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COMPUTING MEDIANS AND MEANS IN HADAMARD SPACES

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ABSTRACT. The geometric median as well as the Fréchet mean of points in an Hadamard space are important in both theory and applications. Surprisingly, no algorithms for their computation are hitherto known. To address this issue, we use a split version of the proximal point algorithm for minimizing a sum of convex functions and prove that this algorithm produces a sequence converging to a minimizer of the objective function, which extends a recent result of D. Bertsekas (2001) into Hadamard spaces. The method is quite robust and not only does it yield algorithms for the median and the mean, but it also applies to various other optimization problems. We moreover show that another algorithm for computing the Fréchet mean can be derived from the law of large numbers due to K.-T. Sturm (2002).

In applications, computing medians and means is probably most needed in tree space, which is an instance of an Hadamard space, invented by Billera, Holmes, and Vogtmann (2001) as a tool for averaging phylogenetic trees. It turns out, however, that it can be also used to model numerous other tree-like structures. Since there now exists a polynomial-time algorithm for computing geodesics in tree space due to M. Owen and S. Provan (2011), we obtain efficient algorithms for computing medians and means, which can be directly used in practice.

1. INTRODUCTION

Given positive weights w_1, \dots, w_N satisfying $\sum w_n = 1$, and a finite set of points a_1, \dots, a_N in a metric space (X, d) , we define its *geometric median* as

$$(1) \quad \Psi(\bar{w}; \bar{a}) := \arg \min_{x \in \mathcal{H}} \sum_{n=1}^N w_n d(x, a_n),$$

and its *Fréchet mean* as

$$(2) \quad \Xi(\bar{w}; \bar{a}) := \arg \min_{x \in \mathcal{H}} \sum_{n=1}^N w_n d(x, a_n)^2,$$

where we denote $\bar{w} := (w_1, \dots, w_N)$ and $\bar{a} := (a_1, \dots, a_N)$. These definitions will not function well in an arbitrary metric space, but as far as geodesic metric spaces of nonpositive curvature (so-called Hadamard spaces) are concerned, they become highly appropriate. Hadamard spaces include, apart from Hilbert spaces, Euclidean buildings, and some Riemannian manifolds, also the Billera-Holmes-Vogtmann tree

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space (BHV tree space), which is a nonpositively curved cubical complex constructed in [9] as a model space for phylogenetic trees. Computing means and medians in this setting can therefore be of great importance in practice. We would like to mention one more application. Very recently, Fréchet means have also emerged in connection with so-called *consensus algorithms* in Hadamard spaces [19].

Our goal in the present paper is to introduce algorithms for computing medians and means in Hadamard spaces. Research in this direction has already started, see for instance [14, 15]. Three different methods (centroid, Birkhoff's shortening, and weighted averages) for computing the Fréchet mean were introduced in [14, Section 4]. Unfortunately, none of these methods gives a correct result, as will be demonstrated in Remark 1.1 below. To the best of our knowledge, no algorithms for computing medians and means hitherto exist.

Remark 1.1. Let (\mathcal{H}, d) be an Hadamard space consisting of three geodesic rays issuing from the origin 0. This is an \mathbb{R} -tree, and as a matter of fact the BHV tree space \mathcal{T}_3 . Consider three points $x, y, z \in \mathcal{H}$ lying in distinct rays issuing from the origin 0 such that $d(0, z) = 5$, and $d(0, x) = d(0, y) = 1$. Then it is easy to see that the Fréchet mean Ξ of x, y, z with the uniform weights $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$ lies on the geodesic $[0, z]$ and $d(0, \Xi) = 1$. On the other hand if we apply Birkhoff's shortening or the centroid method, we will get a point $c \in [0, z]$ such that $d(0, c) > \frac{5}{4}$. Finally, the weighted average method of the points x, y, z yields a point $w \in [0, z]$ with $d(0, w) = \frac{5}{3}$, or the origin 0, depending on the order we choose.

It is worth mentioning that in spite of an apparent similarity between (1) and (2), there is a substantial difference in the complexity of computing the median and the mean even in Euclidean spaces. While it is trivial to find a mean in finitely many steps, see (13) below, there exists no formula for computing a median in \mathbb{R}^d , and we can use only approximation algorithms; see [10] and the references therein.

The BHV tree space and statistical biology. To increase the motivation and whet the appetite even more we will now take a closer look at the BHV space and its applications. The BHV tree space is a CAT(0) cubical complex whose elements are (combinatorial) trees with a fixed number of terminal nodes and lengths assigned to all edges [9].

Combinatorial trees with a fixed number of terminal vertices can represent evolutionary trees in phylogenetics. Then, given a finite collection of such trees, it is desirable to find an average tree. Most naturally this average is the Fréchet mean. On the other hand until now, no algorithm for its computation was available in the BHV tree space and therefore alternative and less satisfactory concepts of an average were used, for instance, a centroid [9], or a majority consensus [34]. However, with our algorithms at hand, one can efficiently compute the Fréchet mean itself. For a general mathematical background of contemporary phylogenetics, the reader is referred to [13, 37, 39].

Apart from phylogenetics, tree-like structures emerge naturally in other subject fields of biology and computing an average tree is again of interest. Let us mention applications to airway systems in human lungs [14, 15] and blood vessels [46].

All the algorithms presented in this paper require computing geodesics in the underlying Hadamard space. While it can be a difficult task in a general Hadamard space, there exists an efficient polynomial-time algorithm for computing geodesics in the BHV tree space due to M. Owen and S. Provan [35, 36], which makes our

algorithms directly applicable in practice. See Section 5 at the end of the present paper.

The proximal point algorithm and its applications. Let (\mathcal{H}, d) be an Hadamard space. Our algorithms for computing medians and means are based on a special version of the proximal point algorithm (PPA) for minimizing a convex function f of the form

$$(3) \quad f := \sum_{n=1}^N f_n,$$

where $f_n : \mathcal{H} \rightarrow (-\infty, \infty]$ are all convex and lower semicontinuous (lsc). The main trick here is that instead of applying iteratively the resolvent

$$J_\lambda(x) := \arg \min_{y \in \mathcal{H}} \left[f(y) + \frac{1}{2\lambda} d(x, y)^2 \right]$$

of the function f , we apply the resolvents

$$J_\lambda^n(x) := \arg \min_{y \in \mathcal{H}} \left[f_n(y) + \frac{1}{2\lambda} d(x, y)^2 \right]$$

of its components f_n either in cyclic or random order; see Definitions 3.3 and 3.5 for the precise formulations of the algorithms. Such algorithms have been recently shown by D. Bertsekas [7] to converge to a minimizer of f when the underlying space is \mathbb{R}^d . We extend these results into locally compact Hadamard spaces and then apply them with f_n being $f_n = w_n d(\cdot, a_n)$ in the case of the median (1), and with $f_n = w_n d(\cdot, a_n)^2$ in the case of the mean (2). In either case, it is then easy to find explicit formulas for the resolvents J_λ^n , and one hence obtains simple algorithms for computing means and medians, respectively. The detailed description is given in Section 4.

The proximal point algorithm in turn relies upon the supermartingale convergence theorem, which is recalled in Section 2 below.

The function (3) can apparently accommodate more general problems. For instance, we can put

$$(4) \quad f(x) := \sum_{n=1}^N w_n d(x, a_n)^p, \quad x \in \mathcal{H},$$

where $p \in [1, \infty)$. Then f is convex continuous and encompasses medians and means as special cases:

- (i) If $p = 1$, then f becomes the objective function in the *Fermat-Weber problem* for optimal facility location and its minimizer is the median of the points a_1, \dots, a_N with weights w_1, \dots, w_N .
- (ii) If $p = 2$, then a minimizer of f is the *barycenter* of the probability measure

$$\pi := \sum_{n=1}^N w_n \delta_{a_n},$$

where δ_{a_n} stands for the Dirac measure at the point a_n . For further details on barycenters, the reader is referred to [23, Chapter 3] and [44]. Thus the mean of the points a_1, \dots, a_N can be equivalently viewed as the barycenter of π .

Another way of generalizing (1), and (2) alike, is to replace the points a_1, \dots, a_N by convex closed sets $C_1, \dots, C_N \subset \mathcal{H}$. Since the distance functions to such sets are convex continuous (see Example 1.3 below) in Hadamard spaces, the objective function in this problem is of the form (3). Namely, we are to minimize the function

$$(5) \quad f(x) := \sum_{n=1}^N w_n d(x; C_n), \quad x \in \mathcal{H}.$$

Problems of this type have been recently studied from various perspectives [31, 32, 33]. Our approach, based on the proximal point algorithm, seems to be however novel even in linear spaces. See also Remark 4.5 where we give an explicit algorithm.

We will now present several natural examples of convex lsc functions in Hadamard spaces to see that the proximal point algorithm is applicable in many more situations. Let still (\mathcal{H}, d) be an Hadamard space.

Example 1.2 (Indicator functions). Let $K \subset \mathcal{H}$ be a convex set. Define the *indicator function* of K by

$$\iota_K(x) := \begin{cases} 0, & \text{if } x \in K, \\ \infty, & \text{if } x \notin K. \end{cases}$$

Then ι_K is a convex function and it is lsc if and only if K is closed.

Example 1.3 (Distance functions). The function

$$(6) \quad x \mapsto d(x, x_0), \quad x \in \mathcal{H},$$

where x_0 is a fixed point of \mathcal{H} , is convex and continuous. The function $d(\cdot, x_0)^p$ for $p > 1$ is strictly convex. More generally, the *distance function* to a closed convex subset $C \subset \mathcal{H}$, defined as

$$d(x; C) := \inf_{c \in C} d(x, c), \quad x \in \mathcal{H},$$

is convex and 1-Lipschitz [12, p.178].

Example 1.4 (Displacement functions). Let $T : \mathcal{H} \rightarrow \mathcal{H}$ be an isometry. The *displacement function* of T is the function $\delta_T : \mathcal{H} \rightarrow [0, \infty)$ defined by

$$\delta_T(x) := d(x, Tx), \quad x \in \mathcal{H}.$$

It is convex and Lipschitz [12, p.229].

Example 1.5 (Busemann functions). Let $c : [0, \infty) \rightarrow \mathcal{H}$ be a geodesic ray. The function $b_c : \mathcal{H} \rightarrow \mathbb{R}$ defined by

$$b_c(x) := \lim_{t \rightarrow \infty} [d(x, c(t)) - t], \quad x \in \mathcal{H},$$

is called the *Busemann function* associated to the ray c . Busemann functions are convex and 1-Lipschitz. Concrete examples of Busemann functions are given in [12, p. 273]. Another explicit example of a Busemann function in the Hadamard space of positive definite $n \times n$ matrices with real entries can be found in [12, Proposition 10.69]. The sublevel sets of Busemann functions are called *horoballs* and carry a lot of information about the geometry of the space in question, see [12] and the references therein.

Example 1.6 (Energy functional). The energy functional is a convex lsc function on an Hadamard space of \mathcal{L}^2 -mappings. It has been studied extensively at a varying level of generality [16, 17, 20, 21, 22, 25]. Minimizers of the energy functional are called *harmonic maps* and are of importance in both geometry and analysis. For a probabilistic approach to harmonic maps in Hadamard spaces, see [41, 42, 45].

The indicator function of Example 1.2 is often employed to convert a constrained minimization problem into an unconstrained one. Indeed, the minimization of the function (3) on a closed convex set $C \subset \mathcal{H}$ is equivalent to the minimization of

$$\tilde{f} := \iota_C + \sum_{n=1}^N f_n,$$

over the whole space \mathcal{H} .

We explicitly mention yet another application of the above version of the proximal point algorithm. Namely, if C_1, \dots, C_N are closed convex subsets of \mathcal{H} such that

$$C_1 \cap \dots \cap C_N \neq \emptyset,$$

and if we set $f_n = \iota_{C_n}$ in (3), then it is easy to see that

$$J_\lambda^n(x) = P_{C_n}(x),$$

for any $x \in \mathcal{H}$, $\lambda > 0$, and $n = 1, \dots, N$. Here P_{C_n} stands for the metric projection onto C_n ; see Section 2. The proximal point algorithm hence becomes the method of cyclic and random projections, respectively, and converges to a point $c \in \bigcap_{n=1}^N C_n$. Such algorithms play an important role in optimization, for instance in convex feasibility problems; see [4, 5, 6] and the references therein. If $N = 2$, both cyclic and random orders of the projections give the same approximating sequence, and we get the so-called alternating projections, which were in Hadamard spaces studied in [3].

The Lie-Trotter-Kato formula. There is also a tight connection to gradient flow semigroups for a function of the form (3), since the proximal point algorithm is a discrete time version of the gradient flow.

We need the following notation. If $F : \mathcal{H} \rightarrow \mathcal{H}$ is a mapping, we denote its k^{th} power, with $k \in \mathbb{N}$, by

$$F^{(k)}x := (F \circ \dots \circ F)x, \quad x \in \mathcal{H},$$

where F appears k -times on the right hand side.

Recall that the *gradient flow semigroup* $(S_t)_{t \geq 0}$ of f is given as

$$(7) \quad S_t x := \lim_{k \rightarrow \infty} \left(J_{\frac{t}{k}} \right)^{(k)}(x), \quad x \in \overline{\text{dom } f}.$$

The limit in (7) is uniform with respect to t on bounded subintervals of $[0, \infty)$, and $(S_t)_{t \geq 0}$ is a strongly continuous semigroup of nonexpansive mappings, see [24, Theorem 1.3.13] and [29, Theorem 1.13]. In the same way we define the semigroups S_t^n of the components f_n , using the appropriate resolvents J_λ^n , for $n = 1, \dots, N$.

The following nonlinear version of the Lie-Trotter-Kato formula was proved in [40]. It shows that, given a function f of the form (3), we can approximate the semigroup of f by the resolvents (or semigroups) of the components f_n . The symbol P_n will denote the metric projection onto $\overline{\text{dom } f_n}$, where $n = 1, \dots, N$.

Theorem 1.7 (Stojkovic). *Let (\mathcal{H}, d) be an Hadamard space, and $f_n : \mathcal{H} \rightarrow (-\infty, \infty]$ be convex lsc, where $n = 1, \dots, N$. Denote $f = \sum_1^N f_n$. Then, for any $t \in [0, \infty)$, and $x \in \overline{\text{dom}} f$, we have*

$$(8a) \quad S_t(x) = \lim_{n \rightarrow \infty} \left(J_{\frac{t}{n}}^N \circ \dots \circ J_{\frac{t}{n}}^1 \right)^{(n)}(x),$$

and,

$$(8b) \quad S_t(x) = \lim_{n \rightarrow \infty} \left(S_{\frac{t}{n}}^N \circ P_N \cdots \circ S_{\frac{t}{n}}^1 \circ P_1 \right)^{(n)}(x).$$

The convergence in both (8a) and (8b) is uniform with respect to t on any compact subinterval of $[0, \infty)$.

Proof. The proof given in [40] uses ultralimits of Hadamard spaces. A simpler proof relying on weak convergence appeared in [1]. \square

The law of large numbers in Hadamard space. The law of large numbers is related to the Fréchet mean in a classical linear setting as well as in Hadamard spaces. For a historical remark, we refer the interested reader to [43, Remark 2.7a]. As we shall see in the sequel, more precisely in Section 5, the probabilistic point of view enables us to find an alternative algorithm for computing the Fréchet mean. We shall also the algorithms based on the PPA with this algorithm based on the law of large numbers.

The author was informed that the algorithm relying upon the law of large numbers was independently discovered by E. Miller, M. Owen, and S. Provan [30].

The organization of the paper. The following Section 2 is devoted to the rudiments of Hadamard space theory including a discussion on medians and means. The main results of the present paper are contained in Section 3. We prove that both the cyclic and random order versions of the PPA converge to a minimizer of the function in question. In Section 4 we apply the PPA to the case of medians and means, respectively, and obtain explicit and user-friendly algorithms for their computations. The last part, Section 5, is devoted to an alternative algorithm for Fréchet means which relies upon the law of large numbers.

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2. PRELIMINARIES

Hadamard spaces. We will now recall basic facts on Hadamard spaces. For further details on the subject, we refer the reader to [12] or [23].

A metric space (X, d) is called *geodesic* if for any pair of points $x, y \in X$ there exists a geodesic which connects them. That is, there exists a mapping $\gamma : [0, 1] \rightarrow X$ such that $\gamma(0) = x, \gamma(1) = y$, and

$$d(\gamma(s), \gamma(t)) = d(x, y) |s - t|,$$

for $s, t \in [0, 1]$. If for any point $z \in X$, geodesic $\gamma : [0, 1] \rightarrow X$, and $t \in [0, 1]$, we have

$$(9) \quad d(z, \gamma(t))^2 \leq (1-t)d(z, \gamma(0))^2 + td(z, \gamma(1))^2 - t(1-t)d(\gamma(0), \gamma(1))^2,$$

the space (X, d) is called CAT(0). This property in particular implies that any two points are connected by a *unique* geodesic.

We will moreover assume the Hadamard spaces in our theorems be locally compact. Apart from the BHV tree space described in the Introduction, the class of locally compact Hadamard spaces includes Euclidean spaces, hyperbolic spaces, complete simply connected Riemannian manifolds of nonpositive sectional curvature, Euclidean buildings, locally compact \mathbb{R} -trees and CAT(0) complexes. The algorithm in Section 5 however works without the local compactness assumption.

We will now recall an inequality which goes back to the work of Reshetnyak. Its modern proof can be found in [44, Proposition 2.4], or in [26, Lemma 2.1].

Lemma 2.1. *Let (\mathcal{H}, d) be an Hadamard space. Then we have*

$$d(x, y)^2 + d(u, v)^2 \leq d(x, v)^2 + d(y, u)^2 + 2d(x, u)d(y, v),$$

for any points $x, y, u, v \in \mathcal{H}$.

Given a pair of points $x, y \in \mathcal{H}$, we denote $(1-t)x + ty = \gamma(t)$, where γ is the geodesic connecting x and y . We say that a set $C \subset \mathcal{H}$ is *convex* provided $x, y \in C$ implies $(1-t)x + ty \in C$ for any $t \in [0, 1]$. Furthermore, we say that a function $f : \mathcal{H} \rightarrow (-\infty, \infty]$ is *convex* if the function $f \circ \gamma : [0, 1] \rightarrow (-\infty, \infty]$ is convex for any geodesic $\gamma : [0, 1] \rightarrow \mathcal{H}$.

Let (\mathcal{H}, d) be an Hadamard space and $C \subset \mathcal{H}$ be a convex closed set. Then for any $x \in \mathcal{H}$ there exists a unique point $c \in C$ such that

$$d(x, c) = \inf_{y \in C} d(x, y),$$

and we denote this point c by $P_C(x)$. The mapping $P_C : \mathcal{H} \rightarrow C$ is nonexpansive and we call it the *metric projection* onto the set C .

Given a function $f : \mathcal{H} \rightarrow (-\infty, \infty]$, we say that a point $z \in \mathcal{H}$ is a minimizer of f if

$$f(z) = \inf_{x \in \mathcal{H}} f(x).$$

The set of all minimizers of f will be denoted $\text{Min}(f)$. A *resolvent* of the function f is defined by

$$(10) \quad J_\lambda(x) := \arg \min_{y \in \mathcal{H}} \left[f(y) + \frac{1}{2\lambda} d(x, y)^2 \right],$$

for any $x \in \mathcal{H}$, and parameter $\lambda > 0$. If f is convex and lsc, then $J_\lambda : \mathcal{H} \rightarrow \mathcal{H}$ is a well-defined nonexpansive mapping [24, Lemma 2.5], and [29, Lemma 1.12].

Let us now state the following result from [18, Lemma 2.2], which then implies the existence of a minimizer of a coercive function in Lemma 2.3 below.

Lemma 2.2. *Let (\mathcal{H}, d) be an Hadamard space and (C_n) be a nonincreasing sequence of bounded closed convex subsets of \mathcal{H} . Then*

$$\bigcap_{n \in \mathbb{N}} C_n \neq \emptyset.$$

Proof. See [18, Lemma 2.2]. □

As a consequence, we obtain the following Lemma 2.3. Just recall that a function $f : \mathcal{H} \rightarrow (-\infty, \infty]$ is *coercive* if it satisfies $f(x) \rightarrow \infty$ whenever $d(x, x_0) \rightarrow \infty$, for some $x_0 \in \mathcal{H}$.

Lemma 2.3. *Let (\mathcal{H}, d) be an Hadamard space and $f : \mathcal{H} \rightarrow (-\infty, \infty]$ be a coercive convex lsc function. Then f has a minimizer.*

Proof. We will first observe that f is bounded from below on bounded sets. Let $C \subset \mathcal{H}$ be bounded, and without loss of generality assume that C is closed convex. If $\inf_C f = -\infty$, then the sets $S_N = \{x \in C : f(x) \leq -N\}$ for $N \in \mathbb{N}$ are all nonempty, closed, convex, and bounded. But then Lemma 2.2 yields a point $z \in \bigcap_{N \in \mathbb{N}} S_N$. Clearly $f(z) = -\infty$, which is not possible.

Since f is bounded from below on bounded sets, it is bounded from below on \mathcal{H} , by the coercivity assumption. Therefore $\inf_{\mathcal{H}} f > -\infty$, and the sublevel sets

$$C_n := \left\{ x \in \mathcal{H} : f(x) \leq \inf_{\mathcal{H}} f + \frac{1}{n} \right\},$$

form a nonincreasing sequence of nonempty, bounded, closed, convex subsets of \mathcal{H} . Such a family has according to Lemma 2.2 nonempty intersection and any point in this intersection is obviously a minimizer of f . \square

Means. Given a finite set of points $a_1, \dots, a_N \in \mathcal{H}$, recall that the (weighted) Fréchet mean with positive weights w_1, \dots, w_N satisfying $\sum w_n = 1$, was in (2) defined as

$$(11) \quad \Xi := \Xi(\bar{w}; \bar{a}) := \arg \min_{x \in \mathcal{H}} \sum_{n=1}^N w_n d(x, a_n)^2,$$

where again we denote $\bar{w} = (w_1, \dots, w_N)$ and $\bar{a} = (a_1, \dots, a_N)$. Some authors alternatively use the name Karcher mean. The existence and uniqueness of the minimizer in the definition is a consequence of nonpositive curvature. It is guaranteed by the following theorem, which is a combination of [23, Theorem 3.2.1], [44, Proposition 4.4], and [26, Lemma 4.2].

Theorem 2.4. *Let (\mathcal{H}, d) be an Hadamard space, let $a_1, \dots, a_N \in \mathcal{H}$ be a finite set of points, and w_1, \dots, w_N be positive weights satisfying $\sum w_n = 1$. Then there exists a unique point $\Xi \in \mathcal{H}$ defined in (11). Furthermore, this Ξ satisfies the variance inequality*

$$(12) \quad d(z, \Xi)^2 + \sum_{n=1}^N w_n d(\Xi, a_n)^2 \leq \sum_{n=1}^N w_n d(z, a_n)^2,$$

for any $z \in \mathcal{H}$. Finally, the function $\Xi(\bar{w}; \cdot)$ satisfies

$$d(\Xi(\bar{w}; \bar{a}), \Xi(\bar{w}; \bar{a}')) \leq \sum_{n=1}^N w_n d(a_n, a'_n),$$

for any $a_1, \dots, a_N \in \mathcal{H}$, and $a'_1, \dots, a'_N \in \mathcal{H}$.

Proof. We are to show that there exists a unique minimizer of the function

$$\varphi : x \mapsto \sum_{n=1}^N w_n d(x, a_n)^2, \quad y \in \mathcal{H}.$$

The function φ is bounded from below by 0. Take a minimizing sequence $(y_k) \subset \mathcal{H}$, that is, a sequence such that $\varphi(y_k) \rightarrow \inf \varphi$. The inequality (9) yields that (y_k) is Cauchy. Indeed, if y_{kl} denotes the midpoint of y_k and y_l , then (9) with $t = \frac{1}{2}$ gives

$$d(y_{kl}, a_n)^2 \leq \frac{1}{2}d(y_k, a_n)^2 + \frac{1}{2}d(y_l, a_n)^2 - \frac{1}{4}d(y_k, y_l)^2.$$

Multiplying this inequality by w_n and summing from $n = 1$ to N easily gives that the sequence (y_k) is Cauchy. Since φ is continuous, the sequence (y_k) converges to a minimizer of φ . The uniqueness of this minimizer follows again from (9). It remains to show (12). Employing (9) yields

$$\begin{aligned} \sum_{n=1}^N w_n d(\gamma(t), a_n)^2 - \sum_{n=1}^N w_n d(\Xi, a_n)^2 &\leq (1-t) \sum_{n=1}^N w_n \left[d(\gamma(0), a_n)^2 - d(\Xi, a_n)^2 \right] \\ &\quad + t \sum_{n=1}^N w_n \left[d(\gamma(1), a_n)^2 - d(\Xi, a_n)^2 \right] \\ &\quad - t(1-t) d(\gamma(0), \gamma(1))^2, \end{aligned}$$

for any geodesic $\gamma : [0, 1] \rightarrow \mathcal{H}$. Setting $\gamma(0) = \Xi$ and $\gamma(1) = z$ gives

$$\begin{aligned} 0 &\leq \sum_{n=1}^N w_n d(\gamma(t), a_n)^2 - \sum_{n=1}^N w_n d(\Xi, a_n)^2 \\ &\leq t \left[\sum_{n=1}^N w_n d(z, a_n)^2 - \sum_{n=1}^N w_n d(\Xi, a_n)^2 \right] - t(1-t) d(\Xi, z)^2 \end{aligned}$$

for any $t \in (0, 1)$. Dividing by t and letting $t \rightarrow 0$ yields (12).

If we denote $\Xi = \Xi(\bar{w}; \bar{a})$ and $\Xi' = \Xi(\bar{w}; \bar{a}')$, then Lemma 2.1 yields

$$d(a_n, \Xi')^2 + d(a'_n, \Xi)^2 \leq d(a_n, \Xi)^2 + d(a'_n, \Xi')^2 + 2d(\Xi, \Xi') d(a_n, a'_n),$$

multiplying by w_n and summing up over n from 1 to N further gives

$$\begin{aligned} \sum_{n=1}^N w_n \left[d(a_n, \Xi')^2 + d(a'_n, \Xi)^2 \right] &\leq \sum_{n=1}^N w_n \left[d(a_n, \Xi)^2 + d(a'_n, \Xi')^2 \right] \\ &\quad + 2d(\Xi, \Xi') \sum_{n=1}^N w_n d(a_n, a'_n). \end{aligned}$$

By the variance inequality (12) we have

$$\begin{aligned} \sum_{n=1}^N w_n \left[d(a_n, \Xi')^2 + d(a'_n, \Xi)^2 \right] &\geq \sum_{n=1}^N w_n \left[d(a_n, \Xi)^2 + d(a'_n, \Xi')^2 \right] \\ &\quad + 2d(\Xi, \Xi')^2. \end{aligned}$$

Altogether we obtain

$$d(\Xi, \Xi') \leq \sum_{n=1}^N w_n d(a_n, a'_n),$$

which finishes the proof. \square

If $a_1, \dots, a_N \in \mathbb{R}^d$, then of course

$$(13) \quad \Xi(\bar{w}; \bar{a}) = \frac{w_1 a_1 + \dots + w_N a_N}{N}.$$

In other words, the Fréchet mean coincides with the usual (weighted) arithmetic mean.

Medians. The (weighted) geometric median of a finite set of points $a_1, \dots, a_N \in \mathcal{H}$, was in (1) defined as

$$(14) \quad \Psi(\bar{w}; \bar{a}) := \arg \min_{x \in \mathcal{H}} \sum_{n=1}^N w_n d(x, a_n).$$

Since the function

$$x \mapsto \sum_{n=1}^N w_n d(x, a_n)$$

is convex continuous and coercive, it has a minimizer due to Lemma 2.3. It is easy to check that a minimizer is unique unless all the points a_1, \dots, a_N lie on a geodesic. As we have already mentioned in the Introduction, a median is an optimal solution to the Fermat-Weber problem in facility location theory.

Supermartingale convergence theorem. The main results of the present paper rely upon the following theorem.

Theorem 2.5. *Let $(Y_k), (Z_k)$, and (W_k) be sequences of random variables and let \mathcal{F}_k be sets of random variables such that $\mathcal{F}_k \subset \mathcal{F}_{k+1}$, for $k \in \mathbb{N}$. Assume that*

- (i) *All random variables are nonnegative, and are functions of the random variables in \mathcal{F}_k .*
- (ii) *For any $k \in \mathbb{N}$ we have*

$$\mathbb{E}(Y_{k+1}, \mathcal{F}_k) \leq Y_k - Z_k + W_k.$$

- (iii) *We have $\sum W_k < \infty$ almost surely.*

Then the sequence (Y_k) converges to a random variable Y , and $\sum Z_k < \infty$, almost surely.

Proof. See for instance [8, Proposition 2]. □

A deterministic version of the above theorem will be used in the proof of Theorem 3.4. We include its proof from [8, Lemma 3.4] for the reader's convenience.

Lemma 2.6. *Let $(a_k), (b_k)$, and (c_k) be sequences of nonnegative real numbers. Assume that*

$$(15) \quad a_{k+1} \leq a_k - b_k + c_k,$$

for any $k \in \mathbb{N}$, and,

$$\sum_{k=1}^{\infty} c_k < \infty.$$

Then the sequence (a_k) converges, and $\sum_{k=1}^{\infty} b_k < \infty$.

Proof. Fix $l \in \mathbb{N}$. Sum (15) over $k \geq l$ and take $\limsup_{k \rightarrow \infty}$ to obtain

$$\limsup_{k \rightarrow \infty} a_k \leq a_l + \sum_{k=l}^{\infty} c_k.$$

Taking $\liminf_{l \rightarrow \infty}$ yields

$$\limsup_{k \rightarrow \infty} a_k \leq \liminf_{l \rightarrow \infty} a_l,$$

and hence (a_k) converges. Now fix $n \in \mathbb{N}$ and sum (15) from $k = 1$ to $k = n$,

$$\sum_{k=1}^n b_k \leq a_1 + \sum_{k=1}^n c_k - a_{n+1}.$$

Since the last inequality holds for any $n \in \mathbb{N}$, we get $\sum_{k=1}^{\infty} b_k < \infty$. \square

3. THE PROXIMAL POINT ALGORITHM

The proximal point algorithm (PPA) is a method for finding a minimizer of a convex lsc function defined on a Euclidean space. Its origins go back to Martinet, Rockafellar, and Brézis&Lions [11, 28, 38]. Quite recently, this algorithm was extended into Riemannian manifolds of nonpositive sectional curvature [27], and later also into Hadamard spaces [2]. We recall the main result of [2] in Theorem 3.1 below.

Let (\mathcal{H}, d) be an Hadamard space and $f : \mathcal{H} \rightarrow (-\infty, \infty]$ be a lsc convex function. Assume that f has a minimizer, that is, $\text{Min}(f) \neq \emptyset$. The *proximal point algorithm* starting at a point $x_0 \in \mathcal{H}$ generates in the k -th step, $k \in \mathbb{N}$, the point

$$(16) \quad x_k = \arg \min_{y \in \mathcal{H}} \left[f(y) + \frac{1}{2\lambda_k} d(y, x_{k-1})^2 \right].$$

In terms of resolvents, we can equivalently express (16) as

$$(17) \quad x_k = J_{\lambda_k}(x_{k-1}).$$

The convergence of the algorithm was established in [2, Theorem 1.4].

Theorem 3.1. *Let (\mathcal{H}, d) be an Hadamard space and $f : \mathcal{H} \rightarrow (-\infty, \infty]$ be a convex lsc function attaining its minimum on \mathcal{H} . Then, for an arbitrary starting point $x_0 \in \mathcal{H}$ and a sequence of positive reals (λ_n) such that $\sum_1^{\infty} \lambda_n = \infty$, the sequence $(x_n) \subset \mathcal{H}$ defined by (16) weakly converges to a minimizer of f .*

If the underlying space \mathcal{H} is locally compact, or the function f is strongly convex, we obtain strong convergence in the above theorem.

In the present paper, we consider a function $f : \mathcal{H} \rightarrow (-\infty, \infty]$ of the form

$$(18) \quad f := \sum_{n=1}^N f_n,$$

where $f_n : \mathcal{H} \rightarrow (-\infty, \infty]$ are convex lsc, and $N \in \mathbb{N}$. In many cases, it is much easier to find the resolvents

$$(19) \quad J_{\lambda}^n(x) := \arg \min_{y \in \mathcal{H}} \left[f_n(y) + \frac{1}{2\lambda} d(x, y)^2 \right]$$

of the components f_n than the resolvent of the function f itself. This is true for instance for the median and the mean. Then, instead of applying iteratively the resolvent of f as in (17), we will apply the resolvents (19) of the components

f_n . There are essentially two ways of doing that. We either fix an order of the components (that is, a permutation of the numbers $1, \dots, N$, which without loss of generality may be the identity permutation), and at each cycle we will apply the corresponding resolvents in this fixed order, or alternatively, we will at each step pick a number $r \in \{1, \dots, N\}$ at random, and apply the resolvent of f_r . In either case, we get a sequence converging to a minimizer of f . For (\mathcal{H}, d) being the Euclidean space, such results were recently obtained by D. Bertsekas [7], and we follow his proof strategy.

To prove Theorem 3.4 and Lemma 3.6, we will need the following estimate on the function value at a single PPA step.

Lemma 3.2. *Let $h : \mathcal{H} \rightarrow (-\infty, \infty]$ be a convex lsc function on an Hadamard space (\mathcal{H}, d) , and let*

$$J_\lambda^h(x) := \arg \min_{z \in \mathcal{H}} \left[h(z) + \frac{1}{2\lambda} d(x, z)^2 \right]$$

be its resolvent with parameter $\lambda > 0$. Then

$$h(J_\lambda^h(x)) - h(y) \leq \frac{1}{2\lambda} d(x, y)^2 - \frac{1}{2\lambda} d(J_\lambda^h(x), y)^2,$$

for any $x, y \in \mathcal{H}$.

Proof. Choose $x, y \in \mathcal{H}$. From the definition of $J_\lambda^h(x)$ we have

$$h(J_\lambda^h(x)) + \frac{1}{2\lambda} d(J_\lambda^h(x), x)^2 \leq h(p) + \frac{1}{2\lambda} d(p, x)^2,$$

for any $p \in \mathcal{H}$. In particular, let $t \in [0, 1)$ and $p_t = (1-t)y + tJ_\lambda^h(x)$, then

$$\frac{1}{2\lambda} d(J_\lambda^h(x), x)^2 - \frac{1}{2\lambda} d(p_t, x)^2 \leq h(p_t) - h(J_\lambda^h(x)).$$

Applying (9) to the above inequality gives

$$\begin{aligned} (1-t) [h(y) - h(J_\lambda^h(x))] &\geq -\frac{1-t}{2\lambda} d(y, x)^2 \\ &\quad + \frac{1-t}{2\lambda} d(J_\lambda^h(x), x)^2 \\ &\quad + \frac{t(1-t)}{2\lambda} d(J_\lambda^h(x), y)^2, \end{aligned}$$

or, after taking into account that $t \neq 1$,

$$h(J_\lambda^h(x)) - h(y) \leq \frac{1}{2\lambda} d(y, x)^2 - \frac{1}{2\lambda} d(J_\lambda^h(x), x)^2 - \frac{t}{2\lambda} d(J_\lambda^h(x), y)^2.$$

After passing to the limit $t \rightarrow 1$, we conclude that

$$h(J_\lambda^h(x)) - h(y) \leq \frac{1}{2\lambda} d(y, x)^2 - \frac{1}{2\lambda} d(J_\lambda^h(x), x)^2 - \frac{1}{2\lambda} d(J_\lambda^h(x), y)^2,$$

which finishes the proof. \square

Cyclic order version. We will now prove the first main result, namely, that the proximal point algorithm with cyclic order of applying the marginal resolvent gives a sequence which converges to a minimizer. Let us first precisely define the procedure.

Definition 3.3. Consider a function f of the form (18). Let (λ_k) be a sequence of positive reals satisfying

$$(20) \quad \sum_{k=0}^{\infty} \lambda_k = \infty, \quad \text{and} \quad \sum_{k=0}^{\infty} \lambda_k^2 < \infty.$$

Let $x_0 \in \mathcal{H}$ be an arbitrary starting point. For each $k \in \mathbb{N}_0$ we set

$$\begin{aligned} x_{kN+1} &= J_{\lambda_k}^1(x_{kN}), \\ x_{kN+2} &= J_{\lambda_k}^2(x_{kN+1}), \\ &\vdots \\ x_{kN+N} &= J_{\lambda_k}^N(x_{kN+N-1}), \end{aligned}$$

where the resolvents are defined by (19) above.

Note that the step size parameter λ_k is constant throughout each cycle. The convergence of the above algorithm is assured by the following theorem. The assumption (21) will be commented on later in Remark 3.8.

Theorem 3.4 (Cyclic order version of the PPA). Let (\mathcal{H}, d) be a locally compact Hadamard space, and $f : \mathcal{H} \rightarrow (-\infty, \infty]$ be of the form (18) with $\text{Min}(f) \neq \emptyset$. Given a starting point $x_0 \in \mathcal{H}$, let (x_j) be the sequence defined in Definition 3.3. Assume there exists $L > 0$ such that

$$(21) \quad f_n(x_{kN}) - f_n(x_{kN+n}) \leq Ld(x_{kN}, x_{kN+n}),$$

for any $k \in \mathbb{N}$, and $n = 1, \dots, N$. Then (x_j) converges to a minimizer of f .

Proof. We divide the proof into two steps. Step 1: We claim that

$$(22) \quad d(x_{kN+N}, y)^2 \leq d(x_{kN}, y)^2 - 2\lambda_k [f(x_{kN}) - f(y)] + 2\lambda_k^2 L^2 N(N+1),$$

for any $y \in \mathcal{H}$. Indeed, apply Lemma 3.2 with $h = f_n$ and $x = x_{kN+n-1}$ to obtain

$$d(x_{kN+n}, y)^2 \leq d(x_{kN+n-1}, y)^2 - 2\lambda_k [f_n(x_{kN+n}) - f_n(y)],$$

for every $y \in \mathcal{H}$, and $n = 1, \dots, N$. By summing up we obtain

$$\begin{aligned} d(x_{kN+N}, y)^2 &\leq d(x_{kN}, y)^2 - 2\lambda_k \sum_{n=1}^N [f_n(x_{kN+n}) - f_n(y)], \\ &\leq d(x_{kN}, y)^2 - 2\lambda_k [f(x_{kN}) - f(y)] \\ &\quad + 2\lambda_k \sum_{n=1}^N [f_n(x_{kN}) - f_n(x_{kN+n})]. \end{aligned}$$

By the assumption (21), we have

$$f_n(x_{kN}) - f_n(x_{kN+n}) \leq Ld(x_{kN}, x_{kN+n}),$$

where the right hand side can be further estimated as

$$d(x_{kN}, x_{kN+n}) \leq d(x_{kN}, x_{kN+1}) + \dots + d(x_{kN+n-1}, x_{kN+n}).$$

By the definition of the algorithm we have

$$f_m(x_{kN+m}) + \frac{1}{2\lambda_k} d(x_{kN+m-1}, x_{kN+m})^2 \leq f_m(x_{kN+m-1}),$$

for any $m = 1, \dots, N$, which then gives

$$\begin{aligned} d(x_{kN+m-1}, x_{kN+m}) &\leq 2\lambda_k \frac{f_m(x_{kN+m-1}) - f_m(x_{kN+m})}{d(x_{kN+m-1}, x_{kN+m})} \\ &\leq 2\lambda_k L, \end{aligned}$$

where we again employed the assumption (21). Hence,

$$f_n(x_{kN}) - f_n(x_{kN+n}) \leq 2\lambda_k L^2 n,$$

and finally,

$$d(x_{kN+N}, y)^2 \leq d(x_{kN}, y)^2 - 2\lambda_k [f(x_{kN}) - f(y)] + 2\lambda_k^2 L^2 N(N+1),$$

which finishes the proof of (22).

Step 2: Let now $z \in \text{Min}(f)$, and apply (22) with $y = z$. Then

$$d(x_{kN+N}, z)^2 \leq d(x_{kN}, z)^2 - 2\lambda_k [f(x_{kN}) - f(z)] + 2\lambda_k^2 L^2 N(N+1),$$

which according to Lemma 2.6 implies that the sequence

$$(d(x_{kN}, z))_{k \in \mathbb{N}_0}$$

converges, (and in particular, the sequence (x_{kN}) is bounded), and

$$(23) \quad \sum_{k=0}^{\infty} \lambda_k [f(x_{kN}) - f(z)] < \infty.$$

From (23) we immediately obtain that there exists a subsequence $(x_{k_l N})$ of (x_{kN}) for which

$$f(x_{k_l N}) \rightarrow f(z), \quad \text{as } l \rightarrow \infty.$$

Since the sequence $(x_{k_l N})$ is bounded, it has a subsequence which converges to a point $\hat{z} \in \mathcal{H}$. By the lower semicontinuity of f we obtain $\hat{z} \in \text{Min}(f)$. Then we know that

$$(d(x_{kN}, \hat{z}))_{k \in \mathbb{N}_0}$$

converges, and also that it converges to 0, since a subsequence of (x_{kN}) converges to \hat{z} .

By Lemma 3.2 with $h = f_1$, and $x = x_{kN}$, and $y = \hat{z}$, we get that

$$0 \leq f_1(x_{kN+1}) - f_1(\hat{z}) \leq \frac{1}{2\lambda} d(x_{kN}, \hat{z})^2 - \frac{1}{2\lambda} d(x_{kN+1}, \hat{z})^2,$$

and taking the limit $k \rightarrow \infty$ yields

$$\lim_{k \rightarrow \infty} x_{kN+1} = \hat{z}.$$

In the same way, we obtain

$$\lim_{k \rightarrow \infty} x_{kN+n} = \hat{z},$$

for any $n = 2, \dots, N$. Hence the whole sequence (x_j) converges to \hat{z} , and the proof is complete. \square

Random order version. Instead of applying the marginal resolvents in a cyclic order, one can at each step select a number from $\{1, \dots, N\}$ at random and use the corresponding resolvent. Next we prove that the resulting sequence converges to a minimizer of the function f , too.

Definition 3.5. Consider a function $f : \mathcal{H} \rightarrow (-\infty, \infty]$ of the form (18). Let (λ_k) be a sequence of positive reals satisfying

$$(24) \quad \sum_{k=0}^{\infty} \lambda_k = \infty, \quad \text{and} \quad \sum_{k=0}^{\infty} \lambda_k^2 < \infty.$$

Let (r_k) be a sequence of random variables which attain values from $\{1, \dots, N\}$ according to the uniform distribution, independently of previous steps. For every $k \in \mathbb{N}_0$, define

$$(25) \quad x_{k+1} := J_{\lambda_k}^{r_k}(x_k),$$

with a starting point $x_0 \in \mathcal{H}$. Finally, denote x_{k+1}^n the result of the iteration with x_k if $r_k = n$.

The following Lemma 3.6 is a stochastic version of Lemma 3.2. Again, the assumption (26) will be commented on in Remark 3.8.

Lemma 3.6. Let (\mathcal{H}, d) be an Hadamard space and f be of the form (18). Given a starting point $x_0 \in \mathcal{H}$, let (x_k) be the sequence defined in Definition 3.5. Assume there exists $L > 0$ such that

$$(26) \quad f_n(x_k) - f_n(x_{k+1}^n) \leq Ld(x_k, x_{k+1}^n),$$

for any $k \in \mathbb{N}$, and $n = 1, \dots, N$. If we denote $\mathcal{F}_k = \{x_k, \dots, x_0\}$, then

$$\mathbb{E} \left[d(x_{k+1}, y)^2 \mid \mathcal{F}_k \right] \leq d(x_k, y)^2 - \frac{2\lambda_k}{N} [f(x_k) - f(y)] + 4\lambda_k^2 L^2,$$

almost surely, for any $y \in \mathcal{H}$.

Proof. By Lemma 3.2 we have

$$d(x_{k+1}, y)^2 \leq d(x_k, y)^2 - 2\lambda_k [f_{r_k}(x_k) - f_{r_k}(y)].$$

Taking the conditional expectation with respect to \mathcal{F}_k gives

$$\mathbb{E} \left[d(x_{k+1}, y)^2 \mid \mathcal{F}_k \right] \leq d(x_k, y)^2 - 2\lambda_k \mathbb{E} [f_{r_k}(x_k) - f_{r_k}(y) \mid \mathcal{F}_k].$$

If we denote x_{k+1}^n the result of the iteration with x_k when $r_k = n$, we get

$$\begin{aligned} \mathbb{E} \left[d(x_{k+1}, y)^2 \mid \mathcal{F}_k \right] &\leq d(x_k, y)^2 - \frac{2\lambda_k}{N} \sum_{n=1}^N [f_n(x_{k+1}^n) - f_n(y)] \\ &= d(x_k, y)^2 - \frac{2\lambda_k}{N} [f(x_k) - f(y)] \\ &\quad + \frac{2\lambda_k}{N} \sum_{n=1}^N [f_n(x_k) - f_n(x_{k+1}^n)]. \end{aligned}$$

By the assumption (26) we have

$$\sum_{n=1}^N [f_n(x_k) - f_n(x_{k+1}^n)] \leq L \sum_{n=1}^N d(x_k, x_{k+1}^n) \leq 2L^2 \lambda_k N,$$

since

$$d(x_k, x_{k+1}^n) \leq 2\lambda_k \frac{f_n(x_k) - f_n(x_{k+1}^n)}{d(x_k, x_{k+1}^n)} \leq 2\lambda_k L.$$

We hence finally obtain

$$\mathbb{E} \left[d(x_{k+1}, y)^2 \mid \mathcal{F}_k \right] \leq d(x_k, y)^2 - \frac{2\lambda_k}{N} [f(x_k) - f(y)] + 4\lambda_k^2 L^2,$$

which finishes the proof. \square

We now get to the second convergence theorem.

Theorem 3.7 (Random order version of the PPA). *Let (\mathcal{H}, d) be a locally compact Hadamard space and f be of the form (18) with $\text{Min}(f) \neq \emptyset$. Assume that the Lipschitz condition (26) holds true. Then, given a starting point $x_0 \in \mathcal{H}$, the sequence (x_k) defined in Definition 3.5 converges to a minimizer of f almost surely.*

Proof. Let (v_i) be a countable dense subset of $\text{Min}(f)$. For each $i \in \mathbb{N}$ apply Lemma 3.6 with $y = v_i$ to obtain

$$\mathbb{E} \left[d(x_{k+1}(\omega), v_i)^2 \mid \mathcal{F}_k \right] \leq d(x_k(\omega), v_i)^2 - \frac{2\lambda_k}{N} [f(x_k(\omega)) - f(v_i)] + 4\lambda_k^2 L^2.$$

Theorem 2.5 immediately gives that $d(v_i, x_k(\omega))$ converges, and

$$\sum_{k=0}^{\infty} \lambda_k [f(x_k(\omega)) - \inf f] < \infty,$$

for all $\omega \in \Omega_{v_i}$, where Ω_{v_i} is a set of full measure. Next denote

$$\Omega = \bigcap_{i \in \mathbb{N}} \Omega_{v_i},$$

which is obviously again a set of full measure. The last inequality yields that for $\omega \in \Omega$, we have $\liminf_{k \rightarrow \infty} f(x_k(\omega)) = \inf f$, and since $(x_k(\omega))$ is bounded, it has a cluster point $x(\omega) \in \mathcal{H}$. By the lower semicontinuity of f we may assume that $x(\omega) \in \text{Min}(f)$.

For each $\varepsilon > 0$ there exists $v_{i(\varepsilon)} \in (v_i)$ such that $d(x(\omega), v_{i(\varepsilon)}) < \varepsilon$. Because the sequence $d(x_k(\omega), v_{i(\varepsilon)})$ converges and $x(\omega)$ is a cluster point of $x_k(\omega)$, we have

$$\lim_{k \rightarrow \infty} d(x_k(\omega), v_{i(\varepsilon)}) < \varepsilon.$$

This yields $x_k(\omega) \rightarrow x(\omega)$. We obtain that x_k converges to a minimizer almost surely. This finishes the proof. \square

Remark 3.8. The assumptions (21) in Theorem 3.4, and (26) in Theorem 3.7 are satisfied, for instance, if

- (i) the functions f_n are Lipschitz on \mathcal{H} with constant L , or
- (ii) the functions f_n are Lipschitz on bounded sets and coercive.

The latter condition applies for a function f of the form (4). In particular, for both the mean (11), and the median (14).

4. COMPUTING MEDIANS AND MEANS

The algorithms from Definitions 3.3 and 3.5 can be directly applied to compute means and medians in locally compact Hadamard spaces. We will now give an explicit description of these two special cases.

Algorithms for computing means. Given positive weights w_1, \dots, w_N with $\sum w_n = 1$ and points $a_1, \dots, a_N \in \mathcal{H}$, we wish to minimize the function

$$f(x) := \sum_{n=1}^N w_n d(x, a_n)^2, \quad x \in \mathcal{H}.$$

The existence and uniqueness of a minimizer is assured by Theorem 2.4. Furthermore, the function f is of the form (18) and according to Remark 3.8 satisfies both (21) and (26). We can therefore employ proximal point algorithms with $f_n = w_n d(\cdot, a_n)^2$, for $n = 1, \dots, N$. Let us first consider the cyclic order version from Definition 3.3. Let (λ_k) be a sequence of positive reals satisfying (20). We start at some point $x_0 \in \mathcal{H}$, and for each $k \in \mathbb{N}_0$ we set

$$\begin{aligned} x_{kN+1} &= J_{\lambda_k}^1(x_{kN}), \\ x_{kN+2} &= J_{\lambda_k}^2(x_{kN+1}), \\ &\vdots \\ x_{kN+N} &= J_{\lambda_k}^N(x_{kN+N-1}), \end{aligned}$$

where $J_{\lambda_k}^n$ is now the resolvent of the function $f_n = w_n d(\cdot, a_n)^2$, for $n = 1, \dots, N$. It is easy to find these resolvents explicitly. Indeed, fix $k \in \mathbb{N}_0$ and $n = 1, \dots, N$. Then x_{kN+n} is the unique minimizer of the function

$$w_n d(\cdot, a_n)^2 + \frac{1}{2\lambda_k} d(\cdot, x_{kN+n-1})^2,$$

and it is obvious that such a minimizer lies on the geodesic $[x_{kN+n-1}, a_n]$, that is,

$$x_{kN+n} = (1 - t_k^n) x_{kN+n-1} + t_k^n a_n,$$

for some $t_k^n \in [0, 1]$. By an elementary calculation we get

$$(27) \quad t_k^n = \frac{2\lambda_k w_n}{1 + 2\lambda_k w_n}.$$

The above algorithm then reads:

Algorithm 4.1 (Computing mean, cyclic order version). *Given $x_0 \in \mathcal{H}$ and (λ_k) satisfying (20) we set*

$$\begin{aligned} x_{kN+1} &= \frac{1}{1 + 2\lambda_k w_1} x_{kN} + \frac{2\lambda_k w_1}{1 + 2\lambda_k w_1} a_1, \\ x_{kN+2} &= \frac{1}{1 + 2\lambda_k w_2} x_{kN+1} + \frac{2\lambda_k w_2}{1 + 2\lambda_k w_2} a_2, \\ &\vdots \\ x_{kN+N} &= \frac{1}{1 + 2\lambda_k w_N} x_{kN+N-1} + \frac{2\lambda_k w_N}{1 + 2\lambda_k w_N} a_N, \end{aligned}$$

for each $k \in \mathbb{N}_0$ and $n = 1, \dots, N$.

The convergence of the sequence (x_j) produced by Algorithm 4.1 to the weighted mean of the points a_1, \dots, a_N follows by Theorem 3.4 above. Note that if the weights are uniform, that is, $w_n = \frac{1}{N}$ for each $n = 1, \dots, N$, then the coefficients t_k^n are independent of n .

We will now turn to the randomized version from Definition 3.5. By a similar process as above we obtain the following algorithm.

Algorithm 4.2 (Computing mean, random order version). *Let $x_0 \in \mathcal{H}$ be a starting point and (λ_k) satisfy (24). At each step $k \in \mathbb{N}_0$, choose randomly $r_k \in \{1, \dots, N\}$ according to the uniform distribution and put*

$$x_{k+1} = \frac{1}{1 + 2\lambda_k w_{r_k}} x_k + \frac{2\lambda_k w_{r_k}}{1 + 2\lambda_k w_{r_k}} a_{r_k}.$$

The convergence of the sequence (x_k) produced by Algorithm 4.2 to the weighted mean of the points a_1, \dots, a_N follows by Theorem 3.7 above.

Algorithms for computing medians. Given positive weights w_1, \dots, w_N with $\sum w_n = 1$ and points $a_1, \dots, a_N \in \mathcal{H}$, we wish to minimize the function

$$f(x) := \sum_{n=1}^N w_n d(x, a_n), \quad x \in \mathcal{H}.$$

The function f is again of the form (18) with $f_n = w_n d(\cdot, a_n)$, for $n = 1, \dots, N$. It is Lipschitz, and hence satisfies the assumptions (21) and (26). In the cyclic order version, we start at some point $x_0 \in \mathcal{H}$, and for each $k \in \mathbb{N}_0$ we set

$$\begin{aligned} x_{kN+1} &= J_{\lambda_k}^1(x_{kN}), \\ x_{kN+2} &= J_{\lambda_k}^2(x_{kN+1}), \\ &\vdots \\ x_{kN+N} &= J_{\lambda_k}^N(x_{kN+N-1}), \end{aligned}$$

where $J_{\lambda_k}^n$ is the resolvent of the function $f_n = w_n d(\cdot, a_n)$, for $n = 1, \dots, N$, and (λ_k) is a sequence of positive reals satisfying (20). More specifically, if we fix $k \in \mathbb{N}_0$ and $n = 1, \dots, N$, then x_{kN+n} is the unique minimizer of the function

$$w_n d(\cdot, a_n) + \frac{1}{2\lambda_k} d(\cdot, x_{kN+n-1})^2,$$

and it is obvious that such a minimizer lies on the geodesic $[x_{kN+n-1}, a_n]$, that is,

$$x_{kN+n} = (1 - t_k^n) x_{kN+n-1} + t_k^n a_n,$$

for some $t_k^n \in [0, 1]$. These coefficients are again easy to determine. We have to however treat the cyclic and the random case separately.

Algorithm 4.3 (Computing median, cyclic order version). *Given $x_0 \in \mathcal{H}$ and (λ_k) satisfying (20) we set*

$$\begin{aligned} x_{kN+1} &= (1 - t_k^1) x_{kN} + t_k^1 a_1, \\ x_{kN+2} &= (1 - t_k^2) x_{kN+1} + t_k^2 a_2, \\ &\vdots \\ x_{kN+N} &= (1 - t_k^N) x_{kN+N-1} + t_k^N a_N, \end{aligned}$$

with t_k^n defined by

$$t_k^n = \min \left\{ 1, \frac{\lambda_k w_n}{d(a_n, x_{kN+n-1})} \right\},$$

for each $k \in \mathbb{N}_0$ and $n = 1, \dots, N$.

The convergence of the sequence (x_j) produced by Algorithm 4.3 to the median of the points a_1, \dots, a_N with the weights (w_1, \dots, w_N) follows by Theorem 3.4 above. Finally, the randomized version can be derived in a similar way.

Algorithm 4.4 (Computing median, random order version). *Let $x_0 \in \mathcal{H}$ be a starting point and (λ_k) satisfies (24). At each step $k \in \mathbb{N}_0$, choose randomly $r_k \in \{1, \dots, N\}$ according to the uniform distribution and put*

$$(28) \quad x_{k+1} = (1 - t_k)x_k + t_k a_{r_k},$$

with t_k defined by

$$t_k = \min \left\{ 1, \frac{\lambda_k w_{r_k}}{d(a_{r_k}, x_k)} \right\},$$

for each $k \in \mathbb{N}_0$.

The convergence of the sequence (x_k) produced by Algorithm 4.4 to the median of the points a_1, \dots, a_N follows by Theorem 3.7 above.

Remark 4.5. Let now take a look at a more general situation mentioned already in the Introduction. Let C_1, \dots, C_N be convex closed subsets of our locally compact Hadamard space (\mathcal{H}, d) and minimize the function (5), that is,

$$f(x) = \sum_{n=1}^N w_n d(x; C_n), \quad x \in \mathcal{H},$$

where $\bar{w} = (w_1, \dots, w_N)$ are again positive weights with $\sum w_n = 1$. We have to assume that at least one of the sets C_1, \dots, C_N be bounded in order to fulfill the assumption $\text{Min}(f) \neq \emptyset$ in Theorems 3.4 and 3.7. It is also clear that f is convex and 1-Lipschitz and thus satisfies both (21) and (26). Denote P_n the metric projection onto the set C_n , where $n = 1, \dots, N$. We describe the random version of the PPA algorithm only, the cyclic version being completely analogous.

Let $x_0 \in \mathcal{H}$ be a starting point and (λ_k) satisfies (24). At each step $k \in \mathbb{N}_0$, choose randomly $r_k \in \{1, \dots, N\}$ according to the uniform distribution and put

$$x_{k+1} = (1 - t_k)x_k + t_k P_{r_k}(x_k),$$

with t_k defined by

$$t_k = \min \left\{ 1, \frac{\lambda_k w_{r_k}}{d(P_{r_k}(x_k), x_k)} \right\},$$

for each $k \in \mathbb{N}_0$. Then the sequence (x_k) converges to a minimizer of f by Theorem 3.7.

5. COMPUTING MEANS VIA THE LAW OF LARGE NUMBERS

In this last section we give an alternative algorithm for computing the Fréchet mean, which is based on the law of large numbers. The advantage of this approach is that we do not require the underlying Hadamard space be locally compact. We shall also compare this algorithm with Algorithm 4.2.

Let again $a_1, \dots, a_N \in \mathcal{H}$ be a finite set of points and w_1, \dots, w_N be positive weights satisfying $\sum w_n = 1$. Denote the probability measure

$$(29) \quad \pi = \sum_{n=1}^N w_n \delta_{a_n},$$

where δ_{a_n} stands for the Dirac measure at a_n . Assume that Y is a random variable with values in \mathcal{H} distributed according to π . Then the variational inequality (12) can be written as

$$(30) \quad d(z, \Xi)^2 + \mathbb{E}d(\Xi, Y)^2 \leq \mathbb{E}d(z, Y)^2, \quad z \in \mathcal{H},$$

where the expectation \mathbb{E} is of course taken with respect to the distribution π .

Given a sequence of random variables Y_k with values in \mathcal{H} , we define a sequence (S_k) of random variables putting $S_1 = Y_1$, and

$$(31) \quad S_{k+1} = \frac{k}{k+1}S_k + \frac{1}{k+1}Y_{k+1},$$

for $i \in \mathbb{N}$. The random variables Y_k , and hence also S_k , are defined on some probability space Ω , but this space Ω of course plays no role here. The following theorem due to K.-T. Sturm states a nonlinear version of the law of large numbers. It appeared in a much more general form in [43, Theorem 2.6].

Theorem 5.1 (The law of large numbers). *Let (\mathcal{H}, d) be an Hadamard space, and (Y_k) be a sequence of independent random variables $Y_k : \Omega \rightarrow \mathcal{H}$, identically distributed according to the distribution π , defined in (29). Then*

$$S_k \rightarrow \Xi(\bar{w}; \bar{x}), \quad \text{as } k \rightarrow \infty,$$

where the convergence is pointwise.

Proof. First denote

$$\xi = \min_{x \in \mathcal{H}} \sum_{n=1}^N w_n d(x, a_n)^2.$$

We show by induction on $k \in \mathbb{N}$ that

$$(32) \quad \mathbb{E}d(\Xi, S_k)^2 \leq \frac{1}{k}\xi.$$

It obviously holds for $k = 1$ and we assume it holds for some $k \in \mathbb{N}$. We have

$$\mathbb{E}d(\Xi, S_{k+1})^2 = \mathbb{E}d\left(\Xi, \frac{k}{k+1}S_k + \frac{1}{k+1}Y_{k+1}\right)^2,$$

by (9) we get

$$\leq \frac{k}{k+1}\mathbb{E}d(\Xi, S_k)^2 + \frac{1}{k+1}\mathbb{E}d(\Xi, Y_{k+1})^2 - \frac{k}{(k+1)^2}\mathbb{E}d(Y_{k+1}, S_k)^2,$$

and applying independence and (30) gives

$$\begin{aligned} &\leq \frac{k}{k+1}\mathbb{E}d(\Xi, S_k)^2 + \frac{1}{k+1}\mathbb{E}d(\Xi, Y_{k+1})^2 \\ &\quad - \frac{k}{(k+1)^2}\mathbb{E}\left[d(\Xi, S_k)^2 + d(\Xi, Y_{k+1})^2\right] \\ &= \left(\frac{k}{k+1}\right)^2 \mathbb{E}d(\Xi, S_k)^2 + \frac{1}{(k+1)^2}\xi \\ &\leq \frac{1}{k+1}\xi. \end{aligned}$$

This shows that (32) holds, and hence the proof is complete. \square

One can rather straightforwardly convert Theorem 5.1 into an approximation algorithm for computing the Fréchet mean. Let us now describe such an algorithm. It receives the points a_1, \dots, a_N and weights w_1, \dots, w_N as the input, and at each iteration $k \in \mathbb{N}$ it produces a new point $s_k \in \mathcal{H}$, which is an approximate version of the desired mean $\Xi = \Xi(\bar{w}; \bar{x})$ in the sense that $d(s_k, \Xi) \rightarrow 0$ as $k \rightarrow \infty$. The sequence is defined as follows. At each step $k \in \mathbb{N}_0$, choose randomly $r_k \in \{1, \dots, N\}$ according to the distribution $\bar{w} = (w_1, \dots, w_N)$ and put

$$(33) \quad s_{k+1} := \frac{k}{k+1} s_k + \frac{1}{k+1} a_{r_k}.$$

The convergence of this algorithm is guaranteed by Theorem 5.1.

We shall now compare the algorithm (33) with Algorithm 4.2. Let us first consider the *unweighted* case, that is, $w_n = \frac{1}{N}$ for every $n = 1, \dots, N$. At each iteration $k \in \mathbb{N}_0$, the algorithm (33) selects $r_k \in \{1, \dots, N\}$ according to the *uniform* distribution and generates a new point

$$s_{k+1} := \frac{k}{k+1} s_k + \frac{1}{k+1} a_{r_k}.$$

In Algorithm 4.2, at each step $k \in \mathbb{N}_0$ we randomly choose a number $r_k \in \{1, \dots, N\}$ according to the *uniform* distribution and put

$$x_{k+1} := \frac{1}{1+2\lambda_k} x_k + \frac{2\lambda_k}{1+2\lambda_k} a_{r_k}.$$

Thus Algorithm 4.2 produces the same sequence as the algorithm (33) provided we set $\lambda_k := \frac{1}{2k}$ for each $k \in \mathbb{N}$. In other words the algorithm (33) is a special case of Algorithm 4.2.

On the other hand as far as *weighted* Fréchet means are concerned, there exists a difference between these two algorithms. Indeed, if $\bar{w} := (w_1, \dots, w_N)$ are the weights, then the algorithm (33) selects $r_k \in \{1, \dots, N\}$ according to the *distribution* \bar{w} and generates a new point

$$s_{k+1} := \frac{k}{k+1} s_k + \frac{1}{k+1} a_{r_k},$$

that is, with the same coefficients as in the unweighted case. Algorithm 4.2 in contrast still selects $r_k \in \{1, \dots, N\}$ according to the *uniform* distribution, but the new point is given by

$$x_{k+1} = \frac{1}{1+2\lambda_k w_{r_k}} x_k + \frac{2\lambda_k w_{r_k}}{1+2\lambda_k w_{r_k}} a_{r_k},$$

that is, the coefficients now do depend on the weights. In summary, introducing weights effects either the coefficients (Algorithm 4.2), or the probability distribution which is used for selecting the points a_1, \dots, a_N (the algorithm (33)).

Final remarks. Notice that all the algorithms presented in this paper require finding a geodesic at each iteration. For instance, in Algorithm 4.4, we need to find the geodesic $[x_k, a_{r_k}]$ at each step $k \in \mathbb{N}_0$, or more precisely, we need to compute the point x_{k+1} which lies on this geodesic. When employing these algorithms in the BHV tree space, we can use the Owen-Provan algorithm (mentioned in the Introduction) to find this point in finitely many steps in polynomial time.

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