RAIRO
INFORMATIQUE THÉORIQUE

DAN A. SIMOVICI

Computing of graphs of relations using generative grammars

RAIRO – Informatique théorique, tome 14, nº 3 (1980), p. 279-299.

<http://www.numdam.org/item?id=ITA_1980__14_3_279_0>

R.A.I.R.O. Informatique théorique/Theoretical Informatics
(vol. 14, n° 3, 1980, p. 279 à 299)

COMPUTING OF GRAPHS OF RELATIONS USING GENERATIVE GRAMMARS (*)

by Dan A. SIMOVICI (1)

Communicated by J.-F. PERROT

Abstract. — *By considering representations of graphs of relations on the set of natural numbers by bounded languages it is possible to develop a study of relations which are computable in a certain sense by context-sensitive grammars. There are obtained closure proprieties for the class of these relations, which allow to prove the context-sensitiveness of certain languages.*

Résumé. — *En représentant les graphes de relations définies sur les nombres entiers par des langages bornés, on peut étudier les relations qui sont calculables (en un certain sens) par des grammaires « context-sensitive ». On obtient des propriétés de fermeture pour cette famille de relations, qui permettent d'établir que certains langages sont « context-sensitive ».*

1. INTRODUCTION

We shall define an encoding of graphs of relations over the set of natural numbers using bounded languages. Thus, it will be possible "to compute" these graphs using generative grammars. Our attention is focused on relations which are computable by context-sensitive grammars. Among other facts we shall obtain several closure properties of the class of bounded context-sensitive languages, which will imply that certain intricate languages are context-sensitive.

Our study is situated in the stream of the study of relations [1, 3] which has been largely developed in the last years. Similar techniques have been used in [1] or [5].

We use, in general, the notations and results from [4] and [6].

(*) Received April 1979, Revised October 1979.
(1) University of Iasi, Department of Mathematics, Iasi, Romania.

R.A.I.R.O. Informatique théorique/Theoretical Informatics, 0399-0540/1980/279/$ 5.00

A relation f from the set M_1 to the set M_2, denoted by $f : M_1 \to M_2$ is a mapping $\hat{f} : \mathscr{P}(M_1) \to \mathscr{P}(M_2)$ [where $\mathscr{P}(M)$ is the power set of M], which is completely additive, i. e.

$$\hat{f}(\bigcup \{ A_j \mid j \in J \}) = \bigcup \{ \hat{f}(A_j) \mid j \in J \}.$$

The values of $\hat{f}$ on singletons, $\hat{f}(\{x\})$, are denoted by $f(x)$.

The graph of f is the set

$$\gamma f = \{ (x, y) \mid x \in M_1, y \in M_2, y \in f(x) \}.$$

If $f : M_1 \to M_2$ and $g : M_2 \to M_3$ are two relations we shall use their product $f \circ g : M_1 \to M_3$, where $z \in f \circ g(x)$ iff there exists $y \in M_2$ such that $y \in f(x)$ and $z \in g(y)$.

In view of the associativity of the Cartesian product many relations can have the same graph. The domain and the codomain of f are dom $f = \{ x \mid x \in M_1, f(x) \neq \emptyset \}$ and codom $f = \bigcup \{ f(x) \mid x \in \text{dom} f \}$.

Let Ob $\mathscr{C}$ and Mor $\mathscr{C}$ be the class of objects and morphisms, respectively of the category $\mathscr{C}$. $\mathscr{C}(X, Y)$ is the set of all morphisms from Mor $\mathscr{C}$ from X to Y.

$\mathscr{R}$ will be the category defined by Ob $\mathscr{R} = \{ N^n \mid n \in N \}$ and $\mathscr{R}(N^r, N^s) = \{ f \mid f$ is a relation, $f : N^r \to N^s \}$. A subcategory $\mathscr{A}$ of $\mathscr{R}$ is *semiadmissible* if Ob $\mathscr{A}$ = Ob $\mathscr{R}$ and:

(i) from $f \in \mathscr{A}(N^r, N^s)$ it follows that $N \times f \in \mathscr{A}(N^{1+r}, N^{1+s})$, where

$$N \times f(p, q) = \{ (p, r) \mid r \in f(q) \}, \qquad \forall p \in N, \quad q \in N^r$$

(the left cylindrification property);

(ii) the functions $\theta_k : N^k \to N^k$ and $\Delta : N \to N^2$ given by $\theta_k(m, n, p) = (n, m, p)$, $\Delta(n) = (n, n)$, $\forall m, n \in N$, $p \in N^{k-2}$ belong to Mor $\mathscr{A}$.

If $\mathscr{A}$ is semiadmissible and the projection Π belongs to $\mathscr{A}(N, N^0)$ then $\mathscr{A}$ is an *admissible* category.

For $n = (n_1, \ldots, n_k) \in N^k$ the sum $\sum \{ n_j \mid 1 \leq j \leq k \}$ will be denoted by $\| n \|$. A relation $f : N^r \to N^s$ is extending (anti-extending) if there exists a positive constant M_f such that

$$\| n \| \leqq M_f \| m \| \, (\| n \| \geqq M_f \| m \|)$$

for every $m \in f(n)$. By Ex and Ax we shall denote the subcategories of $\mathscr{R}$ defined by

$$\text{Ob Ax} = \text{Ob Ex} = \text{Ob } \mathscr{R}$$

and

$$\mathrm{Ex}(N^k, N^h)=\{f \mid f: N^k \to N^h, f \text{ is extending}\},$$
$$\mathrm{Ax}(N^k, N^h)=\{f \mid f: N^k \to N^h, f \text{ is anti-extending}\}.$$

It is easy to see that Ex is a semiadmissible subcategory and Ax an admissible one.

2. CONTEXT-SENSITIVE RELATIONS AND SUBCATEGORIES OF $\mathscr{R}$

Let Ω be the infinite alphabet

$$\Omega=\{x_1, \ldots, x_n, \ldots\} \cup \{y_1, \ldots, y_n, \ldots\}$$

and let us consider the $(m+n)$-uple

$$X=(x_{j_1}, \ldots, x_{j_m}, y_{h_1}, \ldots, y_{h_n}) \in \Omega^{m+n}.$$

If P is an $(m+n)$-uple from N^{m+n}, $P=(p_1, \ldots, p_m, q_1, \ldots, q_n)$ the word $w=x_{j_1}^{p_1} \ldots x_{j_m}^{p_m} y_{h_1}^{q_1} \ldots y_{h_n}^{q_n}$ will be denoted by $w=X^P$. The bounded language $\{X^P \mid P \in A\}$, where A is a subset of N^{m+n}, will be denoted by X^A.

For a relation $f: N^m \to N^n$, L_f will be the language $L_f=\{X^P \mid P \in \gamma f\}$, where X is a standard $(m+n)$-uple $X=(x_1, \ldots, x_m, y_1, \ldots, y_n)$.

If L is a bounded language, $L \subseteq \{X^P \mid P \in N^{m+n}\}$ we shall denote by $\mathscr{R}_L$ the set of relations $\mathscr{R}_L=\{f \mid L_f=L\}$.

A grammar is defined, as usual, as a 4-uple $G=(V_N, V_T, \xi_0, F)$, where V_N and V_T are disjoint alphabets, $\xi_0 \in V_N$ is the initial symbol and F is a finite set of pairs (u, v) such that u is a word over $V=V_N \cup V_T$ containing al least one symbol from V_N and v is a word over V. The symbols from V_N are called nonterminals and those of V_T terminals; if $(u, v) \in F$ then (u, v) is a rule and this pair is written $u \to v$. The relations "$\Rightarrow$" and "$\overset{*}{\Rightarrow}$" are used in their standard sense (*see*, for instance [6]). $L(G)$ is the language generated by the grammar G, $L(G)=\{p \mid \xi_0 \overset{*}{\Rightarrow} p, p \in V_T^*\}$.

Definition 1: The relation f is computed by the grammar G if $L(G)=L_f$. $\mathscr{R}_j$ will be the class of type-j relations i. e. the class of relations which are computable by type-j grammars (using Chomsky's hierarchy), $j \in \{0, 1, 2, 3\}$. The members of $\mathscr{R}_j$ will be termed with the same name as the classes of corresponding grammars. For instance, we shall speak about "context-sensitive relations".

By restricting the morphisms from $\mathcal{R}(N^n, N^m)$ to those belonging to $\mathcal{R}_1$, we do not obtain a subcategory since the composition of two context-sensitive relations is not in general a context-sensitive relation. Moreover, we have:

THEOREM 1: *The class* $\mathcal{R}_0$ *is equal to* $\mathcal{R}_1 \circ \mathcal{R}_1$.

Proof: If $f \in \mathcal{R}(N^r, N^s) \cap \mathcal{R}_0$ the language $L_f = \{ X^P \mid P \in \gamma f \}$ is a type-0 language, where $X = (x_1, \ldots, x_r, y_1, \ldots, y_s)$. Using a well known result (*see* [6], p. 89) there exists a mapping $\varphi : N^{r+s} \to N$ such that the language

$$L_1 = \{ Z^Q \mid Q = (n_1, \ldots, n_r, p_1, \ldots, p_s, 1, \varphi(n_1, \ldots, n_r, p_1, \ldots, p_s)), (n_1, \ldots, n_r, p_1, \ldots, p_s) \in \gamma f \}$$

is context-sensitive, where

$$Z = (x_1, \ldots, x_r, y_1, \ldots, y_s, y_{s+1}, y_{s+2}).$$

Let f_1 be the relation $f_1 : N^r \to N^{s+2}$ belonging to $\mathcal{R}_{L_1}$.

Since the language

$$L = \{ Y^S \mid S = (q_1, \ldots, q_s, 1, q, q_1, \ldots, q_s) \in N^s \times \{1\} \times N^{1+s} \},$$

with $Y = (y_1, \ldots, y_{2s+2})$ is context-sensitive we shall consider the context-sensitive relation $f_2 : N^{s+2} \to N^s$ from $\mathcal{R}_L$. It follows that $f = f_1 \circ f_2$.

Indeed, let $(n_1, \ldots, n_r, q_1, \ldots, q_s) \in \gamma f$. In view of the definitions of f_1 and f_2 it follows that $(n_1, \ldots, n_r, q_1, \ldots, q_s, 1, \varphi(n_1, \ldots, n_r, q_1, \ldots, q_s)) \in \gamma f_1$ and $(q_1, \ldots, q_s, 1, \varphi(n_1, \ldots, n_r, q_1, \ldots, q_s), q_1, \ldots, q_s) \in \gamma f_2$, hence $(n_1, \ldots, n_r, q_1, \ldots, q_s) \in \gamma(f_1 \circ f_2)$.

Conversely, if $(n_1, \ldots, n_r, q_1, \ldots, q_s) \in \gamma(f_1 \circ f_2)$ there exists an $(s+2)$-uple $(m_1, \ldots, m_s, m_{s+1}, m_{s+2})$ such that $(n_1, \ldots, n_{r-1}, m_1, \ldots, m_s, m_{s+1}, m_{s+2}) \in \gamma f_1$ and $(m_1, \ldots, m_{s+1}, m_{s+2}, q_1, \ldots, q_s) \in \gamma f_2$. Using the same definitions of f_1 and f_2 we obtain $(n_1, \ldots, n_r, m_1, \ldots, m_s) \in \gamma f$ and $m_j = q_j$, for $1 \leqq j \leqq s$. Therefore $(n_1, \ldots, n_r, q_1, \ldots, q_s) \in \gamma f$ and we have $\gamma f = \gamma(f_1 \circ f_2)$, hence $f = f_1 \circ f_2$. We have thus obtained the inclusion $\mathcal{R}_0 \subseteq \mathcal{R}_1 \circ \mathcal{R}_1$.

To prove the converse inclusion, let f_1 and f_2 be two context-sensitive relations, $f_1 : N^r \to N^s$, $f_2 : N^s \to N^t$. Since the languages $\{ X^P \mid P \in \gamma f_1 \}$ and $\{ Y^Q \mid Q \in \gamma f_2 \}$, with

$$X = (x_1, \ldots, x_r, y_1, \ldots, y_s), \qquad Y = (y_1, \ldots, y_s, z_1, \ldots, z_s)$$

are context sensitive it follows that the languages

$$L_1 = \{ X^P \mid P \in \gamma f_1 \} \{ z_1 \}^* \ldots \{ z_t \}^*$$

and

$$L_2=\{x_1\}^*\dots\{x_r\}^*\{Y^Q \mid Q\in\gamma f_2\}$$

are also context-sensitive. The class of context-sensitive languages is closed with respect to intersection hence the language

$$L'=L_1\cap L_2=\{T^U \mid U=(n_1, \dots, n_r, m_1, \dots, m_s, p_1, \dots, p_t),$$
$$(n_1, \dots, n_r, m_1, \dots, m_s)\in\gamma f_1, (m_1, \dots, m_s, p_1, \dots, p_t)\in\gamma f_2\}$$

is context-sensitive, where $T=(x_1, \dots, x_r, y_1, \dots, y_s, z_1, \dots, z_t)$.

The image of the language L' under the homomorphism

$$h:\{x_1, \dots, x_r, y_1, \dots, y_s, z_1, \dots, z_t\}^* \rightarrow \{x_1, \dots, x_r, y_1, \dots, y_s, z_1, \dots, z_t\}^*$$

defined by

$$h(u)=\begin{cases} u, & \text{if } u\in\{x_1, \dots, x_r, z_1, \dots, z_t\}, \\ \lambda, & \text{if } u\in\{y_1, \dots, y_s\}, \end{cases} \tag{1}$$

is the language $L_{f_1\circ f_2}=\{(x_1,\dots,x_r,z_1,\dots,z_t)^V \mid V\in\gamma(f_1\circ f_2)\}$ hence $f_1\circ f_2$ is a type-0 relation. Here λ is the null word.

LEMMA 1: *Let $f_1: N^r\rightarrow N^s$ and $f_2: N^s\rightarrow N^t$ be two context-sensitive relations. If either f_1 is anti-extending or f_2 is extending it follows that $f_1\circ f_2$ is a context-sensitive relation.*

Proof: Let T^U be a word from the language L' defined in the previous Theorem. The length of this word, $l(T^U)$ is

$$\|(n_1, \dots, n_r)\|+\|(m_1, \dots, m_s)\|+\|(p_1, \dots, p_t)\|.$$

The mapping h is a linear erasing with respect to L' if there exists an integer $c\geqq 1$ for which $l(T^U)\leqq l(h(T^U))$, i.e.

$$\|(m_1, \dots, m_s)\|\leqq(c-1)(\|(n_1, \dots, n_r)\|+\|(p_1, \dots, p_t)\|).$$

If f_1 is antiextending we have

$$\|(n_1, \dots, n_r)\|\geqq M_{f_1}\|(m_1, \dots, m_s)\|$$

hence

$$\|(m_1, \dots, m_s)\|\leqq[1/M_{f_1}](\|(n_1, \dots, n_r)\|+\|(p_1, \dots, p_t)\|).$$

When f_2 is extending it follows that

$$\|(m_1, \ldots, m_s)\| \leqq M_{f_2} \|(p_1, \ldots, p_t)\|$$
$$\leqq [M_{f_2}](\|(n_1, \ldots, n_r)\| + \|(p_1, \ldots, p_t)\|).$$

Using the fact that the class of context-sensitive languages is closed with respect to linear erasings we obtain the desired assertion. ■

LEMMA 2: *Let f be a context-sensitive relation, $f : N^r \to N^s$. The relation $N \times f$ obtained from f by left cylindrification, $N \times f : N^{1+r} \to N^{1+s}$ is context-sensitive.*

Proof: Since f is context-sensitive the language

$$L_f = \{X^P \mid P = (n_1, \ldots, n_r, m_1, \ldots, m_s), (m_1, \ldots, m_s) \in f(n_1, \ldots, n_r)\}$$

is context-sensitive, where $X = (x_1, \ldots, x_r, y_1, \ldots, y_s)$. Let us consider the substitution φ defined by

$$\varphi(x_p) = \{x, x_p\}, \qquad 1 \leqq p \leqq r$$

and

$$\varphi(y_q) = \{y, y_q\} \qquad \text{for} \quad 1 \leqq q \leqq s.$$

The language $K = \bigcup \{x^n \{x_1, \ldots, x_r\}^* y^n \{y_1, \ldots, y_s\}^* \mid n \geqq 0\}$ is obviously context-sensitive hence so is the language

$$\varphi(L_f) \cap K$$
$$= \{x^n x_1^{n_1} \ldots x_r^{n_r} y^n y_1^{m_1} \ldots y_s^{m_s} \mid n \in N, (m_1, \ldots, m_s) \in f(n_1, \ldots, n_r)\}.$$

Since $L_{N \times f} = \varphi(L_f) \cap K$ it follows that $N \times f$ is a context-sensitive relation. ■

REMARK 1: If $f : N^r \to N^s$ is a context-sensitive relation it is possible to prove in the same manner that the right cylindrification of f,

$$f \times N : N^{r+1} \to N^{s+1},$$

where $f \times N(q, p) = \{(r, p) \mid r \in f(q)\}$, $\forall p \in N$, $q \in N^r$ is also context-sensitive. Moreover, the repeated left and right cylindrification of f: $N^k \times f$ or $f \times N^h$ are also context-sensitive for every $k, h \in N$.

Let $f : N^r \to N^s$, $g : N^t \to N^u$ be two relations. The relation

$$f \times g : N^{r+t} \to N^{s+u}$$

is defined by

$$(m_1, \ldots, m_s, q_1, \ldots, q_u) \in (f \times g)(n_1, \ldots, n_r, p_1, \ldots, p_t)$$

iff $(m_1, \ldots, m_s) \in f(n_1, \ldots, n_r)$, $(q_1, \ldots, q_u) \in g(p_1, \ldots, p_t)$.

THEOREM 2: *If the relations $f: N^r \rightarrow N^s$ and $g: N^t \rightarrow N^u$ are context-sensitive it follows that $f \times g$ is also context-sensitive.*

Proof: Let G_f and G_g be the length increasing grammars,

$$G_f=(V_{Nf}, \{x_1, \ldots, x_r, y_1, \ldots, y_s\}, \xi_{0f}, F_f),$$
$$G_g=(V_{Ng}, \{x'_1, \ldots, x'_t, y'_1, \ldots, y'_u\}, \xi_{0g}, F_g)$$

which generate the languages

$$L_f=\{X^P \mid P \in \gamma f\} \quad \text{and} \quad L_g=\{X'^Q \mid Q \in \gamma g\},$$

respectively, where

$$X=(x_1, \ldots, x_r, y_1, \ldots, y_s) \quad \text{and} \quad X'=(x'_1, \ldots, x'_t, y'_1, \ldots, y'_u).$$

Without restricting the generality we shall suppose $V_{Nf} \cap V_{Ng}=\emptyset$. We have to prove that the language $L_{f \times g}$ is context sensitive. To this end we shall prove that $L_{f \times g}$ is generated by a length-increasing grammar with regular restrictions (*see* [6], pp. 190-191) using the fact that every such grammar generates a context-sensitive language.

Let $\overline{F}_f$ and $\overline{F}_g$ be the sets of rules obtained from F_f and F_g by replacing $y_1, \ldots, y_s$ by $\eta_1, \ldots, \eta_s$ and, respectively $x'_1, \ldots, x'_t$ by $\xi'_1, \ldots, \xi'_t$, where $\eta_1, \ldots, \eta_s$ and $\xi'_1, \ldots, \xi'_t$ are new symbols. Let us consider the grammar

$$G=(V_N, \{x_1, \ldots, x_r, x'_1, \ldots, x'_t, y_1, \ldots, y_s, y'_1, \ldots, y'_u, \xi_0, F).$$

We assume that the rule $u \rightarrow v$ can be applied only to sentential forms belonging to the language

$$\rho(u \rightarrow v) \subseteq (V_N \cup \{x_1, \ldots, x_r, x'_1, \ldots, x'_t, y_1, \ldots, y_s, y'_1, \ldots, y'_u\})^*,$$

which is a regular one.

The set of rules F consists from the following groups of rules:

(i) the initial rule $\xi_0 \rightarrow \xi_{0f}\xi_{0g}$;

(ii) $\eta_j\xi'_h \rightarrow \xi'_h\eta_j$, $1 \leqq j \leqq s$ and $1 \leqq h \leqq t$;

(iii) $\xi'_h \rightarrow x_h$, $1 \leqq h \leqq t$, $\eta_j \rightarrow y_j$, $1 \leqq j \leqq s$ with

$$\rho(\xi'_h \rightarrow x_h)=\rho(\eta_j \rightarrow y_j)=\{x_1\}^* \ldots \{x_r\}^* \{\xi'_1, x_1\}^* \ldots$$
$$\ldots \{\xi'_t, x_t\}^* \{\eta_1, y_1\}^* \ldots \{\eta_s, y_s\}^* \{y'_1\}^* \ldots \{y'_u\}^*;$$

(iv) the rules from $\overline{F}_f$ and $\overline{F}_q$.

When ρ is not explicitely given we assume that

$$\rho(u \rightarrow v)=(V_N \cup \{x_1, \ldots, x_r, x'_1, \ldots, x'_t, y_1, \ldots, y_s, y'_1, \ldots, y'_u\})^*$$

Let us prove that $L(G)=L_{f\times g}$. If $w\in L_{f\times g}$ we have

$$w=x_1^{m_1}\dots x_r^{m_r}\,x_1'^{n_1}\dots x_t'^{n_t}\,y_1^{p_1}\dots y_s^{p_s}\,y_1'^{q_1}\dots y_u'^{q_u},$$

where $(p_1, \dots, p_s)\in f(m_1, \dots, m_r)$, $(q_1, \dots, q_u)\in g(n_1, \dots, n_t)$. The word w can be obtained in G be the following derivation:

$$\xi_0 \underset{(i)}{\overset{*}{\Rightarrow}} \xi_{0f}\xi_{0g} \underset{(ii)}{\overset{*}{\Rightarrow}} x_1^{m_1}\dots x_r^{m_r}\,\eta_1^{p_1}\dots\eta_s^{p_s}\,\xi_1'^{n_1}\dots\xi_t'^{n_t}\,y_1'^{q_1}\dots y_u'^{q_u}$$

$$\underset{(ii)}{\overset{*}{\Rightarrow}} x_1^{m_1}\dots x_r^{m_r}\,\xi_1'^{n_1}\dots\xi_t'^{n_t}\,\eta_1^{p_1}\dots\eta_s^{p_s}\,y_1'^{q_1}\dots y_u'^{q_u}$$

$$\underset{(iii)}{\overset{*}{\Rightarrow}} x_1^{m_1}\dots x_r^{m_r}\,x_t'^{n_t}\dots x_t'^{n_t}\,y_1^{p_1}\dots y_s^{p_s}\,y_1'^{q_1}\dots y_u'^{q_u}=w,$$

hence $w\in L(G)$. The subscripts indicate the group of rules which was used.

The converse inclusion can be obtained by remarking that a derivation $\xi_0 \underset{G}{\overset{*}{\Rightarrow}} w$ is necessarily splitted in the way just indicated. ■

Let us define now the subcategories Ax_1 and Ex_1 of $\mathscr{R}$ by $\mathrm{Ob}\,\mathrm{Ax}_1=\mathrm{Ob}\,\mathrm{Ex}_1=\mathrm{Ob}\,\mathscr{R}$ and

$$\mathrm{Ax}_1(N^n, N^m)=\mathrm{Ax}(N^n, N^m)\cap\mathscr{R}_1,$$
$$\mathrm{Ex}_1(N^n, N^m)=\mathrm{Ex}(N^n, N^m)\cap\mathscr{R}_1.$$

THEOREM 3: Ax_1 *is an admissible and* Ex_1 *a semiadmissible subcategory of* $\mathscr{R}$.

Proof: If $f_1\in\mathrm{Ax}_1(N^r, N^s)$ and $f_2\in\mathrm{Ax}_1(N^s, N^t)$ it follows that $f_1\circ f_2\in\mathrm{Ax}_1(N^r, N^t)$, using Lemma 1. By the same lemma, from $f_1\in\mathrm{Ex}_1(N^r, N^s)$, $f_2\in\mathrm{Ex}_1(N^s, N^t)$ if follows that $f_1\circ f_2\in\mathrm{Ex}_1(N^r, N^t)$.

If $f\in\mathrm{Ax}_1(N^r,N^s)$ [or $\mathrm{Ex}_1(N^r,N^s)$] the left cylindrification of f is clearly anti-extending (or, respectively extending). Using lemma 2 we infer that $N\times f$ is context-sensitive, hence $N\times f\in\mathrm{Ax}_1(N^r, N^s)$ [respectively $N\times f\in\mathrm{Ex}_1(N^r, N^s)$].

It is clear that Δ belongs to both to $\mathrm{Ax}_1(N, N^2)$ and $\mathrm{Ex}_1(N, N^2)$ because the language

$$L_\Delta=\{X^P \mid X=(x_1, y_1, y_2),\ P=(n, n, n),\ n\in N\}$$

is context-sensitive. For θ_k we have

$$L_{\theta_k}=\{(x_1,\dots,x_k,y_1,\dots,y_k)^Q \mid$$
$$Q=(m, n, p_1, \dots, p_{k-2}, n, m, p_1, \dots, p_{k-2})\in N^{2k}\}$$

which is also context-sensitive.

We conclude that Ax_1 and Ex_1 are semiadmissible subcategories. Moreover, since $\Pi \in \mathrm{Ax}_1(N, N^0)$ it follows that Ax_1 is an admissible subcategory. ■

Let now $[n]$ be the set $\{1, \ldots, n\}$ and $f: [p] \to [q]$ be a function. The logical function generated by f is the function $f^{\#}: N^q \to N^p$ defined by

$$f^{\#}(n_1, \ldots, n_q) = (n_{f(1)}, \ldots, n_{f(p)}),$$

for every $(n_1, \ldots, n_q) \in N^q$.

The proposition 1.4.1 from [4] states that if $\mathscr{A}$ id an admissible subcategory of $\mathscr{R}$ and $f: [p] \to [q]$ is a function then $f^{\#} \in \mathscr{A}(N^q, N^r)$. As a consequence we obtain the following.

COROLLARY 1: *For every function* $f: [p] \to [q]$ *the language*

$$L_{f^{\#}} = \{ (x_1, \ldots, x_q, y_1, \ldots, y_p)^K \mid K = (n_1, \ldots, n_q, n_{f(1)}, \ldots, n_{f(p)}), (n_1, \ldots, n_q) \in N^q \}$$

is context-sensitive.

Example 1: Let us consider the function $f: [m] \to [1]$, where $f(j) = 1$, for $1 \leqq j \leqq m$. The language

$$L_{f^{\#}} = \{ x_1^n y_1^n \ldots y_m^n \mid n \in N \}$$

is context-sensitive. For the function $h: [2r] \to [r]$ defined by

$$h(j) = \begin{cases} j, & \text{if } 1 \leqq j \leqq r, \\ j - r, & \text{if } r+1 \leqq j \leqq 2r, \end{cases}$$

we obtain the context-sensitive language

$$L_{h^{\#}} = \{ x_1^{n_1} \ldots x_r^{n_r} y_1^{n_1} \ldots y_r^{n_r} y_{r+1}^{n_1} \ldots y_{2r}^{n_r} \mid (n_1, \ldots, n_r) \in N^r \}.$$

Let $f_1: N^r \to N^j$, $f_2: N^r \to N^h$ be two relations. We shall define the relation $\langle f_1, f_2 \rangle : N^r \to N^{j+h}$ by taking

$$(p_1, \ldots, p_j, q_1, \ldots, q_h) \in \langle f_1, f_2 \rangle (n_1, \ldots, n_r)$$

iff

$$(p_1, \ldots, p_j) \in f_1(n_1, \ldots, n_r) \text{ and } (q_1, \ldots, q_h) \in f_2(n_1, \ldots, n_r).$$

THEOREM 4: *If* $f_1: N^r \to N^j$ *and* $f_2: N^r \to N^h$ *are two context-sensitive relations then* $\langle f_1, f_2 \rangle$ *is also context-sensitive.*

Proof: The function $h^{\#}: N^r \to N^{2r}$ defined in corollary 1 is antiextending with $M_{h^{\#}} = 1/2$. Using theorem 2 it follows that $f_1 \times f_2 : N^{2r} \to N^{j+h}$ is context-sensitive. In view of lemma 1 the relation $h^{\#} \circ (f_1 \times f_2) : N^r \to N^{j+h}$ is context-sensitive. It is easy to see that $\langle f_1, f_2 \rangle = h^{\#} \circ (f_1 \times f_2)$ is context-sensitive. ■

3. CLOSURE PROPRIETIES OF THE CLASSES $\mathcal{R}_0$ AND $\mathcal{R}_1$

For a relation $f: N^r \to N^r$ we shall denote by f^j its j-th power (with respect to relation product) and by f^+ the relation $f^+ : N^r \to N^r$ given by

$$f^+ = \bigcup \{ f^j \mid j \geqq 1 \}.$$

THEOREM 5: *If f is a relation from $\mathcal{R}_0$ then $f^+ \in \mathcal{R}_0$.*

Proof: Suppose that L_f is generated by the grammar

$$G = (I_N, \{ x_1, \ldots, x_r, y_1, \ldots, y_r \}, \xi_0, F).$$

F' is the set of rules which is obtained from F by replacing $y_1, \ldots, y_r$ by r new nonterminals $\eta_1, \ldots, \eta_r$ respectively. We shall consider the grammar

$$G_+ = (I_N \cup \{ \eta_1, \ldots, \eta_r, \eta, \eta', \theta_0, \theta_1 \}, I_T, \theta_0, F_+),$$

where F_+ consists from several groups of rules which we shall present explicitely in the sequel, such that the language generated by G_+ is L_{f^+}.

Let $K = (n_1, \ldots, n_r, m_1, \ldots, m_r) \in \gamma f$. The r-uples $(n_1, \ldots, n_r)$ and $(m_1, \ldots, m_r)$ will be denoted by B and T, respectively. There exists a natural number p, $p \geqq 1$ such that $K \in \gamma(f^p)$, hence there exists $p-1$ r-uples $H_j = (h_1^j, \ldots, h_r^j)$, $1 \leqq j \leqq p-1$ such that

$$K_1 = (n_1, \ldots, n_r, h_1^1, \ldots, h_r^1) \in \gamma f,$$
$$K_2 = (h_1^1, \ldots, h_r^1, h_1^2, \ldots, h_r^2) \in \gamma f,$$
$$\ldots$$
$$K_p = (h_1^{p-1}, \ldots, h_r^{p-1}, m_1, \ldots, m_r) \in \gamma f.$$

The r-uple T will be denoted also by H_p. Since we include F' in F_+ we have in G_+ the derivations

$$\xi_0 \overset{*}{\Rightarrow} (x_1, \ldots, x_r, \eta_1, \ldots, \eta_r)^{K_j}, \quad 1 \leqq j \leqq p,$$

which will be used in the derivation of the word $(x_1, \ldots, x_r, y_1, \ldots, y_r)^K$ in the grammar G_+.

We shall consider also in F_+ the following set of rules:

(i) the set of *initial rules*

$$\{ \theta_0 \to \xi_0, \theta_0 \to \xi_0 \eta \xi_0 \theta_1, \eta_1 \to y_1, \ldots, \eta_r \to y_r \}$$

allows either to generate the word

$$(x_1, \ldots, x_r, y_1, \ldots, y_r)^H \in L(G) = L_f$$

or to produce the derivation

$$\theta_0 \Rightarrow \xi_0\eta\xi_0\theta_1 \overset{*}{\Rightarrow} (x_1, \ldots, x_r, \eta_1, \ldots, \eta_r)^{K_1} \eta(x_1, \ldots, x_r, \eta_1, \ldots, \eta_r)^{K_2}\theta_1.$$

(ii) The *permutation rules* $\eta_j\eta_k \to \eta_k\eta_j$, $1 \leqq j < k \leqq r$ allow to move the η_j's until they are situated near η.

(iii) The group of *elimination rules* $\eta_j\eta x_j \to \eta$, $1 \leqq j \leqq r$ gives us the possibility to eliminate pairwise the symbols $\eta_1, \ldots, \eta_r$, $x_1, \ldots, x_r$. Only after their complete elimination it is possible to apply one of the rules $x_j\eta\eta_k \to \eta'$, $1 \leqq j$, $k \leqq r$ in order to obtain η'.

At this stage we have the derivation

$$\theta_0 \overset{*}{\Rightarrow} (x_1, \ldots, x_r, \eta_1, \ldots, \eta_r)^{K_1} \eta(x_1, \ldots, x_r, \eta_1, \ldots, \eta_r)^{K_2}\theta_1 \overset{*}{\Rightarrow} (x_1, \ldots, x_r)^B \eta'(\eta_1, \ldots, \eta_r)^{H_2}\theta_1.$$

By using the permutation rules $\eta'\eta_j \to \eta_j\eta'$, $1 \leqq j \leqq r$, η' is moved to right until θ_1. Now we can use the rule $\eta'\theta_1 \to \eta\xi_0\theta_1$ and a new derivation process can be initiated using the rules from F'. The derivation is developped as follows

$$\theta_0 \overset{*}{\Rightarrow} (x_1, \ldots, x_r)^B (\eta_1, \ldots, \eta_r)^{H_2} \eta(x_1, \ldots, x_r, \eta_1, \ldots, \eta_r)^{K_3}\theta_1.$$

Using again the elimination rules we have

$$\begin{aligned}
\theta_0 &\overset{*}{\Rightarrow} (x_1, \ldots, x_r)^B \eta(\eta_1, \ldots, \eta_r)^{H_3}\theta_1 \\
&\Rightarrow (x_1, \ldots, x_r)^B \eta'(\eta_1, \ldots, \eta_r)^{H_3}\theta_1 \\
&\overset{*}{\Rightarrow} (x_1, \ldots, x_r)^B (\eta_1, \ldots, \eta_r)^{H_3}\eta'\theta_1 \\
&\Rightarrow (x_1, \ldots, x_r)^B (\eta_1, \ldots, \eta_r)^{H_3}\eta\xi_0\theta_1 \\
&\Rightarrow (x_1, \ldots, x_r)^B (\eta_1, \ldots, \eta_r)^{H_3}\eta(x_1, \ldots, x_r, \eta_1, \ldots, \eta_r)^{K_4}\theta_1 \\
&\overset{*}{\Rightarrow} (x_1, \ldots, x_r)^B (\eta_1, \ldots, \eta_r)^T \eta'\theta_1 \Rightarrow (x_1, \ldots, x_r, \eta_1, \ldots, \eta_r)^{K_4}\theta_1 \\
&\overset{*}{\Rightarrow} (x_1, \ldots, x_r, y_1, \ldots, y_r)^K.
\end{aligned}$$

The last step was realized using the rule $\eta'\theta_1 \to \lambda$. We have thus obtained the inclusion $L_f \subseteqq L(G_+)$. The converse inclusion can be proved by analysing the structure of derivations from G_+. This proof is left to the reader. ■

REMARK 2: If $w=(x_1, \ldots, x_r, y_1, \ldots, y_r)^K$ is a non-empty word from $L(G_+)$ it is easy to see that for the working space of w in the grammar G_+, denoted by $WS_{G_+}(w)$, we have

$$WS_{G_+}(w) \leqq \max_{\{H_1, \ldots, H_p\}} \left\{ \max_{1 \leqq j \leqq p-1} (\|B\|+2\|H_j\|+\|H_{j+1}\|+3) \right\},$$

where $\max_{\{H_1, \ldots, H_p\}}$ is considered over all possible $(r+s)$-uples $H_1, \ldots, H_p$.

THEOREM 6: *If $f: N^r \to N^r$ is an extending (anti-extending) context sensitive relation with $M_f \leqq 1$ (with $M_f \geqq 1$) then so is the relation $f^+: N^r \to N^r$.*

Proof: It is easy to see that if f is extending with $M_f \leqq 1$ or anti-extending with $M_f \geqq 1$ then f^+ enjoys the same property.

If f is extending we have

$$\|B\| \leqq M_f \|H_1\|, \quad \|H_1\| \leqq M_f \|H_2\|, \quad \ldots, \quad \|H_{p-1}\| \leqq M_f \|T\|,$$

hence $\|H_j\| \leqq \|T\|$ for $1 \leqq j \leqq p-1$, since $M_f \leqq 1$. It follows that

$$WS_{G_+}(w) \leqq \|B\|+3\|T\|+3 \leqq 6(\|B\|+\|T\|)=6\,l(w).$$

When f is anti-extending we have

$$\|B\| \geqq M_f \|H_1\|, \|H_1\| \geqq M_f \|H_2\|, \ldots, \|H_{p-1}\| \geqq M_f \|T\|,$$

hence $WS_{G_+}(w) \leqq 4\|B\|+3 \leqq 7(\|B\|+\|T\|)=7\,l(w)$.

It follows that in both cases $L(G_+)$ is context-sensitive (due to the workspace thorem, *see* [6]) and the proof is completed. ■

If for the relation $f: N^r \to N^s$ we consider the set of numbers $S_f=\{\|n\|/\|m\| \mid m \in f(n),\ n \in N^r\}$ the last theorem can be reformulated as follows:

THEOREM 6′: *Suppose that for the context-sensitive relation $f: N^r \to N^r$ we have either $S_f \subseteqq [0, 1]$ or $S_f \subseteqq [1, +\infty)$. Then f^+ is also a context-sensitive relation and S_{f^+} is also included either in $[0, 1]$ or in $[1, +\infty)$, respectively.*

REMARK 3: Let f, g be two relations from $\mathrm{Ax}_1(N^r, N^s)$. Since the class of context-sensitive languages is closed with respect to union and intersection it follows that the relations $k=f \cup g$ and $h=f \cap g$ belong to $\mathrm{Ax}_1(N^r, N^s)$ by taking $M_k=\min(M_f, M_g)$ and $M_h=\max(M_f, M_g)$. A similar argument points that $\mathrm{Ex}_1(N^r, N^2)$ is closed with respect to union and intersection of relations. Let f be a relation $f: N^r \to N^r$.

If δ_r is the diagonal relation, $\delta_r: N^r \to N^r$ we shall define $f^*: N^r \to N^r$ by $f^*=\delta_r \cup f^+$. It is clear that f^* belongs to $\mathrm{Ax}(N^r, N^r)(\mathrm{Ex}(N^r, N^r))$ iff f^+ belongs to $\mathrm{Ax}(N^r, N^r)$ (or $\mathrm{Ex}(N^r, N^r)$, respectively).

The exponential of a function $f: N^r \to N^r$ is the function $f^{\S}: N^{r+1} \to N^r$ given by

$$f^{\S}(n_1, \ldots, n_r, 0) = (n_1, \ldots, n_r),$$
$$f^{\S}(n_1, \ldots, n_r, p+1) = f(f^{\S}(n_1, \ldots, n_r, p)),$$

for every $n_1, \ldots, n_r, p \in N$.

THEOREM 7: *If $k: N^r \to N^r$ is a context-sensitive function for which $S_k \subseteq [0,1)$ or $S_k \subseteq (1, +\infty)$ then $k^{\S}$ is also a context-sensitive function.*

Proof: Denoting as usual $p \dot{-} 1 = \max(p-1, 0)$, $\forall p \in N$, let us define the functions $f: N^{r+1} \to N^r$ and $g: N^{r+1} \to N^{r+1}$ by

$$\gamma f = \{(n_1, \ldots, n_r, 0, n_1, \ldots, n_r) \mid n_1, \ldots, n_r \in N\},$$

$$\gamma g = \{(n_1, \ldots, n_r, n, m_1, \ldots, m_r, n \dot{-} 1) \mid (m_1, \ldots, m_r) = k(n_1, \ldots, n_r)\}.$$

It is easy to see that $k^{\S} = g^* f$ (*see* [4], for instance).

Using our previous results it would be sufficient to prove that $g^* \in \mathrm{Ax}_1(N^{r+1}, N^{r+1})$ or $g^* \in \mathrm{Ex}_1(N^{r+1}, N^{r+1})$ since f is both extending and anti-extending.

If $S_k \subseteq [0, 1)$ it follows that $\|(n_1, \ldots, n_r)\| < \|(m_1, \ldots, m_r)\|$, for every

$$(m_1, \ldots, m_r) = k(n_1, \ldots, n_r),$$

hence $\|(n_1, \ldots, n_r)\| + n \leqq \|(m_1, \ldots, m_r)\| + n - 1$, which implies

$$\frac{\|(n_1, \ldots, n_r, n)\|}{\|(m_1, \ldots, m_r, m)\|} \leqq 1,$$

for every $(m_1, \ldots, m_r, m) \in g(n_1, \ldots, n_r, n)$.

In an analogous manner it is possible to prove that, if $S_k \subseteq (1, +\infty)$ we have

$$\frac{\|(n_1, \ldots, n_r, n)\|}{\|(m_1, \ldots, m_r, m)\|} \geqq 1,$$

for every $(m_1, \ldots, m_r, m) \in g(n_1, \ldots, n_r, n)$, hence in every case we conclude that g is a context-sensitive relation. ■

Example 2: The function $f: N \to N$ given by $f(n) = n+1$, $\forall n \in N$ is context-sensitive since the language $L_f = \{x_1^n \, y_1^{n+1} \mid n \in N\}$ is context-sensitive. Using theorem 7 it follows that $f^{\S}: N^2 \to N$ is context-sensitive because $S_f = \{n/(n+1) \mid n \in N\} \subseteq [0, 1)$. We have $f^{\S}(n, p) = n + p$, $\forall n, p \in \mathrm{N}$. This function will be denoted by φ_S^2. Applying repeatedly the cylindrification we obtain the context-sensitive function

$$\varphi_S^r = (\ldots((\varphi_S^2 \times N) \circ \varphi_S^2) \times N \ldots) \circ \varphi_S^2,$$

where $\varphi_S^r(n_1, \ldots, n_r) = n_1 + \ldots + n_r$, $\forall (n_1, \ldots, n_r) \in N^r$.

Since $\Pi \times N \times N$ is anti-extending and context-sensitive it follows that $g = (\Pi \times N \times N) \circ \varphi_S^2$ is context sensitive where $g(m, n, q) = n + q$, $\forall m, n, q \in N$. If $k : N \to N$ is the null function $k(n) = 0$, $\forall n \in N$, we can define the function $h : N^2 \to N$ by recursion starting from k and g by $h(m, 0) = 0$ and $h(m, n+1) = n + h(m, n)$, $\forall m, n \in N$. It is cleat that $h(m, n) = mn$, $\forall m, n \in N$. This function will be denoted by φ_P^2. It follows that the function

$$\varphi_P^r = (\ldots((\varphi_P^2 \times N) \circ \varphi_P^2) \times N \ldots) \circ \varphi_P^2,$$

where $\varphi_P^r(n_1, \ldots, n_r) = n_1 \ldots n_r$, $\forall (n_1, \ldots, n_r) \in N^r$ is context-sensitive.

Given the functions $f : N^r \to N^s$ and $g : N^{r+1+s} \to N^s$, the function defined by recursion from f and g is the function $h : N^{r+1} \to N^s$ for which $h(p, 0) = f(p)$ and $h(p, n+1) = g(p, n, h(p, n))$, for $p \in N^r$, $n \in N$.

REMARK 4: Let h be the function defined by recursion from f and g. There exists two functions $\psi : N^{r+1} \to N^{r+1+s+1}$ and $\varphi : N^{r+1+s} \to N^{r+1+s}$ such that the function $\Gamma : N^{r+1} \to N^{r+1+s}$ defined by

$$\Gamma(p, m) = (p, m, h(p, m)),$$

for every $p \in N^r$, $m \in N$ is the product $\Gamma = \psi \circ \varphi^{\S}$. Indeed, by taking $\psi(p, m) = (p, 0, f(p), m)$ and $\varphi(p, n, q) = (p, n+1, g(p, n, q))$, for every $p \in N^r$, $q \in N^s$, $m, n \in N$, the last relation can be easily verified by induction on m.

THEOREM 8: *If $f : N^r \to N^s$ and $g : N^{r+1+s} \to N^s$ are two context-sensitive functions and there exists a positive number ε for which $\| g(p, n, q) \| + 1 > \| q \| + \varepsilon$ then the function h defined by recursion from f and g is context-sensitive.*

Proof: In order to prove that the function Γ previously defined is context-sensitive we have to prove only that ψ is context-sensitive and $\varphi^{\S}$ is extensive and context-sensitive.

Since f is context-sensitive the language

$$L_f = \{ (x_1, \ldots, x_r, y_1, \ldots, y_s)^K \mid K = (p_1, \ldots, p_r, q_1, \ldots, q_s) \in \gamma f \}$$

is context sensitive. It is easy to see that the language

$$L_\psi = \{ (x_1, \ldots, x_r, x_{r+1}, y_1, \ldots, y_{r+s+2})^H \mid H = (p_1, \ldots, p_r, m, p_1, \ldots, p_r, 0, q_1, \ldots, q_r, m) \in \gamma^\psi \}$$

is also context-sensitive.

The function φ is extensive. Moreover, the condition from the 5th theorem is satisfied since we have

$$\frac{\|p\|+n+\|q\|}{\|\varphi(p, n, q)\|}=\frac{\|p\|+n+\|q\|}{\|p\|+n+1+\|g(p, n, g)\|}$$
$$=1-\frac{\|g(p, n, q)\|+1-\|q\|}{\|p\|+n+1+\|g(p, n, q)\|}\leqq 1-\varepsilon<1.$$

Since Γ is context-sensitive the language

$$L_\Gamma=\{(x_1, \ldots, x_r, x_{r+1}, y_1, \ldots, y_r, y_{r+1}, y_{r+2}, \ldots, y_{r+s+1})^Q \mid$$
$$Q=(p_1, \ldots, p_r, m, p_1, \ldots, p_r, m, q_1, \ldots, q_r)\in\gamma\Gamma\}$$

is context-sensitive. By considering the homomorphism

$$\chi:\{x_1, \ldots, x_{r+1}, y_1, \ldots, y_{r+s+1}\}^*\to\{x_1, \ldots, x_r, y_1, \ldots, y_{r+s+1}\}^*$$

defined by

$$\chi(z)=\begin{cases}\lambda, & \text{if } z\in\{x_1, \ldots, x_{r+1}\},\\ x_j, & \text{if } z=y_j,\ 1\leqq j\leqq r+1,\\ y_{j-(r+1)}, & \text{if } z=y_j,\ j\geqq r+2,\end{cases}$$

it is easy to see that χ is a linear erasing with respect to L_Γ hence the language

$$\{X^P \mid P=(p_1, \ldots, p_r, m, q_1, \ldots, q_s), (q_1, \ldots, q_s)=h(p_1, \ldots, p_r, m)\}$$

is context-sensitive, where $X=(x_1, \ldots, x_{r+1}, y_1, \ldots, y_s)$. This implies that h is a context-sensitive function. ■

4. ELEMENTARY EXAMPLES OF CONTEXT-SENSITIVE LANGUAGES

We have developed in our previous papers [7, 8] a systematic way of proving that certain classes of languages are included into the class of context-sensitive languages. Now, we shall recapture some of our results and obtain several new examples of context-sensitive languages.

THEOREM 9: *If $f\in \mathrm{Ex}_1(N^r, N^s)$ then the language*

$$C_f=\{Y^M \mid M=(y_1, \ldots, y_s), M\in \mathrm{codom}\, f\}$$

is also context-sensitive. Also, if $f\in \mathrm{Ax}_1(N^r, N^s)$ the language $D_f=\{X^Q \mid X=(x_1, \ldots, x_r), Q\in \mathrm{dom}\, f\}$ is context-sensitive.

Proof: Since f is extending and context-sensitive the language $L_f = \{(x_1, \ldots, x_r, y_1, \ldots, y_s)^P \mid P \in \gamma f\}$ is context-sensitive and there exists $M_f \in R_+$ such that $\|(n_1, \ldots, n_r)\| \leqq M_f \|(m_1, \ldots, m_s)\|$, for every $(n_1, \ldots, n_r) \in \operatorname{dom} f$, $(m_1, \ldots, m_s) \in f(n_1, \ldots, n_r)$. Let us consider the homomorphism $\psi: \{x_1, \ldots, x_r, y_1, \ldots, y_s\}^* \to \{y_1, \ldots, y_s\}^*$ given by

$$\psi(z) = \begin{cases} z, & \text{if } z \in \{x_1, \ldots, x_r\}, \\ \lambda, & \text{if } z \in \{y_1, \ldots, y_s\}. \end{cases}$$

Since

$$\begin{aligned} l((x_1, \ldots, x_r, y_1, \ldots, y_s)^P) &= \|(n_1, \ldots, n_r)\| + \|(m_1, \ldots, m_s)\| \\ &\leqq (M_f + 1)\|(m_1, \ldots, m_s)\| = (M_f + 1)\, l(\psi(x_1, \ldots, x_r, y_1, \ldots, y_s)^P)), \end{aligned}$$

where $P = (n_1, \ldots, n_r, m_1, \ldots, m_s)$, it follows that ψ is an $(M_f + 1)$ linear erasing with respect to L_f. Hence $C_f = \psi(L_f)$ is also context-sensitive.

For D_f (when f is context-sensitive and anti-extending) the proof is similar. ■

LEMMA 3: *The function* $g: N^2 \to N$ *defined by* $g(m, n) = m \dot{-} n$ *for every* $m, n \in N$, *belongs to* $\mathrm{Ax}_1(N^2, N)$, where $m \dot{-} n = \max(m - n, 0)$, $\forall m, n \in N$.

Proof: The language which represents the graph of g is

$$L_g = \{x_1^m x_2^n \mid m, n \in N, m \leqq n\} \cup \{x_1^m x_2^n y_1^{m-n} \mid m, n \in N, m > n\}.$$

Since the language $\{x_1^m x_2^n \mid m, n \in N, m \leqq n\}$ is obviously context-sensitive we have to prove only that the language

$$E = \{x_1^m x_2^n y_1^{m-n} \mid m, n \in N, m > n\}$$

is context-sensitive. To this end let us consider the substitution φ defined by $\varphi(x_1) = \{x_1\}$ and $\varphi(x) = \{x_2, y\}$ and the context-sensitive language $L_0 = \{x_1^m x_2^m \mid m \in N\}$. It is easy to see that we have $E = \varphi(L_0) \cap \{x_1\}^* \{x_2\}^* \{y_1\}^*$, hence L_g is context-sensitive. It is clear that g is anti-extending. ■

THEOREM 10: *The distance function* $d_1: N^2 \to N$ *defined by* $d_1(m, n) = |m - n|$, $\forall m, n \in N$ *is context-sensitive and anti-extending.*

Proof: Let us consider the function f: $[4] \to [2]$ given by $f(1) = f(4) = 1$ and $f(2) = f(3) = 2$ and the sum-function $\varphi_S^2: N^2 \to N$, where $\varphi_S^2(m, n) = m + n$. We have $d_1(m, n) = g(m, n) + g(n, m)$, hence we can write $d_1 = \varphi_S^2 \circ (g \times g) \circ f^\#$. Since φ_S^2, $g \times g$ and f belong to Mor Ax_1 it follows that d_1 is context-sensitive and anti-extending. ■

Let d_r: $N^{2r} \to N$ be the distance function given by

$$d_r(m_1, \ldots, m_r, n_1, \ldots, n_r) = |m_1 - n_1| + \ldots + |m_r - n_r|.$$

COROLLARY 2: *The function d_r is context-sensitive and anti-extending.*

Proof: It is clear that $d_r = \varphi_s^r \circ (d_1 \times \ldots \times d_1) \circ h^{\#}$, where h: $[2r] \to [r]$ is defined by

$$h(j) = \begin{cases} (j+1)/2, & \text{when } j \text{ is odd}, \\ r + j/2, & \text{when } j \text{ is even}, \end{cases}$$

for all j, $1 \leqq j \leqq 2r$. According to the theorem 10 it follows that d_r is context-sensitive. ■

Example 3: Let $g_0, g_1 \in \mathrm{Ax}_1(N^s, N^r)$ be two relations and let $sg : N \to N$ be the function given by $sg(0) = 0$ and $sg(n) = 1$, for $n \geqq 1$. Since d_r and sg are anti-extending it follows that the relation $g = \langle g_0, g_1 \rangle \circ d_r \circ sg$ is context-sensitive and anti-extending. If f: $N^s \to N$ is an anti-extending context-sensitive relation it follows that the relation $f_{\neq}$ given by

$$f_{\neq}(p) = \begin{cases} f(p), & \text{if } g_0(p) \neq g_1(p), \\ 0, & \text{if } g_0(p) = g_1(p), \end{cases}$$

is context-sensitive since $f_{\neq} = \langle g, f \rangle \circ \varphi_P^2$. This is the context-sensitive « analogue » of the conditional definition of recursive functions.

LEMMA 4: *If $f: N^{r+1} \to N$ is a context-sensitive function then the function $f_k: N^r \to N$ given by $f_k(p_1, \ldots, p_r) = f(p_1, \ldots, p_r, k)$, for every $(p_1, \ldots, p_r) \in N^r$ is context-sensitive.*

Proof: It is easy to see that $L_{f_k} = L_f \cap \{x_1, \ldots, x_r\}^* x_{r+1}^k \{y_1, \ldots, y_s\}^*$ is context-sensitive since the class of context-sensitive languages is closed with respect to intersection. ■

REMARK 5: The same result holds by fixing any argument of f.

Theorem 11: *Let f: $N^{r+1} \to N$ be a context-sensitive function and consider the functions G: $N^{r+1} \to N$ and H: $N^{r+1} \to N$ defined by*

$$G(p, n) = \sum \{ f(p, m) \mid 0 \leqq m \leqq n \},$$
$$H(p, n) = \Pi \{ f(p, m) \mid 0 \leqq m \leqq n \}, \qquad \forall p \in N^r, \quad m \in N.$$

G is a context-sensitive function. If, for $n \geqq 1$ we have $f(p, n) \geqq 1$, $\forall p \in N^r$ it follows that H is also context-sensitive.

Proof: Since the function f is context-sensitive, using lemma 4, it follows that the function f_0, where $f_0(p)=f(p,0)$ is also context-sensitive. For G and H we can write

$$G(p, 0)=H(p, 0)=f_0(p)$$

and $G(p, n+1)=\varphi_S^2(f(p, n+1), G(p, n))$, $H(p, n+1)=\varphi_P^2(f(p, n+1), H(p, n))$. Let us consider the functions g, h: $N^{r+1+1} \to N$ by $g(p, n, q)=f(p, n+1)+q$, $h(p,n,q)=f(p,n+1)\,q$, $\forall\, p \in N^r, n, q \in N$. Since G and H are defined by recursion starting from f_0, g, and, respectively from f_0, h it follows that G and H are context-sensitive, using the 6th theorem. ■

Example 4: The function f: $N^2 \to N$ defined by $f(m, n)=(m+1)^{n+1}$, $\forall\, m, n \in N$ is context-sensitive. Indeed, we have

$$f(m, 0)=m+1,$$
$$f(m, n+1)=\varphi_P^2(m+1, f(m, n)),$$

for every $m, n \in N$. Thus f is defined by recursion from the functions h: $N \to N$, g: $N^3 \to N$, where $h(m)=m+1$, $g(m, n, q)=(m+1)\,q$, $\forall\, m, n, q \in N$. It is clear that $\| g(m, n, q) \| + 1 \geqq \| q \| + 1$, hence we can apply the 8th theorem. It follows that f is context sensitive.

By applying lemma 4 we obtain the context-sensitive functions f_1: $N \to N$, $f_1(n)=a^{n+1}$ and f_2: $N \to N$, $f_2(m)=(m+1)^b$, $\forall\, m, n \in N$, where $a, b \in N$, $a, b \geqq 1$.

Since L_{f_1} and L_{f_2} are context-sensitive languages so are $x_1 L_{f_1}$ and $x_1 L_{f_2}$, where L_{f_1}, L_{f_2} are subsets of $\{x_1, y_1\}^*$. We infer that the functions f'_1: $N \to N$, f'_2: $N \to N$, where $f'(n)=a^n$, $f'(m)=m^b$ for m, $n \geqq 1$ are context-sensitive functions. By adding to the graphs of f'_1 and f'_2 the pairs (0, 1) and (0, 0), respectively, we obtain the context-sensitive functions $\exp_a$: $N \to N$, $\exp_a(n)=a^n$ and pow_b: $N \to N$, $\mathrm{pow}_b(m)=m^b$, for $m, n \in N$.

Example 4: The linear function lin_a: $N \to N$, $\mathrm{lin}_a(n)=an$, $\forall\, n \in$ is context-sensitive since the language describing its graph is $\{x_1^n y_1^{an} \mid n \in N\}$. Every polynomial with integer non-negative coefficients is a context-sensitive function. Indeed, let P be the polynomial $P(n)=a_1 n^{j_1} + \ldots + a_m n^{j_m}$. We have

$$P=f\# \circ [(\mathrm{pow}_{j_1} \circ \mathrm{lin}_{a_1}) \times \ldots \times (\mathrm{pow}_{j_m} \circ \mathrm{lin}_{a_m})] \circ \varphi_S^m,$$

where $f^\#$ is the logical function considered in example 1, hence P is context-sensitive.

Since P is an extending relation it follows that the language $\{y^{P(n)} \mid n \in N\}$ is context-sensitive (*see* [7]).

THEOREM 12: *Let* $f: N \to N$, $g: N \to N$ *be two context-sensitive functions for which* $\lim_{n \to \infty} g(n)/f(n) = +\infty$, *The function* $g \dot{-} f$ *defined by* $(g \dot{-} f)(n) = g(n) \dot{-} f(n)$, $\forall n \in N$, *is context-sensitive.*

Proof: Since f and g are context-sensitive so is the function $\langle f, g \rangle$ hence the language $L = \{ x_1^n \; y_1^{f(n)} \; y_2^{g(n)} \mid n \in N \}$ is context-sensitive. Let h_1 be the substitution defined by $h_1(x_1) = \{ x_1 \}$, $h_1(y_1) = \{ y_1 \}$ and $h_1(y_2) = \{ y_2, y \}$. The language

$$L' = h_1(L) \cap \{ x_1^n \; y_1^m \; y_2^k \; y^m \mid m, k, n \in N \}$$

is context-sensitive and we have

$$L' = \{ x_1^n \; y_1^{f(n)} \; y_2^{g(n) - f(n)} y^{f(n)} \mid g(n) \geqq f(n), \; n \in N \}.$$

The homomorphism h_2 given by $h_2(x_1) = x_1$, $h_2(y_2) = y_2$ and $h_2(y_1) = h_2(y) = e$ is a 2-linear erasing with respect to L'. Indeed, let $w' \in L'$ be

$$w' = x_1^n \; y_1^{f(n)} \; y_2^{g(n) - f(n)} y^{f(n)}$$

and $h_2(w) = x_1^n \; y_2^{g(n) - f(n)}$. Since $\lim_{n \to \infty} g(n)/f(n) = \infty$ there exists an integer n_3 such that, if $n > n_3$ it follows $g(n) > 3f(n)$. Thus, we have $l(w') \leqq 2\, l(h_2(w))$; h_2 is a 2-linear erasing and the language

$$L'' = h_2(L_2) = \{ x_1^n \; y_2^{g(n) - f(n)} \mid g(n) \geqq f(n), \; n \in N \}$$

is context-sensitive. The language L_{g-f} can be thus obtained by adding to L' a finite language, hence $g \dot{-} f$ is a context-sensitive function.

We obtain an extension of the aforementioned result concerning polynomials in the following.

COROLLARY 3: *Every polynomial P with integer coefficients, whose dominant coefficient is positive defines a context-sensitive function* $f: N \to N$, *where* $f(n) = \max \{ P(n), 0 \}$, $\forall n \in N$.

Proof: Every polynomial which satisfies the conditions of this corollary can be written as $P(n) = K(n) - H(n)$, where K and H are polynomials whose degrees are p and q, respectively $(p > q)$ having integer non-negative coefficients. Therefore, we have $\lim_{n \to \infty} (K(n)/H(n) = \infty$. Since K and H are context-sensitive functions, using theorem 12 the function $K \dot{-} H = \max (P(n), 0)$ is context-sensitive. ■

Let $f: N \to N$ be an increasing function.

DEFINITION 2: The completion of f is the relation $\tilde{f}$: $N \to N$ defined by $\gamma \tilde{f} = \{ (n, m) \mid f(n) \leqq m < f(n+1) \}$.

THEOREM 13: *The completion of an increasing context-sensitive function f for which there exists a positive integer k such that for every $n \in N$, $f(n+1)/f(n) \leqq k$ is a context-sensitive relation.*

Proof: The relation $\sigma = \{ (n, n+1) \mid n \in N, \ n \geqq 1 \}$ is anti-extending with $M_\sigma = 1/2$, hence $\sigma \circ f$ is context-sensitive. It follows (using an argument similar to the first part of the proof of theorem 12) that the language

$$L = \{ x_1^n y_1^{f(n)} y_2^{f(n+1)-f(n)} y^{f(n)} \mid n \in N \},$$

is context-sensitive. The homomorphism h defined by $h(z) = z$ if $z \in \{ x_1, y_1, y_2 \}$ and $h(y) = e$ is a 2-linear erasing with respect to L since f is increasing. Therefore the language

$$L' = h(L) = \{ x_1^n y_1^{f(n)} y_2^{f(n+1)-f(n)} \mid n \in N \}$$

is context-sensitive.

Let us consider the substitution ψ, where $\psi(x_1) = \{x_1\}$, $\psi(y_1) = \{y_1\}$ and $\psi(y_2) = \{y_1, y_2\}$. We obtain the context-sensitive language

$$L'' = \psi(L') \cap \{x\}^* \{y_1\}^* \{y_2\}^+$$

and we have

$$L'' = \{ x_1^n y_1^{f(n)+j} y_2^{(f(n+1)-f(n))-j} \mid 0 \leqq j < f(n+1) - f(n), \ n \in N \}.$$

Using the homomorphism h' with $h'(x_1) = x_1$, $h'(y_1) = y_1$ and $h'(y_2) = e$, which is a k-linear erasing for L'' we have the language

$$L_1 = \{ x_1^n y_1^{f(n)+j} \mid 0 \leqq j < f(n+1) - f(n), \ n \in N \},$$

which is context-sensitive and represents the graph of $\tilde{f}$. Therefore the completition $\tilde{f}$ is context-sensitive. ■

REMARK 6: It is clear that the codomain of the completion of a context-sensitive function which satisfies the conditions of theorem 13 coincides with the set $\{ m \mid m \geqq f(0) \}$.

Let g be the function g: codom $\tilde{f} \to N$ defined by $g(m) = n$ if n is the unique number for which $f(n) \leqq m < f(n+1)$. Since the language $\{ x_1^n y_1^m \mid m, \ n \in N, (n, m) \in \gamma \tilde{f} \}$ is context-sensitive it follows that the language $L_g = \{ x_1^m y_1^{g(m)} \mid m \geqq 0 \}$ is context-sensitive.

Example 5: It is clear that the function pow_p with $p \geqq 2$ satisfies the conditions from theorem 13 and $\text{pow}_p(0)=0$, hence the language $\{ x_1^m y_1^{[\sqrt[p]{m}]} \mid m \in N \}$ is context-sensitive. The same conditions are fulfilled by $\exp_a$, with $a \geqq 2$ hence the language $\{ x_1^m y^{[\log_a m]} \mid m \in N \}$ is context-sensitive.

The results obtained in the second section of this paper give the possibility to prove in a rather simple manner that certain complicate languages are context-sensitive. For instance, by applying theorem 6 to the function $\exp_2$ it follows that the language

$$\{ i^{2^{\cdot^{\cdot^{\cdot^{2^n}}}}} {}^{\}m} \mid m, n \in N \},$$

is context-sensitive.

In a next Note we shall discuss the connection between the class of context-sensitive functions and certain subrecursive hierarchies (*see* [2]).

REFERENCES

1. J. Berstel, *Une hiérachie des parties rationnelles de* N^2, Mathematical Systems Theory, Vol. 7, 1973, pp. 114-137.
2. W. S. Brainerd, L. H. Landweber, *Theory of Computation*, J. Wiley and Sons, 1974.
3. S. Eilenberg, *Automata, Languages and Machines*, Vol. A, Academic Press, 1974.
4. S. Eilenberg, C. C. Elgot, *Recursiveness*, Academic Press, 1970.
5. M. Latteux, *Intersection de langages algébriques bornés*, Acta Informatica, Vol. 11, 1979, pp. 233-240.
6. A. Salomaa, *Formal Languages*, Academic Press, 1973.
7. D. Simovici, S. Istrail, *Computing Grammars and Context-Sensitive Languages*, R.A.I.R.O., Informatique théorique, Vol. 12, 1978, pp. 33-48.
8. D. Simovici, *Computing with Scattered Context Grammars with Regular Restrictions*, Foundations of Control Engineering, Vol. 4, 1979, pp. 177-190.