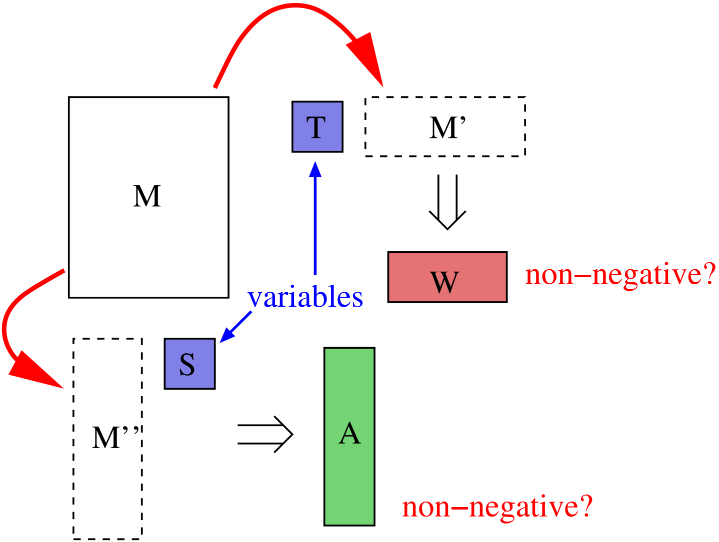

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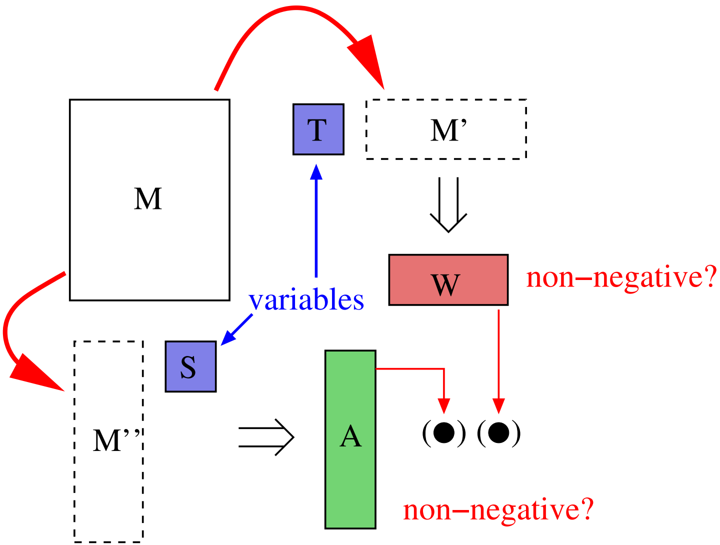

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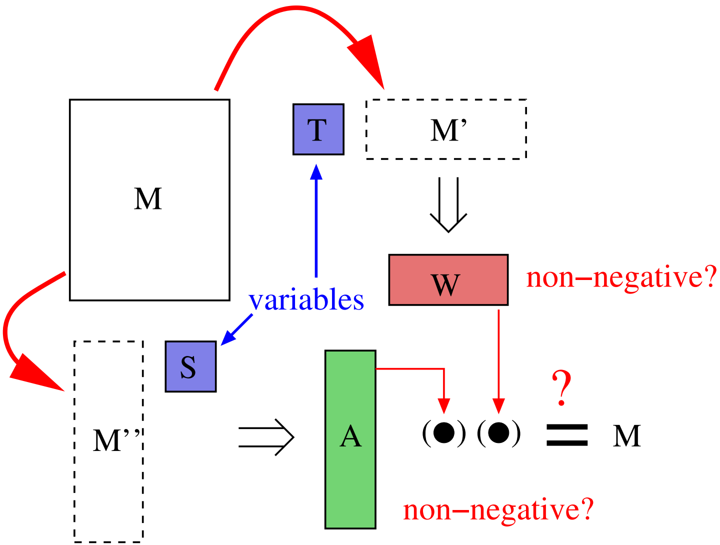

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Recall: For each topic, there is some (anchor) word that only appears in this topic

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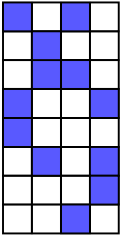

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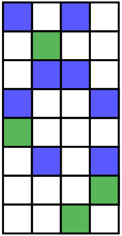

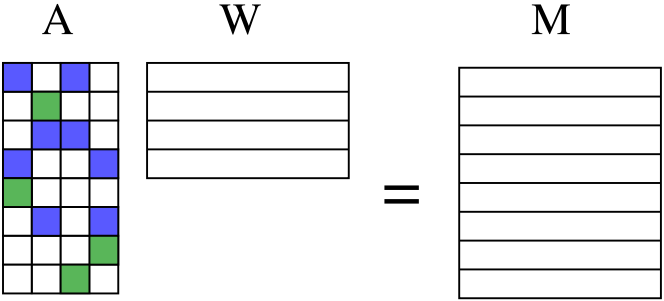
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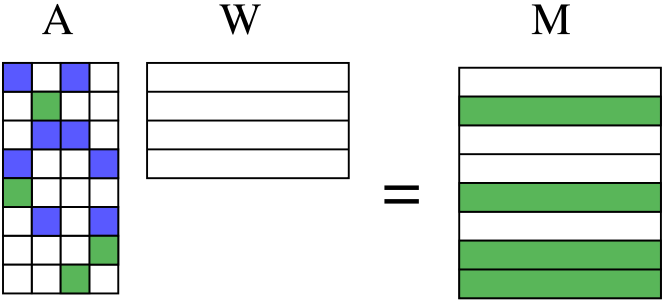
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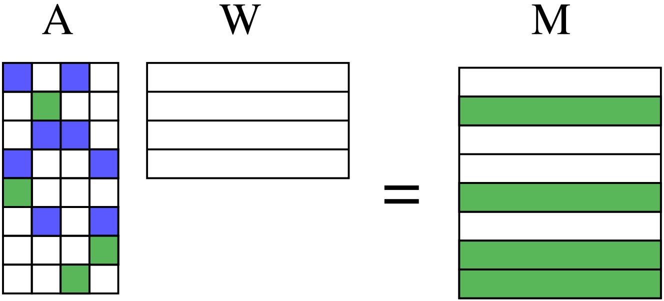
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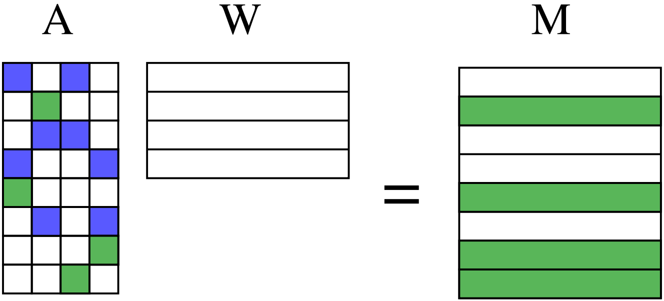

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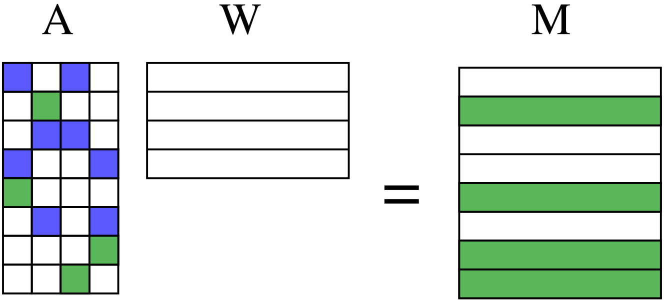

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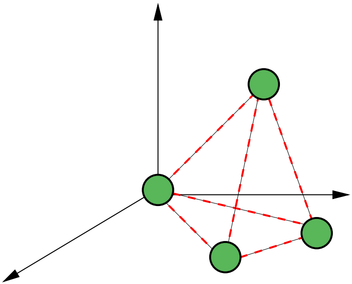

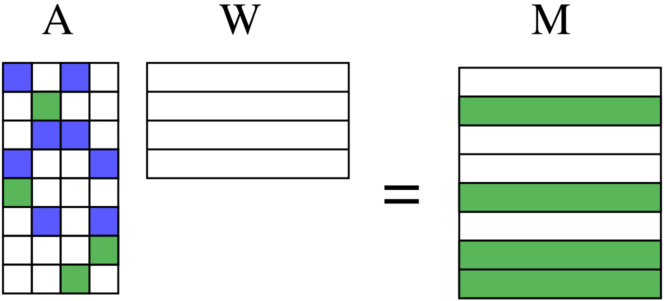

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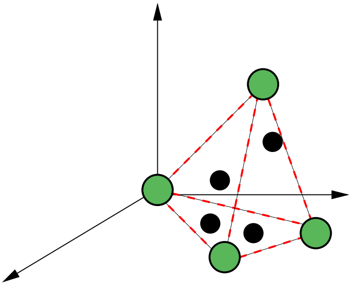

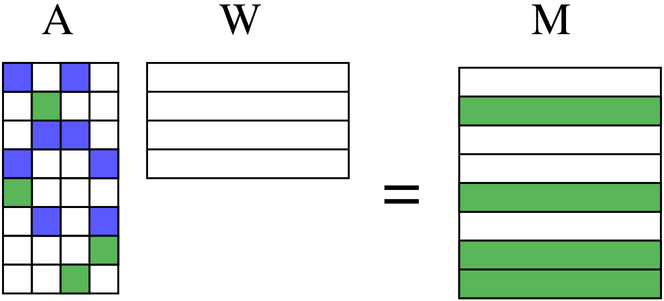

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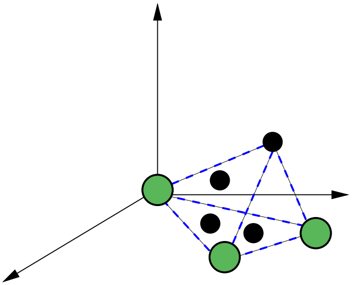

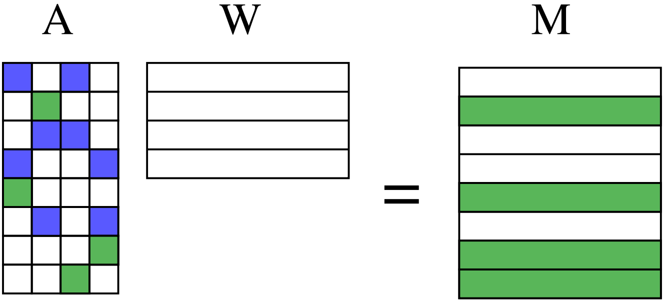

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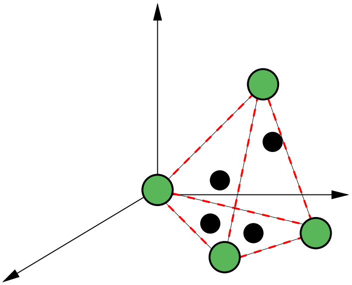

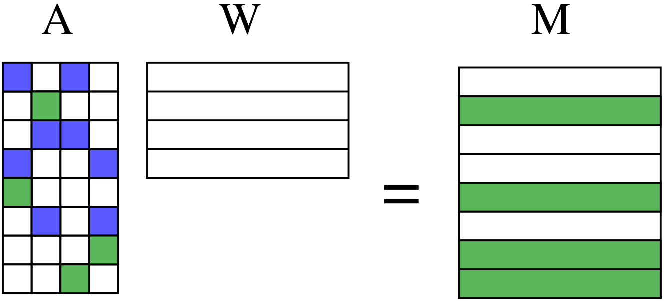

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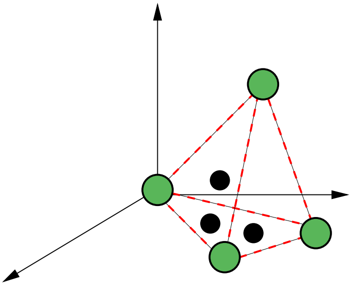

Separable Instances

Recall: For each topic, there is some (anchor) word that only appears in this topic

Observation

Rows of W appear as (scaled) rows of M

Question

How can we identify anchor words?

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Yes! [Arora, Ge, Moitra] we give a provable algorithm based on (noise-tolerant) NMF

Advertisement: Sanjeev will talk about this here in July

Concluding Remarks

This is just part of a broader agenda:

Question

When is machine learning **provably** *easy?*

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Some interesting problems worth further investigation: Topic Models, Independent Component Analysis, Graphical Models, Deep Learning

Thanks!