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Computing Metric Dimension of Certain Families of Toeplitz Graphs

JIA-BAO LIU[1,2], MUHAMMAD FAISAL NADEEM[3], HAFIZ MUHAMMAD AFZAL SIDDIQUI[3], AND WAJIHA NAZIR[3]

[1]School of Mathematics and Physics, Anhui Jianzhu University, Hefei 230601, China
[2]School of Mathematics, Southeast University, Nanjing 210096, China
[3]Department of Mathematics, COMSATS University Islamabad, Lahore Campus, Lahore 54000, Pakistan

Corresponding author: Hafiz Muhammad Afzal Siddiqui (hmasiddiqui@gmail.com)

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ABSTRACT The position of a moving point in a connected graph can be identified by computing the distance from the point to a set of sonar stations which have been appropriately situated in the graph. Let $Q = \{q_1, q_2, \ldots, q_k\}$ be an ordered set of vertices of a graph G and a is any vertex in G, then the code/representation of a w.r.t Q is the k-tuple $(r(a, q_1), r(a, q_2), \ldots, r(a, q_k))$, denoted by $r(a|Q)$. If the different vertices of G have the different representations w.r.t Q, then Q is known as a *resolving set/locating set*. A resolving/locating set having the least number of vertices is the *basis* for G and the number of vertices in the basis is called *metric dimension* of G and it is represented as $dim(G)$. In this paper, the metric dimension of Toeplitz graphs generated by two and three parameters denoted by $T_n\langle 1, t\rangle$ and $T_n\langle 1, 2, t\rangle$, respectively is discussed and proved that it is constant.

INDEX TERMS Metric dimension, basis, resolving set, Toeplitz graph.

I. INTRODUCTION AND PRELIMINARY RESULTS

A real world problem is the study of networks whose structure has not been imposed by a central authority but arisen from local and distributed processes. It is very difficult and costly to obtain a map of all nodes and the links between them. A commonly used technique is to obtain local view of the network from various locations and combine them to obtain a good approximation for the real network. The metric dimension of graphs is very useful to solve such sorts of problems.

The notion of the metric dimension was first time introduced by Slater [16] and then later by Harary and Melter [5] independently. There are many applications of resolvability in graph theory, for example it has applications in pattern recognition, pharmaceutical chemistry [3], processing of images, networks, combinatorial optimization, tricky games and tasks on coin-weighing, robot navigation [9], facility location problems and many more. Solutions of many other practical applications can be found with the help of metric dimension and metric basis of connected graphs. For more details in this regard see [10], [13], [14], [17], [22], [23]. The concept of metric dimension is also useful for solving the problems of percolation in a hierarchical lattice. For more details see [15]. The distance between any pair $t, s \in V(G)$ of vertices of G is the number of edges in a shortest path between them, denoted by $d(t, s)$. Any vertex $p \in V(G)$ is said to resolve or distinguish a pair $t, s \in V(G)$ if $d(p, t) \neq d(p, s)$. An ordered set $Q = \{q_1, q_2, \ldots, q_k\} \subseteq V(G)$ of a connected graph G is considered as resolving set for G if any pair of vertices of G is distinguished by some vertices of Q. A resolving set with least number of vertices is referred as *metric basis* for G and the cardinality of such resolving set is know as *metric dimension* denoted by $dim(G)$.

Dimension of graph is one if and only if $G = P_n$ [12]. The dimension of graph G shown in Figure 1 is two. For $Q_1 = \{q_1, q_2\}$, $r(q_3|Q_1) = r(q_4|Q_1) = (1, 1)$. So, Q_1 is not a resolving set. For $Q_2 = \{q_1, q_3\}$, $r(q_2|Q_2) = (1, 1)$ and $r(q_4|Q_2) = (1, 2)$. So, Q_2 is the resolving set. Hence $dim(G) = 2$.

The reader are advised to see the following papers for better understanding and detailed study about this notion [1]–[3], [7], [8], [11], [18], [19].

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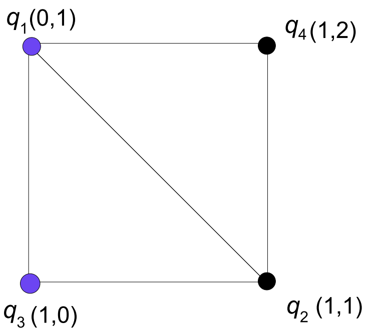

FIGURE 1. Graph with metric dimension 2.

The following lemma works as a useful property for finding the metric basis and metric dimension for any connected graph G.

Lemma 1 [21]: Let Q be a resolving set for a connected graph G and $u, v \in V(G)$. If $d(u, w) = d(v, w)$ for all vertices $w \in V(G) \setminus \{u, v\}$, then $\{u, v\} \cap Q \neq \emptyset$.

Let $\mathcal{E}$ be a family of connected graphs defined as G_m : $\mathcal{E} = (G_m)_{m \geq 1}$ which depends on m, the order $|V(G)| = \varphi(m)$ and $\lim_{m\to\infty} \varphi(m) = \infty$. Let $K > 0$ be any constant, such that $dim(G_m) \leq K$ for each $m \geq 1$, then we say that family of graphs $\mathcal{E}$ has *bounded metric dimension*; otherwise $\mathcal{E}$ has *unbounded metric dimension*.

If all graphs in $\mathcal{E}$ have the same metric dimension (which does not depend on m), then $\mathcal{E}$ has *constant metric dimension*.

Theorem 2 [12]: Let G be a simple connected graph of order $n \geq 2$. Then

(a) $dim(G) = 1$ if and only if $G = P_n$.

(b) $dim(G) = n - 1$ if and only if $G = K_n$.

(c) $dim(G) = n - 2$ if and only if $G = K_{r,s}$, where $r, s \geq 1$ and $n \geq 4$

Theorem 3 [20]: Let G be a simple connected graph with metric dimension 2 and let $\{v_1, v_2\} \subseteq V(G)$ be a metric basis in G, then the degree of both v_1 and v_2 is at most 3 and there exists a unique shortest path between v_1 and v_2.

For a graph G with m vertices labelled as $\{1, 2, 3, \ldots, m\}$ its adjacency matrix A is $m \times m$ matrix whose ij^{th} entry is 1 if the vertex i and vertex j joined by an edge and 0 otherwise. A $m \times m$ matrix $B = b_{ij}$ is known as Toeplitz matrix if $b_{ij} = b_{i+1,j+1}$ for each $i, j = 1, \ldots, m - 1$.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Toeplitz Matrix

A simple undirected graph T_m is *Toeplitz graph* if matrix $m \times m$ which is $B = b_{ij}$ is the symmetric Toeplitz matrix and for all $i, j = 1, \ldots, m$ satisfied the following: edge $\{i, j\}$ is in $E(G)$ iff $b_{ij} = b_{ji} = 1$. An $m \times m$ matrix B will be labelled $0, 1, 2, \ldots, m - 1$ which has m distinct diagonals. The main diagonal has $b_{ii} = 0$ for all $i = 1, \ldots, m$, so Toeplitz graph has no loop. The diagonals $x_1, x_2, \ldots, x_p$ containing ones $0 < x_1 < x_2 < \ldots < x_p < m$. The Toeplitz graph $T_m < x_1, x_2, \ldots, x_p >$ with vertex set $\{1, 2, 3, \ldots, m\}$ has edge $\{i, j\}$, $1 \leq i \leq j \leq m$, occurs if and only if $j - i = x_q$ for some q, $1 \leq q \leq p$.

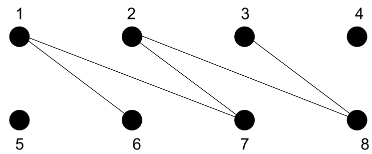

FIGURE 2. Toeplitz graph $T_8\langle 4, 5\rangle$, with gcd(4, 5) = 1.

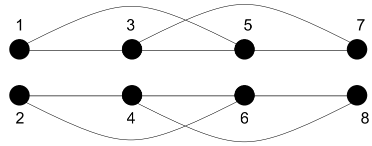

FIGURE 3. Toeplitz graph $T_8\langle 2, 4\rangle$ with two connected components.

As we know that the Toeplitz graphs are those graphs that are derived from Toeplitz matrices. So, the importance of the Toeplitz matrices is also the importance of the Toeplitz graphs. Toeplitz matrices appear in the discretization of the differential and integral equations, they also play a significant role in physical data-processing applications. Moreover, these matrices arise in moment problem, stationary process and the theories of orthogonal polynomials. For more details see [6].

Important properties about the connectivity of Toeplitz graph are as follows:

Theorem 4 [4]: $T_n\langle t_1, \ldots, t_k\rangle$ has at least $gcd(t_1, \ldots, t_k)$ components.

The above theorem shows that a Toeplitz graph can have more than $\gcd(t_1, \ldots, t_k)$ components. The graph $T_8\langle 4, 5\rangle$ shown in Figure 2 has three components whereas gcd $(4, 5) = 1$.

Theorem 5 [4]: If $gcd(t_1, \ldots, t_k) > 1$, then $T_n\langle t_1, \ldots, t_k\rangle$ is disconnected.

We are interested only in connected graphs therefore we will consider only those families of Toeplitz graphs which has only one connected component. We show some of these families of Toeplitz graphs have constant metric dimension.

II. METRIC DIMENSION OF TOEPLITZ GRAPHS $T_n\langle 1, t\rangle$

In this section we study the Toeplitz graph $T_n\langle 1, t\rangle$, where 1 and t are its generators. We classify the dimension of this graph on the bases of the value of t. If t is even, then dimension is 2. If t is odd, then dimension is 3.

Theorem 6: Let $T_n\langle 1, 2\rangle$ be the Toeplitz graph. Then $dim(T_n\langle 1, 2\rangle) = 2$, where $n \geq 4$.

Proof 6: We will show that only two elements in basis set suffice to resolve all the vertices in $V(T_n\langle 1, 2\rangle)$. Let $Q = \{v_1, v_2\}$ resolve the vertices of $T_n\langle 1, 2\rangle$. Then we have the following two cases:

Case 1: If i is odd, then we have the following representation of v_i where $i \geq 3$ with respect to Q;

$$r(v_i|Q) = \left(\frac{i-1}{2}, \frac{i-1}{2}\right)$$

As, all the representation of different vertices are distinct. So, this shows that $Q = \{v_1, v_2\}$ resolve the vertices of $T_n\langle 1, 2\rangle$ which means

$$dim(T_n\langle 1, 2\rangle) \leq 2.$$

Case 2: If i is even, then we have the following representation of v_i where $i \geq 3$ with respect to Q;

$$r(v_i|Q) = \left(\frac{i}{2}, \frac{i}{2} - 1\right)$$

As, all the representation of different vertices are distinct. So, this shows that $Q = \{v_1, v_2\}$ resolve the vertices of $T_n\langle 1, 2\rangle$ which means

$$dim(T_n\langle 1, 2\rangle) \leq 2. \quad (1)$$

Conversely: Now, we show that $dim(T_n\langle 1, 2\rangle) \geq 2$. Suppose on contrary that $dim(T_n\langle 1, 2\rangle) = 1$.

As, we know that $dim(G) = 1$ if and only if $G = P_n$ by Theorem 2. This is not possible for our Toeplitz graph.

So,

$$dim(T_n\langle 1, 2\rangle) \geq 2. \quad (2)$$

Hence from Eq. (1) and (2) we have

$$dim(T_n\langle 1, 2\rangle) = 2. \qquad \square$$

Next theorem is about the metric dimension of Toeplitz graph $T_n\langle 1, t\rangle$ where even $t \geq 4$.

Theorem 7: Let $T_n\langle 1, t\rangle$ be the Toeplitz graph with even $t \geq 4$. Then $dim(T_n\langle 1, t\rangle) = 2$, where $n \geq t + 2$.

Proof 7: We will show that only two elements in basis set suffice to resolve all the vertices in $V(T_n\langle 1, t\rangle)$. Let $Q = \{v_1, v_{\frac{t+2}{2}}\}$ resolve the vertices of $T_n\langle 1, t\rangle$. Then we have the following three cases:

Case 1: If $k \equiv 2, 3, \ldots, \frac{t}{2} (\text{mod } t)$, then v_k has the following representation with respect to Q;

$$r(v_k|Q) = \left(k - pt + p - 1, \frac{(2p+1)t - 2k}{2} + p + 1\right)$$

where $p = \lfloor \frac{k}{t} \rfloor$. As, all the representation of different vertices are distinct. So, this shows that $Q = \left\{v_1, v_{\frac{t+2}{2}}\right\}$ resolve the vertices of $T_n\langle 1, t\rangle$ which means

$$dim(T_n\langle 1, t\rangle) \leq 2.$$

Case 2: If $k \equiv \frac{t}{2} + 1 (\text{mod } t)$, then v_k has the following representation with respect to Q;

$$r(v_k|Q) = \left(k - pt + p - 1, \frac{2k - (2p+1)t}{2} + p - 1\right)$$

where $p = \lfloor \frac{k}{t} \rfloor$. As, all the representation of different vertices are distinct. So, this shows that $Q = \left\{v_1, v_{\frac{t+2}{2}}\right\}$ resolve the vertices of $T_n\langle 1, t\rangle$ which means

$$dim(T_n\langle 1, t\rangle) \leq 2.$$

Case 3: If $k \equiv \frac{t}{2} + 2, \frac{t}{2} + 3, \ldots, t + 1 (\text{mod } t)$ then v_k has the following representation with respect to Q;

$$r(v_k|Q) = \left(pt - k + p + 1, \frac{2k - (2p-1)t}{2} + p - 2\right)$$

where $p = \lfloor \frac{2k+t}{2t} \rfloor$. As, all the representation of different vertices are distinct. So, this shows that $Q = \left\{v_1, v_{\frac{t+2}{2}}\right\}$ resolve the vertices of $T_n\langle 1, t\rangle$ which means

$$dim(T_n\langle 1, t\rangle) \leq 2. \quad (3)$$

Conversely: Now, we show that $dim(T_n\langle 1, t\rangle) \geq 2$. Suppose on contrary that $dim(T_n\langle 1, t\rangle) = 1$.

As, we know that $dim(G) = 1$ if and only if $G = P_n$ by Theorem 2. This is not possible for our graph which is Toeplitz graph.

So,

$$dim(T_n\langle 1, t\rangle) \geq 2. \quad (4)$$

Hence from Eq. (3) and (4) we have

$$dim(T_n\langle 1, t\rangle) = 2. \qquad \square$$

Next theorem is about the metric dimension of Toeplitz graph $T_n\langle 1, t\rangle$ when $t = 3$.

Theorem 8: Let $T_n\langle 1, 3\rangle$ be the Toeplitz graph. Then $dim(T_n\langle 1, 3\rangle) = 3$, where $n \geq 5$.

Proof 8: We will show that only three elements in basis set suffice to resolve all the vertices in $V(T_n\langle 1, 3\rangle)$. Let $Q = \{v_1, v_2, v_3\}$ resolve the vertices of $T_n\langle 1, 3\rangle$. Then we have the following two cases:

Case 1: If $k \equiv 1, 2 (\text{mod } 3)$, then v_k has the following representation with respect to Q;

$$r(v_k|Q) = (k - 2p - 1, 4p - k + 2, k - 2p - 1)$$

where $p = \lfloor \frac{k}{3} \rfloor$. As, all the representation of different vertices are distinct. So, this shows that $Q = \{v_1, v_2, v_3\}$ resolve the vertices of $T_n\langle 1, 3\rangle$ which means

$$dim(T_n\langle 1, 3\rangle) \leq 3.$$

Case 2: If $k \equiv 0 (\text{mod } 3)$, then v_k has the following representation with respect to Q;

$$r(v_k|Q) = (p + 1, p, p - 1)$$

where $p = \frac{k}{3}$. As, all the representation of vertices are distinct. So, this shows that $Q = \{v_1, v_2, v_3\}$ resolve the vertices of $T_n\langle 1, 3\rangle$ which means

$$dim(T_n\langle 1, 3\rangle) \leq 3. \quad (5)$$

Conversely: Now, we show that $dim(T_n\langle 1, 3\rangle) \geq 3$. Suppose on contrary that $dim(T_n\langle 1, 3\rangle) = 2$.

There are two possible cases according to n which are given below:

Case 1: When $n = 5, 6$

Following are two possible basis sets by Theorem 3.

If $Q_1 = \{v_i, v_{i+1}\}$ where $1 \leq i \leq n-1$, then the following vertices have the same representation;

$$r(v_{i+2}|Q_1) = r(v_{i+4}|Q_1) = (2, 1)$$

If $Q_2 = \{v_i, v_a\}$ where $1 \leq i \leq n-1$ and $a \equiv p(\text{mod } 3)$ where $p = 1, 2, \ldots, n-3$ and $k = \lfloor \frac{n-p}{3} \rfloor$, then the following vertices have the same representation;

$$r(v_{i+2}|Q_2) = r(v_{i+4}|Q_2) = (2, 1).$$

Our supposition is wrong, the dimension is not 2 because there exist the same representations.

So,

$$dim(T_n\langle 1, 3\rangle) \geq 3.$$

Case 2: When $n \geq 7$

Following are two possible basis sets by Theorem 3.

If $Q_1 = \{v_i, v_{i+1}\}$ where $i = 1, 2$ and $i = n-2, n-1$, then the following vertices have the same representation;

$$r(v_{i+2}|Q_1) = r(v_{i+4}|Q_1) = (2, 1).$$

If $Q_2 = \{v_i, v_a\}$ where $i = 1, 2, 3$ and $a \equiv p(\text{mod } 3)$ where $p = 0, 1, 2$ and $k = \lfloor \frac{n-p}{3} \rfloor$, then the following vertices have the same representation;

$$r(v_{i+2}|Q_2) = r(v_{i+4}|Q_2) = (2, 1).$$

Our supposition is wrong, the dimension is not 2 because there exist same representations.

So,

$$dim(T_n\langle 1, 3\rangle) \geq 3. \tag{6}$$

Hence, from Eq. (5) and (6) we have

$$dim(T_n\langle 1, 3\rangle) = 3. \qquad \square$$

Next theorem is about the metric dimension of Toeplitz graph $T_n\langle 1, t\rangle$ when odd $t \geq 5$.

Theorem 9: Let $T_n\langle 1, t\rangle$ be the Toeplitz graph with odd $t \geq 5$. Then $dim(T_n\langle 1, t\rangle) = 3$, where $n \geq t+2$.

Proof 9: We will show that only three elements in the basis set suffice to resolve all the vertices in $V(T_n\langle 1, t\rangle)$. Let $Q = \{v_1, v_2, v_{\frac{t+3}{2}}\}$ resolve the vertices of $T_n\langle 1, t\rangle$. Then we have the following four cases:

Case 1: If $k \equiv 3, 4, \ldots, \frac{t+1}{2}(\text{mod } t)$, then v_k has the following representation with respect to Q;

$$r(v_k|Q) = \left(k - pt + p - 1, k - pt + p - 2, \frac{(2p+1)t - 2k + 2p + 3}{2}\right)$$

where $p = \lfloor \frac{k}{t} \rfloor$. As, all the representation of different vertices are distinct. So, this shows that $Q = \left\{v_1, v_{\frac{t+3}{2}}\right\}$ resolve the vertices of $T_n\langle 1, t\rangle$ which means

$$dim(T_n\langle 1, t\rangle) \leq 3.$$

Case 2: If $k \equiv \frac{t+3}{2}(\text{mod } t)$, then v_k has the following representation with respect to Q;

$$r(v_k|Q) = \left(k - pt + p - 1, k - pt + p - 2, \frac{2k - (2p+1)t + 2p - 3}{2}\right)$$

where $p = \lfloor \frac{k}{t} \rfloor$. As, all the representation of different vertices are distinct. So, this shows that $Q = \left\{v_1, v_{\frac{t+3}{2}}\right\}$ resolve the vertices of $T_n\langle 1, t\rangle$ which means

$$dim(T_n\langle 1, t\rangle) \leq 3$$

Case 3: If $k \equiv \frac{t+5}{2}, \frac{t+7}{2}, \ldots, t+1(\text{mod } t)$, then v_k has the following representation with respect to Q;

$$r(v_k|Q) = \left(pt - k + p + 1, pt - k + p + 2, \frac{2k - (2p-1)t + 2p - 5}{2}\right)$$

where $p = \lfloor \frac{2k+t}{2t} \rfloor$. As, all the representation of different vertices are distinct. So, this shows that $Q = \left\{v_1, v_{\frac{t+3}{2}}\right\}$ resolve the vertices of $T_n\langle 1, t\rangle$ which means

$$dim(T_n\langle 1, t\rangle) \leq 3.$$

Case 4: If $k \equiv 2(\text{mod } t)$, then v_k has the following representation with respect to Q;

$$r(v_k|Q) = \left(k - pt + p - 1, k - pt + p - 2, \frac{2k - (2p-1)t + 2p - 5}{2}\right)$$

where $p = \lfloor \frac{k}{t} \rfloor$. As, all the representation of different vertices are distinct. So, this shows that $Q = \left\{v_1, v_{\frac{t+3}{2}}\right\}$ resolve the vertices of $T_n\langle 1, t\rangle$ which means

$$dim(T_n\langle 1, t\rangle) \leq 3. \tag{7}$$

Conversely: Now, we show that $dim(T_n\langle 1, t\rangle) \geq 3$. Suppose on contrary that $dim(T_n\langle 1, t\rangle) = 2$. There are two possible cases according to n which are given below:

Case 1: When $t+2 \leq n \leq 2t$.

Following are three possible basis sets by Theorem 3.

If $Q_1 = \{v_i, v_{i+a}\}$ where $1 \leq i \leq n-a$ and $1 \leq a \leq \lfloor \frac{t}{2} \rfloor$, then the following vertices have the same representation;

$$r(v_{i+a+1}|Q_1) = r(v_{t+i+a}|Q_1) = (a+1, 1).$$

If $Q_2 = \{v_i, v_{i+a}\}$ where $i = 1$ and $i = n-a$, and $\frac{t+3}{2} \leq a \leq t-1$, then the following vertices have the same representation;

$$r(v_{i+a-\frac{t-1}{2}}|Q_2) = r(v_{i+a+\frac{t-1}{2}}|Q_2) = \left(a - \frac{t-1}{2}, \frac{t-1}{2}\right).$$

If $Q_3 = \{v_i, v_a\}$ where $1 \leq i \leq n-t$ and $a \equiv p(\text{mod } t)$ where $p = 1, 2, \ldots, n-t$ and $k = \lfloor\frac{n-p}{t}\rfloor$, then the following vertices have the same representation;

$$r(v_{i+\frac{t+1}{2}}|Q_3) = r(v_{i+\frac{3t-1}{2}}|Q_3) = \left(\frac{t+1}{2}, k + \frac{t-3}{2}\right)$$

Our supposition is wrong, the dimension is not 2 because there exist same representations.

So,

$$dim(T_n\langle 1, t\rangle) \geq 3.$$

Case 2: When $n \geq 2t+1$.

Following are four possible basis sets by Theorem 3.

If $Q_1 = \{v_i, v_{i+a}\}$ where $1 \leq i \leq t-a$ and $n-(t-a) \leq i \leq n-a$, and $1 \leq a \leq \lfloor\frac{t}{2}\rfloor$, then the following vertices have the same representation;

$$r(v_{i+a+1}|Q_1) = r(v_{t+i+a}|Q_1) = (a+1, 1).$$

If $Q_2 = \{v_i, v_{i+a}\}$ where $i = t-c+1$ with $1 \leq c \leq a-1$ and $n = 2t+d$ with $1 \leq d \leq a-c$, and $2 \leq a \leq \lfloor\frac{t}{2}\rfloor$, then the following vertices have the same representation;

$$r(v_{i+a+1}|Q_2) = r(v_{t+i+a}|Q_2) = (a+1, 1).$$

If $Q_3 = \{v_i, v_{i+a}\}$ where $i = 1$ and $i = n-a$, and $\frac{t+3}{2} \leq a \leq t-1$, then the following vertices have the same representation;

$$r(v_{i+a-\frac{t-1}{2}}|Q_3) = r(v_{i+a+\frac{t-1}{2}}|Q_3) = \left(a - \frac{t-1}{2}, \frac{t-1}{2}\right).$$

If $Q_4 = \{v_i, v_a\}$ where $1 \leq i \leq t$ and $a \equiv p(\text{mod } t)$ with $p = 0, 1, 2, \ldots, t-1$ and $k = \lfloor\frac{n-p}{t}\rfloor$, then the following vertices have the same representation;

$$r(v_{i+\frac{t+1}{2}}|Q_4) = r(v_{i+\frac{3t-1}{2}}|Q_4) = \left(\frac{t+1}{2}, k + \frac{t-3}{2}\right).$$

Our supposition is wrong, the dimension is not 2 because there exist same representation.

So,

$$dim(T_n\langle 1, t\rangle) \geq 3. \quad (8)$$

Hence from Eq. (7) and (8) we have

$$dim(T_n\langle 1, t\rangle) = 3. \quad \square$$

III. METRIC DIMENSION OF TOEPLITZ GRAPHS $T_n\langle 1, 2, t\rangle$

In this section we study the Toeplitz graph $T_n\langle 1, 2, t\rangle$, where 1, 2 and t are its generators. We find the metric dimension of Toeplitz graph $T_n\langle 1, 2, t\rangle$, where $t = 3, 4, 5, 6$.

Theorem 10: Let $T_n\langle 1, 2, 3\rangle$ be the Toeplitz graph. Then $dim(T_n\langle 1, 2, 3\rangle) = 3$, where $n \geq 5$.

Proof 10: We will show that only three elements in basis set suffice to resolve all the vertices in $V(T_n\langle 1, 2, 3\rangle)$. Let $Q = \{v_1, v_2, v_3\}$ resolve the vertices of $T_n\langle 1, 2, 3\rangle$. Then we have the following two cases:

Case 1: If $k \equiv 1, 2(\text{mod } t)$, then v_k has the following representation with respect to Q;

$$r(v_k|Q) = (k - pt + p, p, p)$$

where $p = \lfloor\frac{k}{t}\rfloor$. As, all the representation of different vertices are distinct. So, this shows that $Q = \{v_1, v_2, v_3\}$ resolve the vertices of $T_n\langle 1, 2, 3\rangle$ which means

$$dim(T_n\langle 1, 2, 3\rangle) \leq 3.$$

Case 2: If $k \equiv 0(\text{mod } t)$, then v_k has the following representation with respect to Q;

$$r(v_k|Q) = (k - pt + p, k - pt + p, p - 1)$$

where $p = \lfloor\frac{k}{t}\rfloor$. As, all the representation of different vertices are distinct. So, this shows that $Q = \{v_1, v_2, v_3\}$ resolve the vertices of $T_n\langle 1, 2, 3\rangle$ which means

$$dim(T_n\langle 1, 2, 3\rangle) \leq 3. \quad (9)$$

Conversely: Now, we show that $dim(T_n\langle 1, 2, 3\rangle) \geq 3$. Suppose on contrary that $dim(T_n\langle 1, 2, 3\rangle) = 2$. There are two possible cases according to n which are given below:

Case 1: For $n \equiv 1(\text{mod } 3)$ and $n = 3k+1$ where $k = \lfloor\frac{n}{3}\rfloor$. In $T_n\langle 1, 2, 3\rangle$ the only possible basis set by Theorem 3 is given:

If $Q = \{v_1, v_n\}$, then the following vertices have the same representation;

$$r(v_2|Q) = r(v_3|Q) = (1, k).$$

Our supposition is wrong, the dimension is not 2 because there exist same representations.

So,

$$dim(T_n\langle 1, 2, 3\rangle) \geq 3.$$

Case 2: For the remaining values of n in $T_n\langle 1, 2, 3\rangle$ the only possible basis set by Theorem 3 is $Q = \{v_1, v_n\}$, but v_1 and v_n does not have the unique shortest path. Our supposition is wrong, the dimension is not 2.

So,

$$dim(T_n\langle 1, 2, 3\rangle) \geq 3. \quad (10)$$

Hence from Eq. (9) and (10) we have

$$dim(T_n\langle 1, 2, 3\rangle) = 3. \quad \square$$

Next theorem is about the metric dimension of Toeplitz graph $T_n\langle 1, s, t\rangle$ when $t = 4$ and $s = 2$.

Theorem 11: Let $T_n\langle 1, 2, 4\rangle$ be the Toeplitz graph. Then $dim(T_n\langle 1, 2, 4\rangle) = 3$, where $n \geq 6$.

Proof 11: We will show that only three elements in basis set suffice to resolve all the vertices in $V(T_n\langle 1, 2, 4\rangle)$. Let $Q = \{v_1, v_2, v_3\}$ resolve the vertices of $T_n\langle 1, 2, 4\rangle$. Then we have the following three cases:

Case 1: If $k \equiv 3 (\text{mod } 4)$, then v_k has the following representation with respect to Q;

$$r(v_k|Q) = (k - 3p - 3, k - 3p - 3, p)$$

where $p = \lfloor \frac{k}{4} \rfloor$. As, all the representation of different vertices are distinct. So, this shows that $Q = \{v_1, v_2, v_3\}$ resolve the vertices of $T_n\langle 1, 2, 4\rangle$ which means

$$dim(T_n\langle 1, 2, 4\rangle) \leq 3.$$

Case 2: If $k \equiv 0, 1 (\text{mod } 4)$, then v_k has the following representation with respect to Q;

$$r(v_k|Q) = (5p - k + 1, k - 3p, p)$$

where $p = \lfloor \frac{2k+4}{8} \rfloor$. As, all the representation of different vertices are distinct. So, this shows that $Q = \{v_1, v_2, v_3\}$ resolve the vertices of $T_n\langle 1, 2, 4\rangle$ which means

$$dim(T_n\langle 1, 2, 4\rangle) \leq 3.$$

Case 3: If $k \equiv 2 (\text{mod } t)$, then v_k has the following representation with respect to Q;

$$r(v_k|Q) = (p + 1, p, p + 1)$$

where $p = \lfloor \frac{k}{4} \rfloor$. As, all the representation of different vertices are distinct. So, this shows that $Q = \{v_1, v_2, v_3\}$ resolve the vertices of $T_n\langle 1, 2, 4\rangle$ which means

$$dim(T_n\langle 1, 2, 4\rangle) \leq 3. \tag{11}$$

Conversely: Now, we show that $dim(T_n\langle 1, 2, 4\rangle) \geq 3$. Suppose on contrary that $dim(T_n\langle 1, 2, 4\rangle) = 2$. There are two possible cases according to n which are given below:

Case 1: For $n \equiv 1 (\text{mod } 4)$ and $n = 4k + 1$ where $k = \lfloor \frac{n}{4} \rfloor$. In $T_n\langle 1, 2, 4\rangle$ the only possible basis set by Theorem 3 is given:

If $Q = \{v_1, v_n\}$, then the following vertices have the same representation;

$$r(v_4|Q) = r(v_6|Q) = (2, k).$$

Our supposition is wrong, the dimension is not 2 because there exist same representation.

So,

$$dim(T_n\langle 1, 2, 4\rangle) \geq 3.$$

Case 2: For the remaining values of n in $T_n\langle 1, 2, 4\rangle$ the only possible basis set by Theorem 3 is $Q = \{v_1, v_n\}$, but v_1 and v_n does not have the unique shortest path. Our supposition is wrong, the dimension is not 2.

So,

$$dim(T_n\langle 1, 2, 4\rangle) \geq 3. \tag{12}$$

Hence, from Eq. (11) and (12) we have

$$dim(T_n\langle 1, 2, 4\rangle) = 3. \qquad \square$$

Next theorem is about the metric dimension of Toeplitz graph $T_n\langle 1, s, t\rangle$ when $t = 5, 6$ and $s = 2$.

Theorem 12: Let $T_n\langle 1, 2, t\rangle$ where $t = 5, 6$ be the Toeplitz graph. Then $dim(T_n\langle 1, 2, t\rangle) = 3$, where $n \geq t + 3$.

Proof 12: We will show that only three elements in basis set suffice to resolve all the vertices in $V(T_n\langle 1, 2, t\rangle)$. Let $Q = \{v_1, v_2, v_{t-1}\}$ resolve the vertices of $T_n\langle 1, 2, t\rangle$. Then we have the following four cases:

Case 1: If $k \equiv 3, \ldots, t - 2 (\text{mod } t)$, then v_k has the following representation with respect to Q;

$$r(v_k|Q) = (k - pt + p - 2, p + 1, p + 1)$$

where $p = \lfloor \frac{k}{t} \rfloor$. As, all the representation of different vertices are distinct. So, this shows that $Q = \{v_1, v_2, v_{t-1}\}$ resolve the vertices of $T_n\langle 1, 2, t\rangle$ which means

$$dim(T_n\langle 1, 2, t\rangle) \leq 3.$$

Case 2: If $k \equiv t - 1 (\text{mod } t)$, then v_k has the following representation with respect to Q;

$$r(v_k|Q) = (p + 2, k - pt + p - 3, p)$$

where $p = \lfloor \frac{k}{t} \rfloor$. As, all the representation of different vertices are distinct. So, this shows that $Q = \{v_1, v_2, v_{t-1}\}$ resolve the vertices of $T_n\langle 1, 2, t\rangle$ which means

$$dim(T_n\langle 1, 2, t\rangle) \leq 3.$$

Case 3: If $k \equiv 0, 1 (\text{mod } t)$, then v_k has the following representation with respect to Q;

$$r(v_k|Q) = (pt - k + p + 1, p + 1, p)$$

where $p = \lfloor \frac{2k+t}{2t} \rfloor$. As, all the representation of different vertices are distinct. So, this shows that $Q = \{v_1, v_2, v_{t-1}\}$ resolve the vertices of $T_n\langle 1, 2, t\rangle$ which means

$$dim(T_n\langle 1, 2, t\rangle) \leq 3.$$

Case 4: If $k \equiv 2 (\text{mod } t)$, then v_k has the following representation with respect to Q;

$$r(v_k|Q) = (p + 1, p, p + 1)$$

where $p = \lfloor \frac{k}{t} \rfloor$. As, all the representation of different vertices are distinct. So, this shows that $Q = \{v_1, v_2, v_{t-1}\}$ resolve the vertices of $T_n\langle 1, 2, t\rangle$ which means

$$dim(T_n\langle 1, 2, t\rangle) \leq 3. \tag{13}$$

Conversely: Now, we show that $dim(T_n\langle 1, 2, t\rangle) \geq 3$. Suppose on contrary that $dim(T_n\langle 1, 2, t\rangle) = 2$ where $t = 5, 6$. There are two possible cases according to n which are given below:

Case 1: For $n \equiv 1 (\mathrm{mod}\ t)$ and $n = tk + 1$ where $k = \lfloor \frac{n}{t} \rfloor$. In $T_n\langle 1, 2, t\rangle$ the only possible basis set by Theorem 3 is given:

If $Q = \{v_1, v_n\}$, then the following vertices have the same representation;

$$r(v_2|Q) = r(v_3|Q) = (1, k + 1).$$

Our supposition is wrong, the dimension is not 2 because there exist same representation.

So,

$$dim(T_n\langle 1, 2, t\rangle) \geq 3.$$

Case 2: For the remaining values of n in $T_n\langle 1, 2, t\rangle$ the only possible basis set by Theorem 3 is $Q = \{v_1, v_n\}$, but v_1 and v_n does not have the unique shortest path. Our supposition is wrong, the dimension is not 2.

So,

$$dim(T_n\langle 1, 2, t\rangle) \geq 3. \quad (14)$$

Hence from Eq. (13) and (14) we have

$$dim(T_n\langle 1, 2, t\rangle) = 3. \quad \square$$

IV. CONCLUSION

We studied the metric dimension of certain families of Toeplitz graphs, $T_n\langle 1, t\rangle$ when t is even and odd and $T_n\langle 1, 2, t\rangle$ when $t = 3, 4, 5, 6$. We have also concluded that these graphs have constant metric dimension.

V. OPEN PROBLEMS

It is natural to ask about the characterization of connected graphs on the bases of the nature of the metric dimension. For the characterization of the Toeplitz graphs the readers are invited to study the following open problems.

1) Is $dim\,(T_n\langle 2, t\rangle)$ for $t \geq 3$ odd constant, bounded or unbounded?
2) Is $dim\,(T_n\langle 1, 2, t\rangle)$ for $t \geq 7$ constant, bounded or unbounded?
3) Is $dim\,(T_n\langle 1, 3, t\rangle)$ for $t \geq 4$ constant, bounded or unbounded?
4) If anyone work on the general result for $dim\,(T_n\langle t_1, t_2, t_3, \ldots, t_k\rangle)$, then it will be an interesting result.

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JIA-BAO LIU received the B.S. degree in mathematics and applied mathematics from Wanxi University, Anhui, China, in 2005, and the M.S. and Ph.D. degrees in mathematics and applied mathematics from Anhui University, Anhui, in 2009 and 2016, respectively. From September 2013 to July 2014, he was a Visiting Researcher with the School of Mathematics, Southeast University, China, where he was a Postdoctoral Fellow, in March 2017. He is currently an Associate Professor with the School of Mathematics and Physics, Anhui Jianzhu University, Hefei, China. He is the author or coauthor of more than 100 journal articles and two edited books. His current research interests include graph theory and its applications, fractional calculus theory, neural networks, and complex dynamical networks. He is a Reviewer of *Mathematical Reviews* and *Zentralblatt-Math*.

MUHAMMAD FAISAL NADEEM received the Ph.D. degree in mathematics from the Abdus Salam School of Mathematical Sciences, Government College University, Lahore, in 2014. He is currently an Assistant Professor with COMSATS University Islamabad, Lahore Campus. His current research interest includes graph theory and its applications. He is a Reviewer of *Mathematical Reviews* and *Zentralblatt-Math*.

HAFIZ MUHAMMAD AFZAL SIDDIQUI received the Ph.D. degree in mathematics from the National University of Science and Technology, Islamabad, in 2015. He is currently an Assistant Professor with the COMSATS University Islamabad, Lahore Campus. His current research interests include graph theory and combinatorics.

WAJIHA NAZIR was born in Lahore, Pakistan. She did the matriculation and intermediate from BISE Lahore. She received the B.S. degree in mathematics from Queen Marry College (P.U.) and the M.S. degree in mathematics from COMSATS University Islamabad, Lahore Campus. She is currently an ESE with the Government Junior Model School, Mochi Gate, Lahore.