Computing the Domination Number of Grid Graphs

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Abstract

Let $\gamma_{m,n}$ denote the size of a minimum dominating set in the $m \times n$ grid graph. For the square grid graph, exact values for $\gamma_{n,n}$ have earlier been published for $n \leqslant 19$. By using a dynamic programming algorithm, the values of $\gamma_{m,n}$ for $m, n \leqslant 29$ are here obtained. Minimum dominating sets for square grid graphs up to size 29×29 are depicted.

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1 Introduction

An $m \times n$ *grid graph* G has the vertex set $V = \{v_{i,j} : 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$ with two vertices $v_{i,j}$ and $v_{i',j'}$ being adjacent if $i = i'$ and $|j - j'| = 1$ or if $j = j'$ and $|i - i'| = 1$. The $m \times n$ grid graph can also be presented as a Cartesian product $P_m \square P_n$ of a path of length $m - 1$ and a path of length $n - 1$.

A *dominating set* of a graph $G = (V, E)$ is a subset $V' \subseteq V$ such that every vertex not in V' is adjacent to at least one vertex in V'. The *domination number* $\gamma(G)$ of a graph G is the cardinality of a smallest dominating set. When G is the $m \times n$ grid graph, we denote the domination number by $\gamma_{m,n} = \gamma(G)$.

The domination number of grid graphs has been studied since the 1980s. For the general case, efforts have been made to obtain lower and upper bounds on $\gamma_{m,n}$. Studies of general bounds on $\gamma_{m,n}$ include [2, 4, 6, 7, 10]. Several studies have also been carried out on specific bounds for small values of (either one or both of) the parameters.

Jacobson and Kinch [16] established $\gamma_{m,n}$ for $1 \leqslant m \leqslant 4$ and all n. This work was later extended to the cases of $m = 5, 6$ and all n by Chang and Clark [3]. Hare [11] used a computational approach to determine $\gamma_{m,n}$ for $m = 7, 8$, $n \leqslant 500$; $m = 9$, $n \leqslant 233$; and $m = 10$, $n \leqslant 125$. Some of the early results were later confirmed in [18]. In the 1990s Fisher developed a new method for calculating domination numbers for grid graphs. This work remained unpublished but is described in Spalding's PhD thesis [21], where the values of $\gamma_{m,n}$ for $m \leqslant 19$ and all n are given. We summarize these results in Figure 1.

The values of $\gamma_{n,n}$ are at the moment of writing recorded for $n \leqslant 14$ in the On-Line Encyclopedia Integer Sequences (OEIS) [20] as sequence A104519. However, it follows from the discussion above that the range of settled cases of $\gamma_{n,n}$ is actually $n \leqslant 19$. In the current work, this range will be extended to $n \leqslant 29$.

The explanation of the sequence A104519 in OEIS is that it is the smallest number of cells in an $n \times n$ array that need to be occupied to make it impossible to add an X-pentomino to the array that does not intersect the occupied cells. Indeed, there is a direct correspondence between this formulation of the problem and minimum dominating sets of the $(n-2) \times (n-2)$ grid via the correspondence between an X-pentomino and the vertices dominated by a vertex in a grid graph.

In this study the problem of determining $\gamma_{n,m}$ for as large parameters as possible will be attacked by a dynamic programming algorithm. Algorithms based on dynamic programming have also earlier been developed for this problem [11, 12, 14, 17] (algorithms have earlier been studied also in, for example, [18, 19, 22]).

In Section 2 we give some definitions and theorems that are necessary in the development of the new algorithm, which is presented in Section 3. Using this algorithm we have calculated $\gamma_{m,n}$ for the cases $m \leqslant 27$, $n \leqslant 1000$ and for $m = n = 28$ and $m = n = 29$. The values of $\gamma_{m,n}$ for $m, n \leqslant 29$ are tabulated in Section 4, where minimum dominating sets for $\gamma_{n,n}$, $n \leqslant 29$ are also depicted.

$$\gamma_{1,n} = \left\lfloor \frac{n+2}{3} \right\rfloor$$

$$\gamma_{2,n} = \left\lfloor \frac{n+2}{2} \right\rfloor$$

$$\gamma_{3,n} = \left\lfloor \frac{3n+4}{4} \right\rfloor$$

$$\gamma_{4,n} = \begin{cases} n+1, & \text{if } n = 5, 6, 9 \\ n, & \text{otherwise} \end{cases}$$

$$\gamma_{5,n} = \begin{cases} \left\lfloor \frac{6n+6}{5} \right\rfloor, & \text{if } n = 7 \\ \left\lfloor \frac{6n+8}{5} \right\rfloor, & \text{otherwise} \end{cases}$$

$$\gamma_{6,n} = \begin{cases} \left\lfloor \frac{10n+10}{7} \right\rfloor, & \text{if } n \equiv 1 \pmod 7 \\ \left\lfloor \frac{10n+12}{7} \right\rfloor, & \text{otherwise} \end{cases}$$

$$\gamma_{7,n} = \left\lfloor \frac{5n+3}{3} \right\rfloor$$

$$\gamma_{8,n} = \left\lfloor \frac{15n+14}{8} \right\rfloor$$

$$\gamma_{9,n} = \left\lfloor \frac{23n+20}{11} \right\rfloor$$

$$\gamma_{10,n} = \begin{cases} \left\lfloor \frac{30n+37}{13} \right\rfloor, & \text{if } n \neq 13, 16 \text{ or } n \equiv 0, 3 \pmod{13} \\ \left\lfloor \frac{30n+24}{13} \right\rfloor, & \text{otherwise} \end{cases}$$

$$\gamma_{11,n} = \begin{cases} \left\lfloor \frac{38n+21}{15} \right\rfloor, & \text{if } n = 11, 18, 20, 22, 33 \\ \left\lfloor \frac{38n+36}{15} \right\rfloor, & \text{otherwise} \end{cases}$$

$$\gamma_{12,n} = \left\lfloor \frac{80n+66}{29} \right\rfloor$$

$$\gamma_{13,n} = \begin{cases} \left\lfloor \frac{98n+111}{33} \right\rfloor, & \text{if } n \equiv 14, 15, 17, 20 \pmod{33} \\ \left\lfloor \frac{98n+78}{33} \right\rfloor, & \text{otherwise} \end{cases}$$

$$\gamma_{14,n} = \begin{cases} \left\lfloor \frac{35n+40}{11} \right\rfloor, & \text{if } n \equiv 18 \pmod{22} \\ \left\lfloor \frac{35n+29}{11} \right\rfloor, & \text{otherwise} \end{cases}$$

$$\gamma_{15,n} = \begin{cases} \left\lfloor \frac{44n+27}{13} \right\rfloor, & \text{if } n \equiv 5 \pmod{26} \\ \left\lfloor \frac{44n+40}{13} \right\rfloor, & \text{otherwise} \end{cases}$$

$$\gamma_{m,n} = \left\lfloor \frac{(m+2)(n+2)}{5} \right\rfloor, \quad \text{for } 16 \leqslant m \leqslant 19$$

Figure 1: Known formulas for $\gamma_{m,n}$, $m \leqslant n$

2 Preliminaries

To simplify the description of the algorithm, we first define an order of the vertices of an $m \times n$ grid graph with vertices $v_{i,j}$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, as defined in the Introduction. This notation gives a lexicographic order of the vertices, where $v_{i,j}$ is smaller than $v_{k,l}$ if $i < k$ or if $i = k$ and $j < l$.

We will next introduce some notations that are useful in the sequel.

Definition 2.1. *Consider a grid graph $G = (V, E)$.*

- *For a vertex $v \in V$, the set of vertices dominated by v is denoted by $D(v)$.*
- *For a set $V' \subseteq V$, the set of vertices dominated by (the vertices in) V' is denoted by $D(V')$. In other words, $D(V') = \cup_{v \in V'} D(v)$.*
- *For a set $S \subseteq V$, the lexicographically smallest vertex in $V \setminus S$ is denoted by $s(S)$.*

We are now ready to present the theorems that will help us in developing the algorithm. There are many similarities between our approach and that in [12] but also many differences, so a detailed treatment of the details is required.

Theorem 2.1. *Every minimum dominating set can be constructed by an exhaustive search where in each step any undominated vertex is picked, after which all possible ways of dominating this vertex are considered in turn.*

Proof. Every vertex must be dominated. The order in which vertices are added to the dominating set is irrelevant. □

As we can pick the vertices to be dominated in any order, we have chosen to always consider the lexicographically smallest undominated vertex. Each vertex can be dominated in five different ways. We shall now show that it is not necessary to consider all five possibilities in the exhaustive algorithm. We need the following observation in the proofs, cf. the concept of *beatable dominating sets* in [12].

Theorem 2.2. *Consider an $m \times n$ grid graph $G = (V, E)$, and let $V_1 \subseteq V$ and $V_2 \subseteq V$ such that $|V_1| = |V_2|$ and $D(V_1) \subseteq D(V_2)$. To find a minimum dominating set of G, one may ignore V_1 and only consider dominating sets that extend V_2.*

Proof. For any dominating set $V_1 \cup V_3$ of G, $V_2 \cup V_3$ is a dominating set. Hence, as $|V_1| = |V_2|$, it suffices to consider V_2 in the search for a minimum dominating set. □

In the subsequent theorems, we focus on sets S of *dominated* vertices rather than sets of *dominating* vertices. Recall that $s(S)$ denotes the lexicographically smallest undominated vertex (Definition 2.1).

Theorem 2.3. *Consider an $m \times n$ grid graph $G = (V, E)$ and let $S \subseteq V$. When considering vertices for dominating $v_{i,j} = s(S)$, the candidates $v_{i-1,j}$ and $v_{i,j-1}$ can be ignored (whenever such vertices exist, that is, $i \geqslant 2$ and $j \geqslant 2$, respectively).*

Proof. By the definition of $s(S)$, $v_{i,j}$ is the only undominated vertex that can be dominated by $v_{i-1,j}$. Similarly, the only undominated vertices that can be dominated by $v_{i,j-1}$ (assuming $j \geqslant 2$) are $v_{i,j}$ and $v_{i+1,j-1}$ (if $i \leqslant m-1$). However, when $i \leqslant m-1$, $v_{i+1,j}$ dominates the same vertices, and when $i = m$, $v_{i,j+1}$ (or $v_{i,j}$, if $j = n$) dominates them. The result now follows from Theorem 2.2. □

Further reductions of candidates are possible in special cases.

Theorem 2.4. *Consider an $m \times n$ grid graph $G = (V, E)$ and let $S \subseteq V$. When considering vertices for dominating $v_{i,j} = s(S)$ when $v_{i,j+1} \in S$, $j \leqslant n-1$, the candidate $v_{i,j}$ can be ignored.*

Proof. If $v_{i,j+1} \in S$, the only undominated vertices that can be dominated by $v_{i,j}$ are $v_{i,j}$ and, if $i \leqslant m-1$, $v_{i+1,j}$. For $i \leqslant m-1$, $v_{i+1,j}$ dominates both of these vertices. For $i = m$, $v_{i,j+1}$ dominates $v_{i,j}$, and the result now follows from Theorem 2.2. □

In the final special case, there is only one candidate left.

Theorem 2.5. *Consider an $m \times n$ grid graph $G = (V, E)$ and let $S \subseteq V$. When considering vertices for dominating $v_{i,j} = s(S)$ when $v_{i,j+1}, v_{i,j+2} \in S$, $i \leqslant m-1$, $j \leqslant n-2$, the candidate $v_{i,j+1}$ can be ignored.*

Proof. If $v_{i,j+1}, v_{i,j+2} \in S$, the only undominated vertices that can be dominated by $v_{i,j+1}$ are $v_{i,j}$ and, if $i \leqslant m-1$, $v_{i+1,j+1}$. However, for $i \leqslant m-1$, $v_{i+1,j}$ dominates both of these vertices. The result then follows from Theorem 2.2. □

Notice that the requirement that $i \leqslant m-1$ is not necessary in Theorem 2.5, but we need it to avoid a conflict with Theorem 2.4 (if $i = m$, $v_{i,j} = s(S)$, and $v_{i,j+1}, v_{i,j+2} \in S$, then it suffices to consider only one vertex to dominate $v_{i,j}$, but we need to decide which one). The three cases in Theorems 2.3 to 2.5 are shown in Figure 2, where the dominated vertices are black (the indices of the vertices increase when going down and to the right).

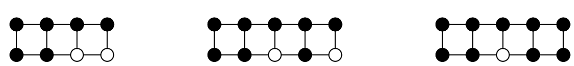

Figure 2: The three possible situations

The automorphism (symmetry) group Aut(G) of an $m \times n$ grid graph has order 4 if $m \neq n$ and order 8 if $m = n$. However, due to the way the search proceeds, we find only the subgroup of order 2 generated by the mapping of $v_{i,j}$ to $v_{i,n+1-j}$ useful. This symmetry—the term *mirror images* is used in [12]—should be taken into account for improved performance.

Theorem 2.6. *Consider an $m \times n$ grid graph $G = (V, E)$ and a mapping $f : V \to V$ such that $f(v_{i,j}) = v_{i,n+1-j}$ for all i, j. Let $V_1 \subseteq V$ and $V_2 \subseteq V$ such that f maps the set V_1 to V_2. To find a minimum dominating set of G, one may ignore V_1 and only consider dominating sets that extend V_2.*

Proof. We denote $f(V') = \cup_{v \in V'}\{f(v)\}$. For any dominating set $V_1 \cup V_3$ of G, $f(V_1 \cup V_3) = f(V_1) \cup f(V_3) = V_2 \cup f(V_3)$ is a dominating set of G. Hence, as $|V_1| = |V_2|$, it suffices to consider V_2 in the search for a minimum dominating set. □

3 The Algorithm

Our exhaustive search algorithm, the input parameters of which are the size parameters m and n of the considered grid graph, is a breadth-first search (BFS) algorithm with the features of *dynamic programming* [8, Chapter 15]. During the search, we maintain sets of dominated (rather than dominating) vertices.

On each level of the BFS, we have a collection $\mathcal{S}$ of sets (starting from the empty set), and for each $S \in \mathcal{S}$ we consider all vertices that dominate $s(S)$, except for those vertices that can be excluded by Theorems 2.3 to 2.5. The algorithm terminates when the entire grid graph has been dominated.

When we form a new collection $\mathcal{S}$ of dominated vertices from an old collection $\mathcal{S}'$, we use Theorem 2.2 whenever possible to reject solutions. Also a combination of Theorems 2.2 and 2.6 can be used to reject solutions. This rejection criterion is similar to the one used by Hare and Fisher in [12] to speed up the algorithm introduced by Hare in [11]. Using Theorem 2.2 takes up most of the CPU time, but is essential for minimizing the total cpu time for the search. An efficient implementation of this part is crucial; we shall now briefly elaborate on this issue.

In a collection $\mathcal{S}$ of sets that we maintain, we may store a set S either as S or as $f(S)$ (cf. Theorem 2.6). This choice is made based on the maximum of $s(S)$ and $s(f(S))$. Moreover, the collection $\mathcal{S}$ is kept sorted so that if S comes before T, then $s(S) \geqslant s(T)$. The fact that all vertices $v_{i,j}$ up to some value of i have been dominated in a set $S \in \mathcal{S}$ can be used to encode S efficiently.

Consider the situation when a new set S—if necessary, we first apply the mapping f to get a pair S, $f(S)$ that fulfills $s(S) \geqslant s(f(S))$—is to be considered for inclusion in a collection $\mathcal{S}$. Now we start comparing S and $f(S)$ with the elements of $\mathcal{S}$, starting from the beginning of the collection.

As long as $s(S)$ is smaller than $s(S')$ ($S' \in \mathcal{S}$, also in the sequel), we test whether $S \subseteq S'$ or $f(S) \subseteq S'$, and reject S if this happens (and stop the search). When $s(S)$ equals $s(S')$, we need to test both whether $S \subseteq S'$ or $f(S) \subseteq S'$ and whether $S' \subseteq S$ or $S' \subseteq f(S)$, and if one of these situations occurs reject S or S', respectively. If S has survived to the point where $s(S)$ becomes larger than $s(S')$, or sooner if some element in $\mathcal{S}$ is rejected, we know that S is to be inserted in the list, but we also need to test whether $S' \subseteq S$ or $S' \subseteq f(S)$ (which would lead to deletion of old sets) for all sets $S' \in \mathcal{S}$ to the end of the collection. Implementing a linked list for $\mathcal{S}$ and data structures for the sets in the list are standard tasks.

Whenever we encounter a situation where the entire grid graph is dominated, we may terminate the search, and the level of the BFS gives the size of the smallest dominating set. When we determine $\gamma_{m,n}$ in this way, we also get $\gamma_{m',n}$ for all $m' < m$ as a by-product by checking when the first set S is created for which $s(S)$ is larger than $v_{m',n}$. Observe that we only get the size of a minimum dominating set, not the dominating set itself (but such sets can be found relatively easily, for example, by local search [1]). As a final comment, all experiments with branch-and-bound type arguments—based on bounds regarding the domination of undominated vertices—led to a deterioration of the current approach.

4 Results

The values of $\gamma_{m,n}$ for $m, n \leqslant 29$ can be found in Table 1, and minimum dominating sets attaining $\gamma_{n,n}$ for $n \leqslant 29$ are shown in Figures 3 to 10. As for the case of determining $\gamma_{n,n}$, the computing time grows by a factor of roughly 4 for consecutive instances, whereas the memory requirement grows by a factor of just under 2. For the largest square case solved, $\gamma_{29,29}$, 31 CPU-days (using a 3-GHz Intel Core2 Duo CPU E8400) and 75MB of memory were needed. It is not clear to the authors how to implement a distributed version of the developed algorithm (cf. [19]); such an implementation would be necessary for pushing the range of calculated values of $\gamma_{m,n}$ several steps further.

Practical experiments show that the computing time grows approximately linearly in one of the parameters when the other parameter is fixed; we were able to apply the algorithm to determine all values of $\gamma_{m,n}$ for $m \leqslant 27$ and $n \leqslant 1000$. The following observation gives a concise description of these values.

In [2] the upper bound

$$\gamma_{m,n} \leqslant \left\lfloor \frac{(m+2)(n+2)}{5} \right\rfloor - 4$$

is proved for $m, n \geqslant 8$. It is also conjectured [2] that this upper bound gives the exact value for $m, n \geqslant 16$. The current work shows that this conjecture holds for $m, n \leqslant 29$ as well as for $16 \leqslant m \leqslant 27$, $16 \leqslant n \leqslant 1000$. The intermediate data from the computations can be used to develop exact formulas for $\gamma_{m,n}$ with one of the parameters fixed (cf. [17]); this issue will be studied further in subsequent work.

Further issues that can be addressed with variants of the current algorithm include the study of possible components in the subgraph induced by a minimum dominating set. In particular, the *independent domination number*—where the components are single vertices—can be studied to determine when this number and the domination number coincide. One could also study the domination number for graphs that are products of other graphs than paths as well as other similar graphs [5, 9, 13, 15].

Acknowledgements

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Table 1: Domination numbers $\gamma_{m,n}$ for $m, n \leqslant 29$

$m \diagdown n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
1	1																												
2	1	2																											
3	1	2	3																										
4	2	3	4	4																									
5	2	3	4	6	7																								
6	2	4	5	7	8	10																							
7	3	4	6	7	9	11	12																						
8	3	5	7	8	11	12	14	16																					
9	3	5	7	10	12	14	16	18	20																				
10	4	6	8	10	13	16	17	20	22	24																			
11	4	6	9	11	14	17	19	22	24	27	29																		
12	4	7	10	12	16	18	21	24	26	29	32	35																	
13	5	7	10	13	17	20	22	26	29	31	35	38	40																
14	5	8	11	14	18	21	24	28	31	34	37	40	44	47															
15	5	8	12	15	19	22	26	29	33	36	40	43	47	50	53														
16	6	9	13	16	20	24	27	31	35	38	42	46	49	53	57	60													
17	6	9	13	17	22	26	29	33	37	41	45	49	53	56	60	64	68												
18	6	10	14	18	23	27	31	35	39	43	47	51	55	60	64	68	72	76											
19	7	10	15	19	24	28	32	37	41	45	50	54	58	63	67	71	75	80	84										
20	7	11	16	20	25	30	34	39	43	48	52	57	62	66	70	75	79	84	88	92									
21	7	11	16	21	26	31	36	41	45	50	55	60	64	69	74	78	83	88	92	97	101								
22	8	12	17	22	28	32	37	43	47	52	57	62	67	72	77	82	87	92	96	101	106	111							
23	8	12	18	23	29	34	39	44	49	54	60	65	70	75	80	86	91	96	101	106	111	116	121						
24	8	13	19	24	30	36	41	46	52	57	63	68	73	79	84	89	94	100	105	110	115	120	126	131					
25	9	13	19	25	31	37	42	48	54	59	65	71	76	82	87	93	98	104	109	114	120	125	131	136	141				
26	9	14	20	26	32	38	44	50	56	62	68	74	79	85	91	96	102	108	113	119	124	130	136	141	146	152			
27	9	14	21	27	34	40	46	52	58	64	70	76	82	88	94	100	106	112	117	123	129	135	141	146	152	158	164		
28	10	15	22	28	35	41	47	54	60	66	73	79	85	91	97	104	110	116	122	128	134	140	146	152	158	164	170	176	
29	10	15	22	29	36	42	49	56	62	69	75	82	88	94	101	107	113	120	126	132	138	144	151	157	163	169	175	182	188

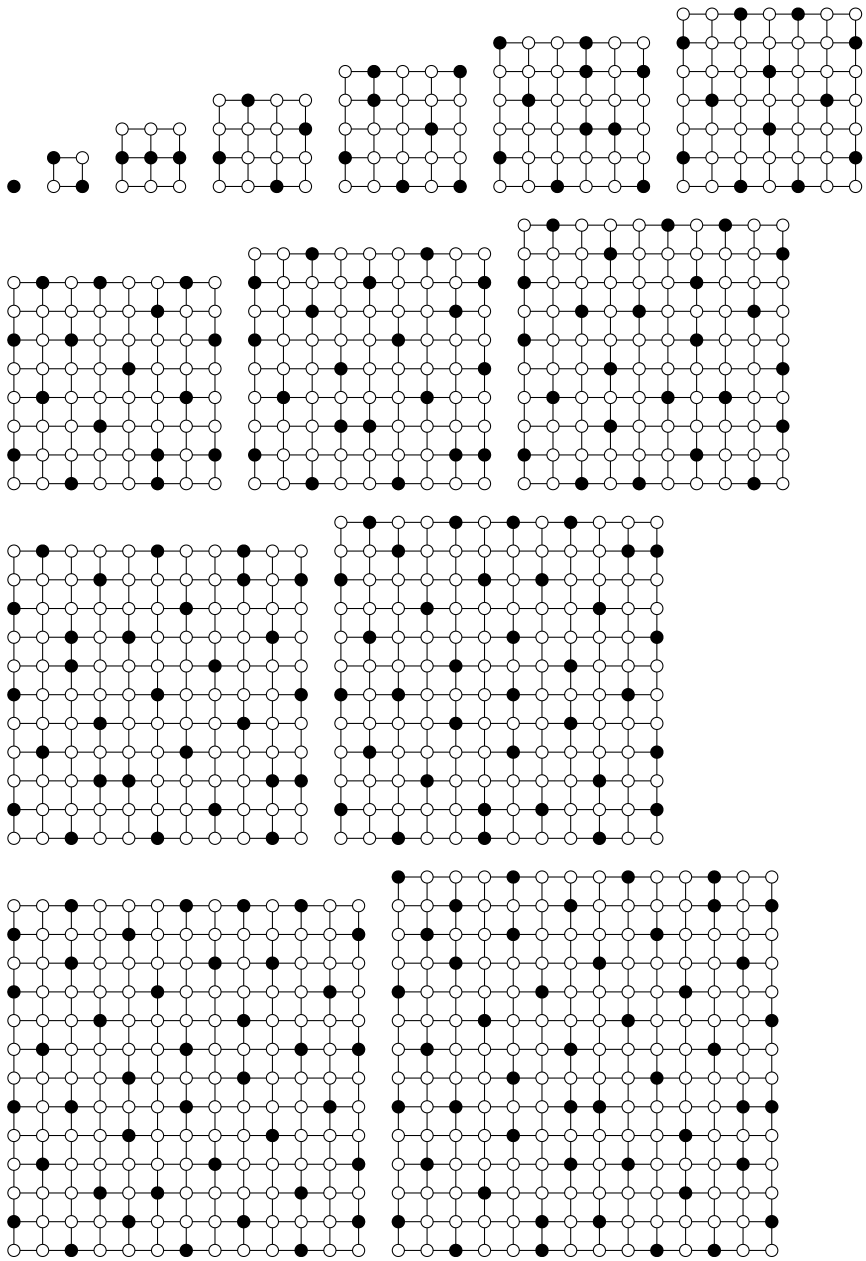

Figure 3: Minimum dominating sets of square grid graphs for $1 \leqslant n \leqslant 14$

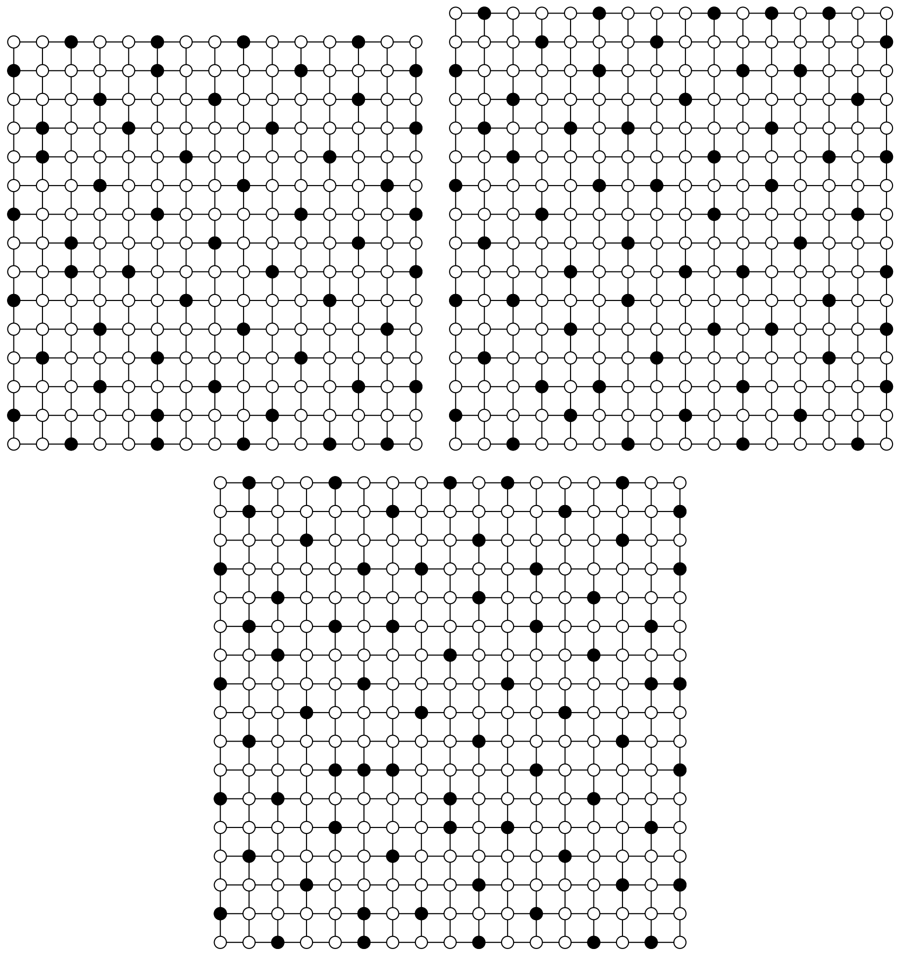

Figure 4: Minimum dominating sets of square grid graphs for $15 \leqslant n \leqslant 17$

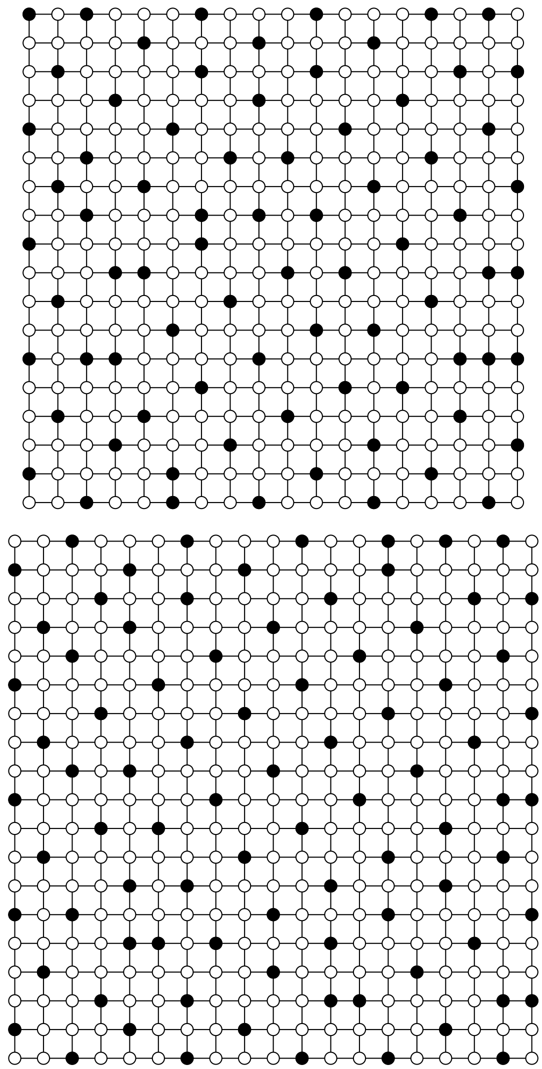

Figure 5: Minimum dominating sets of square grid graphs for $18 \leqslant n \leqslant 19$

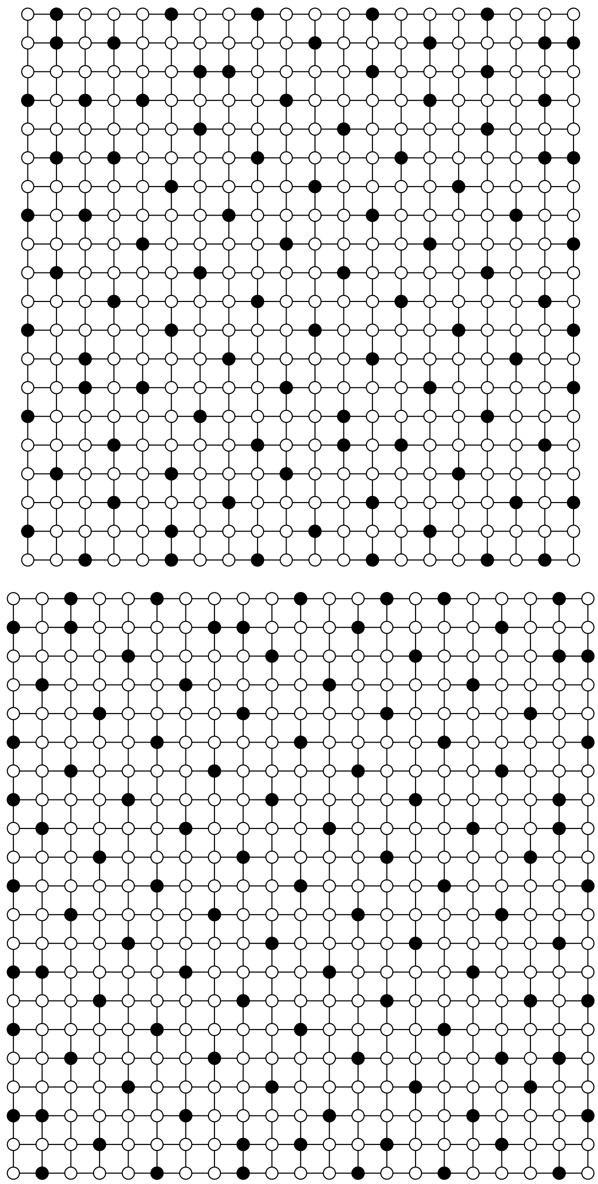

Figure 6: Minimum dominating sets of square grid graphs for $20 \leqslant n \leqslant 21$

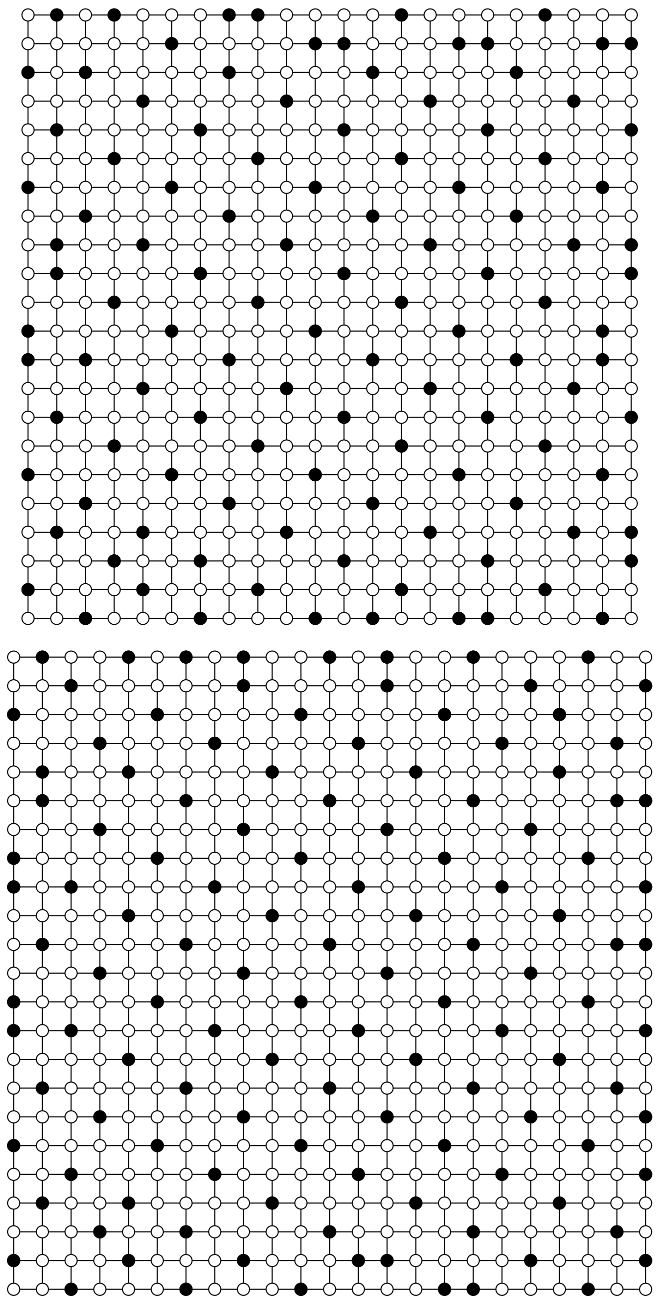

Figure 7: Minimum dominating sets of square grid graphs for $22 \leqslant n \leqslant 23$

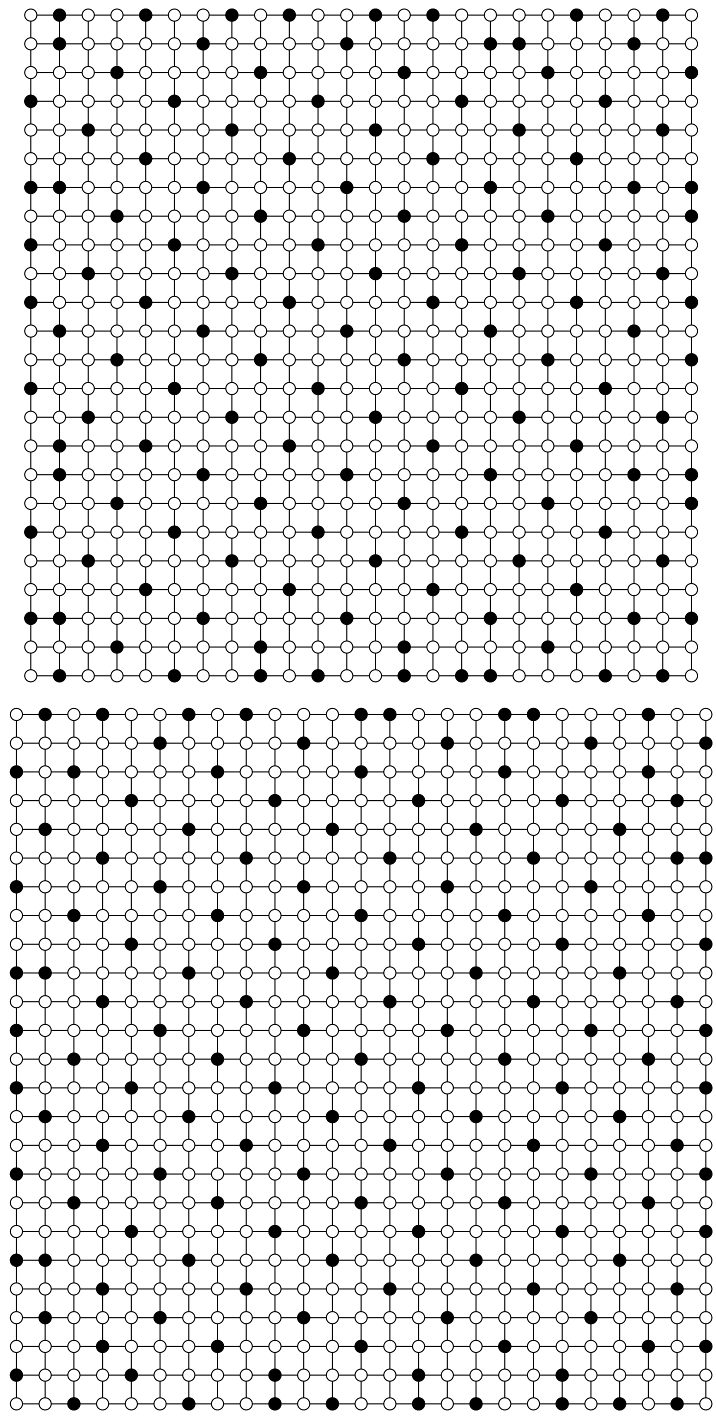

Figure 8: Minimum dominating sets of square grid graphs for $24 \leqslant n \leqslant 25$

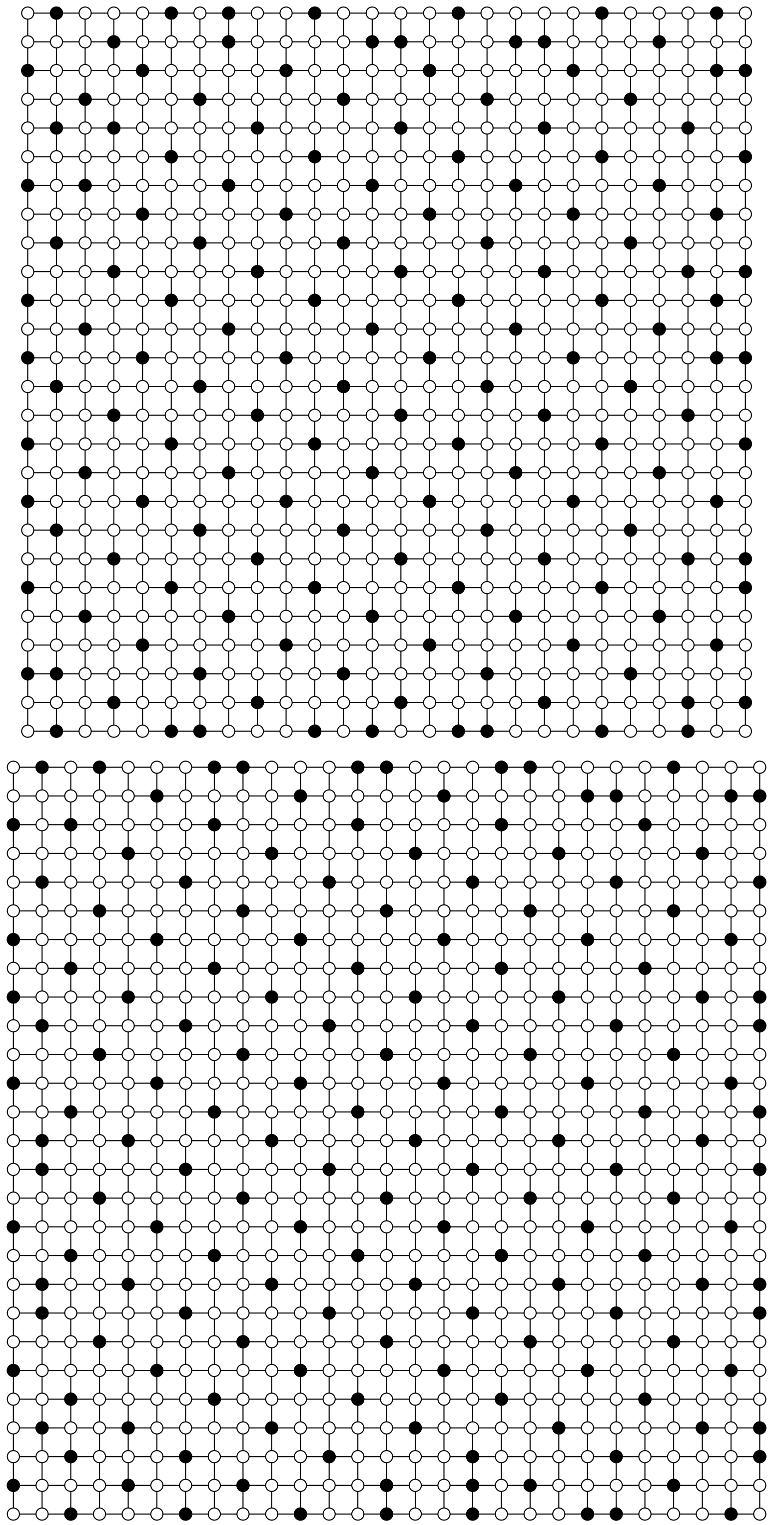

Figure 9: Minimum dominating sets of square grid graphs for $26 \leqslant n \leqslant 27$

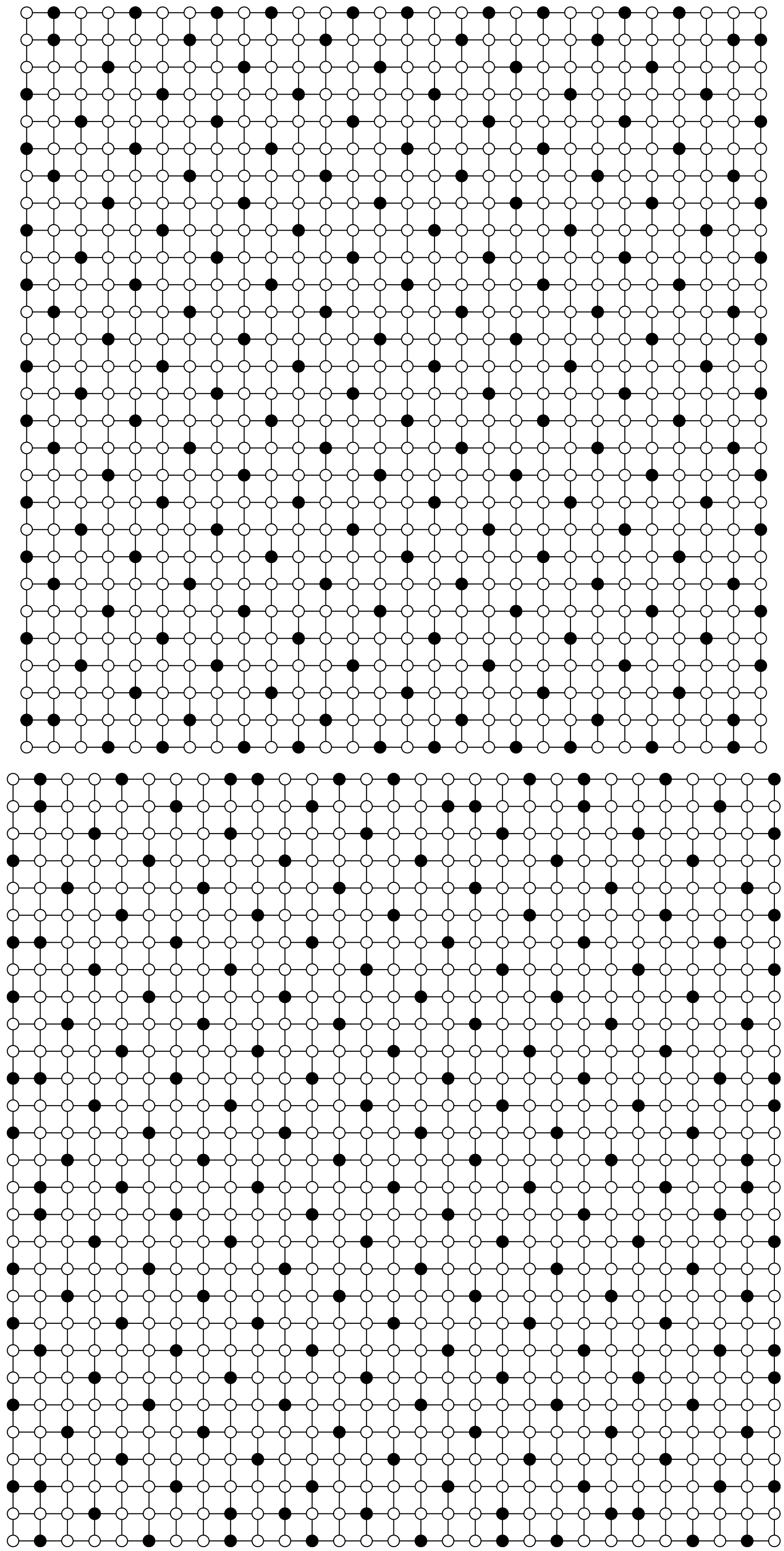

Figure 10: Minimum dominating sets of square grid graphs for $28 \leqslant n \leqslant 29$

References

[1] E. Aarts and J. K. Lenstra (Eds.), *Local Search in Combinatorial Optimization*, Wiley, New York, 1997. Reprinted by Princeton University Press, Princeton, 2003.

[2] T. Y. Chang, *Domination Numbers of Grid Graphs*, Ph.D. thesis, Dept. of Mathematics, University of South Florida, 1992.

[3] T. Y. Chang and W. E. Clark, The domination numbers of the $5 \times n$ and $6 \times n$ grid graphs, *J. Graph Theory*, **17** (1993), 81–107.

[4] T. Y. Chang, W. E. Clark, and E. O. Hare, Domination numbers of complete grid graphs. I. *Ars Combin.* **38** (1994), 97–111.

[5] R. Chérifi, S. Gravier, X. Lagraula, C. Payan and I. Zighem, Domination number of the cross product of paths, *Discrete Appl. Math.* **94** (1999), 101–139.

[6] R. Chérifi, S. Gravier, and I. Zighem, Bounds on domination number of complete grid graphs, *Ars Combin.* **60** (2001), 307–311.

[7] E. J. Cockayne, E. O. Hare, S. T. Hedetniemi, and T. V. Wimer, Bounds for the domination number of grid graphs, *Congr. Numer.* **47** (1985), 217–228.

[8] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, *Introduction to Algorithms*, 3rd ed., MIT Press, Cambridge, 2009.

[9] G. H. Fricke, S. M. Hedetniemi, S. T. Hedetniemi, C. K. Wallis, M. S. Jacobson, H.W. Martin, and W. D. Weakley, Combinatorial problems on chessboards: a brief survey, in Y. Alavi and A. Schwenk (Eds.), *Graph Theory, Combinatorics, and Algorithms, Vol. 1*, Wiley, New York, 1995, pp. 507–528.

[10] D. R. Guichard, A lower bound for the domination number of complete grid graphs, *J. Combin. Math. Combin. Comput.* **49** (2004), 215–220.

[11] E. O. Hare, *Algorithms for Grids and Grid-Like Graphs*, Ph.D. thesis, Dept. of Computer Science, Clemson University, 1989.

[12] E. O. Hare and D. C. Fisher, An application of beatable dominating sets to algorithms for complete grid graphs, in Y. Alavi and A. Schwenk (Eds.), *Graph Theory, Combinatorics, and Algorithms, Vol. 1*, Wiley, New York, 1995, pp. 497–506.

[13] E. O. Hare and W. R. Hare, Domination in graphs similar to grid graphs, *Congr. Numer.* **97** (1993), 143–154.

[14] E. O. Hare, S. T. Hedetniemi, and W. R. Hare, Algorithms for computing the domination number of $k \times n$ complete grid graphs, *Congr. Numer.* **55** (1986), 81–92.

[15] S. M. Hedetniemi, S. T. Hedetniemi, and R. Reynolds, Combinatorial problems on chessboards: II, in T. W. Haynes, S. T. Hedetniemi, and P. J. Slater (Eds.), *Domination in Graphs*, Dekker, New York, 1998, pp. 133–162.

[16] M. S. Jacobson and L. F. Kinch, On the domination number of products of graphs. I. *Ars Combin.* **18** (1984), 33–44.

[17] M. Livingston and Q. F. Stout, Constant time computation of minimum dominating sets, *Congr. Numer.* **105** (1994), 116–128.

[18] K. L. Ma and C. W. H. Lam, Partition algorithm for the dominating set problem, *Congr. Numer.* **81** (1991), 69–80.

[19] H. G. Singh and R. P. Pargas, A parallel implementation for the domination number of a grid graph, *Congr. Numer.* **59** (1987), 297–311.

[20] N. J. A. Sloane, An on-line version of the encyclopedia of integer sequences, *Electron. J. Combin.* **1** (1994), #F1 (electronic, 5pp.).

[21] A. Spalding, *Min-Plus Algebra and Graph Domination*, Ph.D. thesis, Dept. of Applied Mathematics, University of Colorado, 1998.

[22] J. Žerovnik, Deriving formulas for domination numbers of fasciagraphs and rotagraphs, in G. Ciobanu and G. Păun (Eds.), *Fundamentals of Computation Theory*, LNCS 1684, Springer, Berlin, 1999, pp. 559–568.