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Professor Julio B. CLEMPNER, PhD
E-mail: julio@clempner.name
Instituto Politécnico Nacional
Professor Alexander S. POZNYAK, PhD
E-mail: apoznyak@ctrl.cinvestav.mx
Center for Research and Advanced Studies

COMPUTING THE TRANSFER PRICING FOR A MULTIDIVISIONAL FIRM BASED ON COOPERATIVE GAMES

Abstract. *This paper suggests a novel cooperative game-theoretic approach to solve the transfer pricing problem for a multidivisional firm considering that each division purchases goods from an upstream division in the supply chain. The formulation of the transfer pricing problem considers costs and taxes in a vertically integrated supply chain. We conceptualize the transfer pricing as a multi-objective problem and present a method for finding the strong Nash equilibrium that maximizes the utility of the entire multidivisional firm. The approach consists on determining a scalar* λ^* *and the corresponding strategies* $v^*(\lambda^*)$ *fixing specific bounds on the Pareto front. Bounds correspond to restrictions imposed by each division over the Pareto front that determine the maximum and minimum transfer price legally authorized. For ensuring the existence of a unique strong Nash equilibrium, we employ a penalized regularization method for poly-linear functions. We implement a recurrent procedure for finding the extremal point. For finding the strong Nash equilibrium, we select the Pareto optimal strong strategy* $v^*(\lambda^*)$ *with minimal distance to the utopia point. We show that the regularized functional of the game decreases and converges, proving the existence and uniqueness of strong Nash equilibrium. The usefulness of the method is successfully demonstrated by a numerical example.*

Keywords: *Transfer pricing cooperative games strong Nash equilibrium multidivisional firm regularization multi-objective.*

1.Introduction

1.1. Brief review

The problem of transfer pricing employs optimization goals to set internal prices of goods or services that are sold between subsidiaries or divisions of a firm [8]. Several pricing policies are followed in practice: marginal cost, market price, average cost, etc. and particular solutions (negotiations) adapted to specific situations. In a

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decentralized system, each divisional manager identifies independent opportunities other than the opportunity of trading internally with other divisions. Then, each division estimates the performance measures (revenues or profits) of each opportunity and selects the best opportunity. As a result, the transfer pricing system focuses on the maximization of divisional income, leading the divisions to achieve the goals of the firm through sub-optimization. However, sub-optimizations by individual divisions do not result in an optimum for the firm.

The transfer pricing problem needs to solve transfer prices considering that each division independently purchases goods from an upstream division in the supply chain. Transfer prices are determined in such a way that divisional sub-optimizations imply an overall optimum for the firm as whole. Then, transfer pricing problem has to consider the allocation of a global profit between subsidiaries or divisions of a firm. As a result, it is usually employed for tax avoidance by firms that lower profits in subsidiaries or divisions located in high-tax places and increase profits in divisions located in low-tax places. Because it is considered an instrument for allocating the total profit of a multidivisional firm, transfer pricing plays a fundamental role in the decision making for buying or selling subsidiaries or divisions [15].

1.2. Related works

Related works presented in the literature have identified that the transfer pricing problem is inherently a multi-objective problem for optimally and developed transfer pricing methodologies based on mathematical programming and game theory. The seminal paper on transfer pricing was presented by Hirshleifer[15] who considering a system that focus on the maximization of divisional income suggested that the precise transfer price for a good is the marginal cost of the producing division. Horst[16] explored the profit-maximizing strategy for a monopolistic firm selling to two national markets simultaneously. Kassicieh[18] presented a model of transfer pricing that is developed further to include all of the issues that affect the total profits of the multinational corporation through transfer prices and expanded to include economic externalities and interdependencies such as nonlinear cost functions and functional relationships among demand, supply and transfer prices. Luft and Libby[22] examined experienced managers' judgments about the effects of market price and accounting profit information on negotiated transfer prices. Lakhal[19] developed a method using a mathematical programming model and providing companies an opportunity to work proactively with the internal revenue service in a cooperative manner in order to avoid costly audit and litigation. Lakhal et al.[20] proposed a framework and methodology for profit sharing and transfer-pricing between network companies presenting a paradigm that enables maximization of operating profits by the manufacturing-network in its

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larger supply chain, suggesting a departure from the model that maximizes profits for the individual company within the sphere of its own supply chain. Shunko and Gavirneni[25] showed that randomness in a supply chain magnifies the impact of transfer prices and analyzed possible reasons behind this behavior and summarized the impact of various supply chain parameters on the magnitude of profit improvement. Rosenthal[24] developed a cooperative game that provides transfer prices for the intermediate products in a vertically integrated supply chain. Leng and Parlarb [21] considered the transfer pricing decision for a multidivisional firm with an upstream division and multiple downstream divisions solving the problem using a cooperative game which computes the Shapley value-based transfer prices for the firm. Huh and Park[17] compared the supply chain profits for transfer pricing suggesting that the firm-wide and divisional profits tend to be higher under the cost-plus method than they are under the resale-price method. Hammami and Frein[14] developed an optimization model for supply chain addressing decisions, costs, and complexity factors integrating transfer pricing to derive a series of insights that may be helpful for companies and governments. Clempner and Poznyak[8] proposed a solution to the transfer pricing problem from the point of view of the Nash bargaining game theory approach.

1.3. Main results

Transfer pricing is related to the pricing of an intermediate product or service within a firm. The products or services are collaborative transferred or relocated between divisions of the firm. As a consequence, transfer pricing refers to the allocation of profits in a supply chain. Encouraged by the significant impact of transfer pricing methods for tax purposes on operational decisions and the corresponding profits of a supply chain we propose a model for a multidivisional firm in a vertically integrated supply chain. We suggest a new game-theoretic approach to solve transfer prices considering that each division purchases goods from an upstream division in the supply chain, and in which transfer prices are determined in such a way that divisional sub-optimizations imply an overall optimum for the firm as whole. The main results of this paper are as follows:

- We propose a method for computing the transfer pricing for a multidivisional firm.
- We suggest a multi-objective approach for representing a cooperative game.
- For computing the strong Nash equilibrium, the method finds the Pareto optimal point with minimal distance to the utopia point.

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- The Tikhonov's regularization [3, 2, 13] method is used to guarantee the convergence to a single (strong) equilibrium point.
- We propose a method based on a penalized programming approach for computing the regularized strategies of the game.
- We develop a programming method to solve the successive single-objective constrained problems that arise from taking the regularized functional of the game.
- We implement a recurrent procedure based on the gradient method to solve the regularized problem and finding the Pareto optimal strategies.
- We show that in the regularized problem the functional of the game decreases and finally converges, proving the existence and uniqueness of strong Nash equilibrium.
- In addition, we present the convergence conditions and compute the estimate rate of convergence of variables μ and δ corresponding to the step size parameter of the regularized penalty method.
- We provide all the details needed to implement the method in an efficient and numerically stable way.
- The usefulness of the method is successfully demonstrated by a numerical example.

1.4. Organization of the paper

The remainder of the paper is organized as follows. The next Section suggests the formulation of the problem for transfer pricing. Section 3 presents the cooperative game model describing the Pareto front, the restrictions over jurisdictions imposed by governments and the strong Nash equilibrium of the problem. Section 4 describes the regularized penalty function optimization method. Section 5 concludes the paper with some remarks and future work. The appendix presents the proofs of the theorems.

2. Formulation of the problem

Transfer pricing refers to the pricing of an intermediate product or service within a firm. This product or service is transferred between divisions of the firm. Thus, transfer pricing is closely related to the allocation of profits in a supply chain.

Problem: *The transfer pricing problem consists in fixing transfer prices in such a way that divisional sub-optimizations imply an overall optimum for the firm as whole*.

Remark 1. *No division, or subset of divisions, can obtain a greater profit from being outside the supply chain.*

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An example is as follows[1]. American Petroleum (AM) is the world's top producer of petroleum and natural gas hydrocarbons. AM has a refinery called Refinery American Petroleum (RAM) in U.S.A. which produces A-oil. Petroleum products are materials derived from crude oil (petroleum) as it is processed in oil refineries. RAM established a corporation CAM as a subsidiary to distribute product A-oil in Europe. Then, AM sells petroleum to RAM, that sells the A-oil to CAM which then sells it to approximately 1000 third-party retailers in Europe. The function performed by CAM are that of purchase of inventories of product A-oil and the sale of these inventories to final retailers. Both, RAM and CAM buy products (petroleum and A-oil) at negotiated transfer prices in order to maximize the global profit of the firm. In calculating the product's price, it must be considered the costs and taxes involving for the sales transactions of petroleum from company AM to company RAM, those involving the sales transactions of A-oil from company RAM to company CAM and finally, the costs and taxes involving the sales to the retailers.

The formulation of the transfer price for the multidivisional firm considers a vertically integrated supply chain. The game consists of $\mathcal{N}$ divisions or players (denoted by $l=\overline{1,\mathcal{N}}$) which jointly make their decisions to maximize the global profits of the firm. For solving the problem, the multidivisional firm should compute the optimal strategy under an allocation scheme able to get the global profit maximization. We develop a cooperative game model for computing the transfer pricing for a multidivisional firm based on a multi-objective approach.

Let us consider for $l=1,\ldots,\mathcal{N}-1$, intermediate goods are shipped from level *l* to level*l*+1, i.e., downstream along the supply chain. Then, we develop the utility functions for divisions. Division*l* makes its market pricing decision p^l and sells q^l units of its final products. Following [1], we can consider that q^l is determined by a linear demand function, i.e. $q^l=\alpha^l-\beta^l p^l$ where $\alpha,\beta>0$. Because of the existence of economies of scale, we consider that the division's unit production cost is dependent on the production quantity. Then, the unit production cost which is incurred by division l when the division sells q^l units of intermediate products is represented by $c^l(q^l)$. As well, we represent the taxes that a division l has to pay as function depending on the product and the quantity represented by $r^l(p^l q^l)$. We do not consider any specific function for the

[1] This example describes the treatment for transfer pricing cost-taxation purposes and the aim is to illustrate the key points to take into consideration.

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costs and the taxes, and we use the general form $c^l(q^l)$ and $r^l(p^l q^l)$ for our analysis. Then, our variables are defined as follows:

- Market prices p^l for one intermediate good (sold from l to $l+1$).
- Quantities q^l of intermediate good l shipped from l to $l+1$ for $l=1,\ldots,\mathcal{N}-1$;
- Production costs $c^l(q^l) := \theta_c^l q^l$, $\theta_c^l \geq 0$ (e.g.: transactional costs, raw materials, components, and their per period inventory costs in dollars) at each level $l=\overline{1,\mathcal{N}}$;
- Taxes $r^l(p^l q^l) := \theta_r^l p^l q^l$, $\theta_r^l \geq 0$, at each level $l=\overline{1,\mathcal{N}}$.

We suppose that all the division are located in different marketing areas, therefore they face independent demands q^l, costs $c^l(q^l)$ and taxes $r^l(p^l q^l)$. Then, the division's utility U^l for each level $l=\overline{1,\mathcal{N}}$ is given by

$$U^1(p^1,q^1) = p^1 q^1 - c^1(q^1) - r^1(p^1 q^1)$$

$$U^l(p^l,p^{l-1},q^l) = \left(p^l - p^{l-1}\right)q^l - c^l(q^l) - r^l(p^l q^l) \text{ for } l = 2,\ldots,\mathcal{N}$$

where $l=1$ represents the first division and $l=2,\ldots,\mathcal{N}$ the rest of the divisions on the vertically integrated supply chain.

3.The cooperative game model

3.1. Pareto set

The *Pareto set* can be defined as [6, 7,11, 12]

$$\mathcal{P} := \left\{ v^*(\lambda) := \arg \max_{v \in V_{adm} = P_{adm} \otimes Q_{adm}} W(v \mid \lambda), \lambda \in S^{\mathcal{N}} \right\} \tag{1}$$

Where

$$W(v \mid \lambda) := \lambda_1 U^1\left(p^1,q^1\right) + \sum_{l=2}^{\mathcal{N}} \lambda_l U^l\left(p^l,p^{l-1},q^l\right)$$

$$v = \left\{(p,q) : p = (p_1,\ldots,p_{\mathcal{N}})^{\mathsf{T}} \text{ and } q = (q_1,\ldots,q_{\mathcal{N}})^{\mathsf{T}}\right\}$$

$$S^{\mathcal{N}} := \left\{ \lambda \in \mathbb{R}^{\mathcal{N}} : \lambda_l \in [0,1], \sum_{l=1}^{\mathcal{N}} \lambda_l = 1 \right\}$$

with the Pareto front given by

$$\mathbf{U}(v^*(\lambda)) = \left(U^1\left(v^*(\lambda)\right), U^2\left(v^*(\lambda)\right), \ldots, U^{\mathcal{N}}\left(v^*(\lambda)\right)\right)$$

3.2. Constraints

Multidivisional firms look for coordinate its divisions in order to maximize the global profit of the firm after cost-tax through the transfer pricing decision. For instance, if a given upstream division is located in a lower-cost-tax jurisdiction, then increasing the transfer price results in an increase in the firm's after-cost-tax profit. To prevent

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multidivisional firms from intentionally transferring their profits from high-cost-tax jurisdictions to low-cost-tax jurisdictions governments establish bounds over the transfer pricing. Bounds correspond to restrictions imposed over each division on the Pareto front that determine the maximum and minimum transfer price legally authorized. The bounds determine specific decision area where the Pareto optimal strategies can be selected.

The bounds over the Pareto front [7,11] are defined as follows

$$p^l \in \left[p_-^l, p_+^l\right], q^l \in \left[q_-^l, q_+^l\right], \quad P_{adm} := \prod_{l=1}^{\mathcal{N}} \left[p_-^l, p_+^l\right], Q_{adm} := \prod_{l=1}^{\mathcal{N}} \left[q_-^l, q_+^l\right]$$

Remark 2. *For practical reasons and improving the flexibility of the model we propose to establish bounds on both* p *and* q.

3.3. Optimal weights selection

Let define the *min* and *max* allowed bounds as follows

$$F_i^- = \min_{\lambda \in S^{\mathcal{N}}} U^l\left(v^*(\lambda)\right), \qquad F_l^+ = \max_{\lambda \in S^{\mathcal{N}}} U^l\left(v^*(\lambda)\right), \qquad \text{for } l = 1,\ldots,\mathcal{N} \tag{2}$$

Suppose that these bounds are *a* priory given as

$$U^l\left(v^*(\lambda)\right) \in \left[U^{l-}, U^{l+}\right], l = 1,\ldots,\mathcal{N}$$

According to [4,5] define the optimal $\lambda^* \in S^{\mathcal{N}}$ that corresponds to the strong Nash equilibrium of the problem

$$\lambda^* = \arg\min_{\lambda \in \mathbf{S}^{\mathcal{N}}} \left\| \mathbf{U}^* - \mathbf{U}(v^*(\lambda)), \right\| \quad \mathbf{U}^* = \left(U^{1*}, \ldots, U^{\mathcal{N}*}\right)$$
$$U^{l*} = \max_{0 \le p^l, p^{l-1}, 0 \le q^l} U^l\left(p^l, p^{l-1}, q^l\right) \tag{3}$$

U^{l*} corresponds with the *utopian* point) subject to

$$U^l\left(v^*(\lambda)\right) \in \left[U^{l-}, U^{l+}\right], l = 1,\ldots,\mathcal{N}$$

Lemma 3. *The problem (3) formulated above is feasible iff*

1. $\left[U^{l-}, U^{l+}\right] \subseteq \left[F_l^-, F_l^+\right]$ (4)
2. $\left[U^{1-}, U^{1+}\right] \cap \left[U^{2-}, U^{2+}\right] \cap \ldots \cap \left[U^{\mathcal{N}-}, U^{\mathcal{N}+}\right] \cap \mathcal{P} \neq \varnothing$ (5)

3.4. Final result

- The optimal individual utility is given by

$$\mathbf{U}\left(v^*\left(\lambda^*\right)\right) = \left(U^1\left(v^*\left(\lambda^*\right)\right), U^2\left(v^*\left(\lambda^*\right)\right), \ldots, U^{\mathcal{N}}\left(v^*\left(\lambda^*\right)\right)\right)$$

- The optimal global utility

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$$W\left(v^* \mid \lambda^*\right) = \left(\lambda^*\right)^{\intercal} \mathbf{U}(v^*) = \sum_{l=1}^{\mathcal{N}} \lambda_l^* U^l\left(p^{l*}, p^{(l-1)*}, q^{l*}\right)$$

Remark 4. *The proposed model can find the collaborative equilibrium point of both,* p *by fixing the value of* q *and,* p *and* q *simultaneously, depending on the case.*

4. Regularization method

4.1. Regularized poly-linear programming problem

Consider the following poly-linear programming problem

$$\begin{array}{c} f(x) = \alpha_1 \sum_{j_1=1}^{N} c_{j_1} x_{j_1} + \alpha_2 \sum_{j_1=1}^{N} \sum_{j_2=1}^{N} c_{j_1,j_2} x_{j_1} x_{j_2} + \alpha_3 \sum_{j_1=1}^{N} \sum_{j_2=1}^{N} \sum_{j_3=1}^{N} c_{j_1,j_2,j_3} x_{j_1} x_{j_2} x_{j_3} + \Lambda + \\ \alpha_{N-1} \sum_{j_1=1}^{N} \sum_{j_2=1}^{N} \Lambda \sum_{j_{N-1}=1}^{N} c_{j_1,\dots,j_{N-1}} x_{j_1} \Lambda\, x_{j_{N-1}} + \alpha_N \sum_{j_1=1}^{N} \sum_{j_2=1}^{N} \Lambda \sum_{j_N=1}^{N} c_{j_1,\dots,j_N} x_{j_1} \Lambda\, x_{j_N} \to \min_{x \in X_{adm}} \\ \alpha_j = \{0;1\} (j = 1,\dots, N) \text{ are binary variables} \\ X_{adm} := \left\{ x \in \mathbb{R}^N : x \geq 0, V_0 x = b_0 \in \mathbb{R}^{M_0}, V_1 x \leq b_1 \in \mathbb{R}^{M_1} \right\} \text{is a bounded set.} \end{array} \tag{6}$$

Introducing the "slack" vectors $u \in \mathbb{R}^{M_1}$ with nonnegative components, that is, $u_j \geq 0$ for all $j = 1,\dots, M_1$, the original problem (6) can be rewritten as

$$\left.\begin{array}{c} \min_{x \in X_{adm}, u \geq 0} f(x) \\ \text{subject to} \\ X_{adm} := \left\{ x \in \mathbb{R}^N : x \geq 0, V_0 x = b_0, V_1 x - b_1 + u = 0 \right\} \end{array}\right\} \tag{7}$$

Notice that this problem may have non-unique solution and $\det\left(V_0^{\intercal} V_0\right) = 0$. Define by $X^* \subseteq X_{adm}$ the set of all solutions of the problem (7).

Following [9] and [10] consider the *penalty function*

$$\tilde{\mathsf{F}}_{k,\delta}(x,u) := f(x) + k\left[\frac{1}{2}\left\|V_0 x - b_0\right\|^2 + \frac{1}{2}\left\|V_1 x - b_1 + u\right\|^2 + \frac{\delta}{2}\left(\|x\|^2 + \|u\|^2\right)\right] \tag{8}$$

where the parameters k and δ are positive. Obviously, the unconstraint on x the optimization problem

$$\min_{x \in X_{adm}, u \geq 0} \tilde{\mathsf{F}}_{k,\delta}(x,u) \tag{9}$$

has a unique solution since the optimized function (8) is *strongly convex* [23] if $\delta > 0$. Notice also that

$$\arg \min_{x \in X_{adm}, u \geq 0} \tilde{\mathsf{F}}_{k,\delta}(x,u) = \arg \min_{x \in X_{adm}, u \geq 0} \mathsf{F}_{\mu,\delta}(x,u)$$

where $\mu := k^{-1} > 0$ and

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$$\mathsf{F}_{\mu,\delta}(x,u) := \mu f(x) + \frac{1}{2}\|V_0 x - b_0\|^2 + \frac{1}{2}\|V_1 x - b_1 + u\|^2 + \frac{\delta}{2}\left(\|x\|^2 + \|u\|^2\right) \quad (10)$$

Proposition 5. *If the penalty parameter* μ *tends to zero by a particular manner, then we may expect that* $x^*(\mu,\delta)$ *and* $u^*(\mu,\delta)$, *which are the solutions of the optimization problem*

$$\min_{x \in X_{adm}, u \geq 0} \mathsf{F}_{\mu,\delta}(x,u)$$

tend to the set X^* of all solutions of the original optimization problem (7), that is,

$$\rho\{x^*(\mu,\delta), u^*(\mu,\delta); X^*\} \underset{\mu \downarrow 0}{\to} 0 \quad (11)$$

where $\rho\{a; X^*\}$ is the Hausdorff distance defined as

$$\rho\{a; X^*\} = \min_{x^* \in X^*} \|a - x^*\|^2.$$

Below we define exactly how the parameters μ and δ should tend to zero to provide the property (11).

Theorem 6. *Let us assume that*

1. The bounded set X^* *of all solutions of the original optimization problem (7) is not empty and the Slater's condition holds, that is, there exists a point* $x \in X_{adm}$ *such that*

$$V_1 x < b_1. \quad (12)$$

2. The parameters μ *and* δ *are time-varying, i.e.,*

$$\mu = \mu_n, \delta = \delta_n (n = 0,1,2,....)$$

such that

$$0 < \mu_n \downarrow 0, \frac{\mu_n}{\delta_n} \downarrow 0 \text{ when} n \to \infty. \quad (13)$$

Then

$$x_n^* := x^*(\mu_n, \delta_n) \underset{n \to \infty}{\to} x^{**}, u_n^* := u^*(\mu_n, \delta_n) \underset{n \to \infty}{\to} u^{**} \Big\} \quad (14)$$

where $x^{**} \in X^*$ is the solution of the original problem (7) with the minimal weighted norm which is unique, i.e.,

$$\|x^{**}\| \leq \|x^*\| \text{ for all } x^* \in X^* \Big\} \quad (15)$$

and

$$u^{**} = b_1 - V_1 x^{**}. \quad (16)$$

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Proof. See Appendix A.

We also need the following corolary.

Corolary 7. *Under the assumptions of the Theorem 6 above there exist positive constants* C_μ *and* C_δ *such that*

$$\|x_n^* - x_m^*\| + \|u_n^* - u_m^*\| \le C_\mu |\mu_n - \mu_m| + C_\delta |\delta_n - \delta_m|. \tag{17}$$

4.2. The gradient method

Consider the following recurrent procedure for finding the extremal point $z^{**} = \begin{pmatrix} x^{**} \\ u^{**} \end{pmatrix}$:

$$z_n = \left[z_{n-1} - \gamma_n \frac{\partial}{\partial z} \mathsf{F}_{\mu_n,\delta_n}(z_{n-1}) \right]_+, z := \begin{pmatrix} x \\ u \end{pmatrix} \Bigg\} \tag{18}$$

where $[z_i]_+ = \begin{cases} z_i & \text{if } z_i \ge 0 \\ 0 & \text{if } z_i < 0 \end{cases}$ and

$$\frac{\partial}{\partial z} \mathsf{F}_{\mu_n,\delta_n}(z_{n-1}) = \begin{pmatrix} \frac{\partial}{\partial x} \mathsf{F}_{\mu_n,\delta_n}(x_{n-1}, u_{n-1}) \\ \frac{\partial}{\partial u} \mathsf{F}_{\mu_n,\delta_n}(x_{n-1}, u_{n-1}) \end{pmatrix} = \begin{pmatrix} \mu_n \frac{\partial}{\partial x} f(x_{n-1}) + V_0^{\mathsf{T}} [V_0 x_{n-1} - b_0] + \\ V_1^{\mathsf{T}} [V_1 x_{n-1} - b_1 + u_{n-1}] + \delta_n x_{n-1} \\ V_1 x_{n-1} - b_1 + u_{n-1} \end{pmatrix}$$

Remark 8. *(convergence of the gradient method)If*

$$\sum_{n=0}^{\infty} \gamma_n \delta_n = \infty, \frac{\gamma_n}{\delta_n} \underset{n\to\infty}{\to} 0, \frac{|\mu_n - \mu_{n-1}| + |\delta_n - \delta_{n-1}|}{\gamma_n \delta_n} \underset{n\to\infty}{\to} 0 \tag{19}$$

then

$$W_n := \|z_n - z_n^*\|^2 \underset{n\to\infty}{\to} 0. \tag{20}$$

Let us select the parameters of the algorithm (18) as follows:

$$\delta_n = \begin{cases} \delta_0 & \text{if } n \le n_0 \\ \delta_0 \frac{[1 + \ln(n - n_0)]}{(1 + n - n_0)^\delta} & \text{if } n > n_0 \end{cases}, \quad \mu_n = \begin{cases} \mu_0 & \text{if } n < n_0 \\ \frac{\mu_0}{(1 + n - n_0)^\mu} & \text{if } n \ge n_0 \end{cases}$$
$$\gamma_n = \begin{cases} \gamma_0 & \text{if } n < n_0 \\ \frac{\gamma_0}{(1 + n - n_0)^\gamma} & \text{if } n \ge n_0 \end{cases}, \quad \delta, \mu, \gamma > 0, \delta_0, \mu_0, \gamma_0 > 0 \tag{21}$$

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To guarantee the convergence of the suggested procedure, by the property $\frac{\mu_n}{\delta_n} \underset{n\to\infty}{\to} 0$ and by the conditions (19) we should have

$$\delta \le \mu, \gamma \ge \delta, \gamma + \delta \le 1. \tag{22}$$

Let us prove the following simple result.

Remark 9. *Suppose that for a nonnegative sequence $\{u_n\}$ the following recurrent inequality holds*

$$u_n \le u_{n-1}(1-\alpha_n) + \beta_n \tag{23}$$

where numerical sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfies

$$\alpha_n \in (0,1], \beta_n \ge 0, \upsilon_n > 0 \text{ for all } n = 0,1...$$
$$\sum_{n=0}^{\infty} \alpha_n = \infty, \sum_{n=0}^{\infty} \beta_n \upsilon_n < \infty, \lim_{n\to\infty} \frac{\upsilon_n - \upsilon_{n-1}}{\alpha_n \upsilon_n} := \theta < 1 \tag{24}$$

Then

$$u_n = o(\upsilon_n^{-1}) \tag{25}$$

5. Numerical example

5.1. A priori data required for the solution of the problem

- $p^l \in [p_-^l, p_+^l], q^l \in [q_-^l, q_+^l], l = 1,\ldots,\mathcal{N}; p^1 \in [1000,9000]$, $p^2 \in [4100,15000], p^3 \in [10100,340000], q^1 \in [50,300], q^2 \in [100,500], q^3 \in [150,700]$
- $c^l(q^l) := \theta_c^l q^l, \theta^l \ge 0, l = 1,\ldots,\mathcal{N}; \theta_c^1 = 0.1, \theta_c^2 = 0.1, \theta_c^3 = 0.1.$
- $r^l(p^l q^l) := \theta_r^l p^l q^l, l = 1,\ldots,\mathcal{N}. \theta_r^1 = 0.1, \theta_r^2 = 0.2, \theta_r^1 = 0.3.$
- $p_0^1 = 4500; p_0^2 = 10000; p_0^3 = 180000; q_0^1 = 150; q_0^2 = 300; q_0^3 = 450.$

5.2. Linear constraints interpretation

Let us represent the problem above

$$\mathcal{P} := \left\{ v^*(\lambda) := \arg \max_{v \in V_{adm} = P_{adm} \otimes Q_{adm}} W(v \mid \lambda), \lambda \in S^{\mathcal{N}} \right\}$$

Where

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$$W(v\,|\,\lambda) := \lambda_1 U^1\left(p^l, q^l\right) + \sum_{l=2}^{\mathcal{N}} \lambda_l U^l\left(p^l, p^{l-1}, q^l\right)$$
$$v = \left\{(p,q) : p = (p_1,\dots,p_{\mathcal{N}})^{\mathsf{T}} \text{ and} q = (q_1,\dots,q_{\mathcal{N}})^{\mathsf{T}}\right\}$$
$$S^{\mathcal{N}} := \left\{\lambda \in \mathrm{R}^{\mathcal{N}} : \lambda_l \in [0,1], \sum_{l=1}^{\mathcal{N}} \lambda_l = 1\right\}$$

in the form of a poly-linear function optimization with linear constraints, namely,

$$W(v\,|\,\lambda) \to \max_{v} \qquad A_{ineq} v \le b_{eq}, v \ge 0$$

where (for $\mathcal{N} = 3$)

$$A_{ineq} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & 1 & \cdot & & \\ \cdot & & \cdot & 1 & \cdot & \\ \cdot & & & \cdot & 1 & 0 \\ 0 & \cdot & \cdot & \cdot & 0 & 1 \\ -1 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & -1 & \cdot & & & \\ & \cdot & -1 & \cdot & & \\ & & \cdot & -1 & \cdot & \\ \cdot & & & \cdot & -1 & \cdot \\ 0 & \cdot & & & 0 & -1 \end{bmatrix}, \qquad b_{ineq} = \begin{pmatrix} p_+^1 \\ p_+^2 \\ p_+^3 \\ q_+^1 \\ q_+^2 \\ q_+^3 \\ -p_-^1 \\ -p_-^2 \\ -p_-^3 \\ -q_-^1 \\ -q_-^2 \\ -q_-^3 \end{pmatrix}$$

5.3. Regularized penalty function

Let us introduce the following regularized penalty function:

$$\mathsf{F}_{\mu,\delta}(v,u) := \mu W(v\,|\,\lambda) + \tfrac{1}{2}\left\|A_{ineq} v - b_{ineq} + u\right\|^2 + \tfrac{\delta}{2}\left(\|v\|^2 + \kappa\|u\|^2\right)$$

for which we have

$$\frac{\partial}{\partial v}\mathsf{F}_{\mu,\delta}(v,u) = \mu \frac{\partial}{\partial v} W(v\,|\,\lambda) + A_{ineq}^{\text{ú}}\left(A_{ineq} v - b_{ineq} + u\right) + \delta v$$
$$\frac{\partial}{\partial u}\mathsf{F}_{\mu,\delta}(v,u) = A_{ineq} v - b_{ineq} + (1 + \delta\kappa)u$$

with

$$\frac{\partial}{\partial v} W(v\,|\,\lambda) = \begin{pmatrix} \frac{\partial}{\partial p_1} W(v\,|\,\lambda) \\ \frac{\partial}{\partial p_2} W(v\,|\,\lambda) \\ \frac{\partial}{\partial p_3} W(v\,|\,\lambda) \\ \frac{\partial}{\partial q_1} W(v\,|\,\lambda) \\ \frac{\partial}{\partial q_2} W(v\,|\,\lambda) \\ \frac{\partial}{\partial q_3} W(v\,|\,\lambda) \end{pmatrix} = \begin{pmatrix} \lambda_1\left(q^1 - \theta_r^1 q^1\right) + \lambda_2(-q^2) \\ \lambda_2\left(q^2 - \theta_r^2 q^2\right) + \lambda_3(-q^3) \\ \lambda_3\left(q^3 - \theta_r^3 q^3\right) \\ \lambda_1\left(p^1 - \theta_c^1 - \theta_r^1 p^1\right) \\ \lambda_2\left((p^2 - p^1) - \theta_c^2 - \theta_r^2 p^2\right) \\ \lambda_3\left((p^3 - p^2) - \theta_c^3 - \theta_r^3 p^3\right) \end{pmatrix}$$

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5.4. The recurrent procedure

For maximizing the utility let us consider the following recurrent procedure for finding the extremal point $z^{**} = \begin{pmatrix} v^{**} \\ u^{**} \end{pmatrix}$:

$$z_n = \left[z_{n-1} + \gamma_n \frac{\partial}{\partial z} \mathsf{F}_{\mu_n,\delta_n}(z_{n-1}) \right]_+, z := \begin{pmatrix} v \\ u \end{pmatrix} \Big\} \tag{29}$$

where $[z_i]_+ = \begin{cases} z_i & \text{if} \quad z_i \geq 0 \\ 0 & \text{if} \quad z_i < 0 \end{cases}$ and the first-order derivative of $\mathsf{F}_{\mu_n,\delta_n}(z_{n-1})$

$$\frac{\partial}{\partial z} \mathsf{F}_{\mu_n,\delta_n}(z_{n-1}) = \begin{pmatrix} \frac{\partial}{\partial v} \mathsf{F}_{\mu_n,\delta_n}(v_{n-1}, u_{n-1}) \\ \frac{\partial}{\partial u} \mathsf{F}_{\mu_n,\delta_n}(v_{n-1}, u_{n-1}) \end{pmatrix} = \begin{pmatrix} \lambda_1\left(q^1 - \theta_r^1 q^1\right) + \lambda_2(-q^2) \\ \lambda_2\left(q^2 - \theta_r^2 q^2\right) + \lambda_3(-q^3) \\ \lambda_3\left(q^3 - \theta_r^3 q^3\right) \\ p^1 - \theta_c^1 - \theta_r^1 p^1 \\ \lambda_2\left(p^2 - \theta_c^2 - \theta_r^2 p^2\right) \\ \lambda_3\left(p^3 - \theta_c^3 - \theta_r^3 p^3\right) \\ ========== ===== \\ A_{ineq} v - b_{ineq} + (1 + \delta\kappa) u \end{pmatrix}$$

Figure 4. Transfer price for a multidivisional firm.

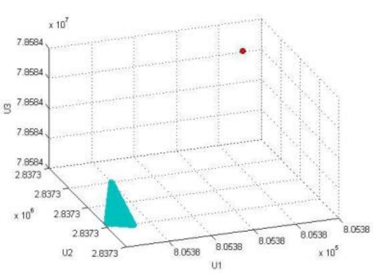

Figure 4 presents the Pareto front of the transfer pricing. Computing the strong Nash equilibrium, we have that $\lambda^* = [0.0256, 0.0256, 0.9487]$, $p^* = [0.0505, 0.1497, 1.9645] \times 10^5$, $q^* = [177.3265, 409.3044, 641.2830]$, $\mathbf{U}^* = [0.0805, 0.2837, 7.8584] \times 10^7$, $W^* = 7.4648 \times 10^7$.

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6. Conclusion and future work

The transfer pricing problem results essential in resource allocation and global firm profit for a multidivisional firm which employs optimization goals to set internal prices of goods that are sold between subsidiaries or divisions of the firm. Usually, each division estimates the revenues or profits of each opportunity and selects the best opportunity. As a result, the transfer prices system focuses on the maximization of divisional income, leading the divisions to achieve the goals of the firm through sub-optimization. However, sub-optimizations by individual divisions do not result in an optimum for the firm.This paper presented a new modeling framework for transfer pricing schemes based on cooperative game theory considering cost and tax policies. The study provided decision makers with a useful tool for supporting the design of global strategies for transfer-pricing, considering different variables of optimization (price and quantity) and establishes a flexible model after cost-taxes profitable configuration. The proposed model integrate financial issues for vertically integrated supply chains. There are relatively few models of global profit maximization reported in the literature for large-scale organizations. Accordingly, we consider that the implementation of transfer pricing models for global profit maximization represents a fundamental opportunity that must be seriously considered by large-scale multidivisional firms.

We proposed a method for computing the transfer pricing for a multidivisional firm based on a multi-objective approach for representing a cooperative game. For computing the strong Nash equilibrium we employed the minimal distance to the utopia point. We suggested the use of the Tikhonov's regularization to guarantee the convergence to a single (strong) equilibrium point. We proposed a method based on a penalized programming approach for computing the regularized strategies of the game. We developed a programming method to solve the successive single-objective constrained problems that arise from taking the regularized functional of the game. In addition, we implemented a recurrent procedure based on the gradient method to solve the regularized problem and finding the Pareto optimal strategies. We proved that in the regularized problem the functional of the game decreases and finally converges, proving the existence and uniqueness of strong Nash equilibrium. We also presented the convergence conditions and compute the estimate rate of convergence of variables μ and δ corresponding to the step size parameter of the regularized penalty method. We provided all the details needed to implement the method in an efficient and numerically stable way.

In terms of future work, there exist a number of challenges left to address. One interesting technical challenge is that of developing a numerical method for implementing the results presented in this paper. We also are considering to extend the

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game theoretic model described in Section 3 for supporting different consideration about transfer pricing. One interesting empirical challenge would be to run a long-term controlled experiment.

Appendix A.

Proof. [**Teorem 6**]

First, let us prove that the Hessian matrix H associated with the penalty function (10) is strictly positive definite for any positive μ and δ, i.e., we prove that for all $x \in \mathrm{R}^N$ and $u \in \mathrm{R}^{M_1}$

$$H = \begin{bmatrix} \frac{\partial^2}{\partial x^2} \mathsf{F}_{\mu,\delta}(x,u) & \frac{\partial^2}{\partial u \partial x} \mathsf{F}_{\mu,\delta}(x,u) \\ \frac{\partial^2}{\partial x \partial u} \mathsf{F}_{\mu,\delta}(x,u) & \frac{\partial^2}{\partial u^2} \mathsf{F}_{\mu,\delta}(x,u) \end{bmatrix} > 0 \tag{A.1}$$

To prove that, by the Schur lemma [23], it is necessary and sufficient to prove that

$$\frac{\partial^2}{\partial x^2} \mathsf{F}_{\mu,\delta}(x,u) > 0, \qquad \frac{\partial^2}{\partial u^2} \mathsf{F}_{\mu,\delta}(x,u) > 0$$

$$\frac{\partial^2}{\partial x^2} \mathsf{F}_{\mu,\delta}(x,u) > \frac{\partial^2}{\partial u \partial x} \mathsf{F}_{\mu,\delta}(x,u) \cdot \left[\frac{\partial^2}{\partial u^2} \mathsf{F}_{\mu,\delta}(x,u) \right]^{-1} \frac{\partial^2}{\partial x \partial u} \mathsf{F}_{\mu,\delta}(x,u) \tag{A.2}$$

We have

$$\frac{\partial^2}{\partial x^2} \mathsf{F}_{\mu,\delta}(x,u) = \mu \frac{\partial^2}{\partial x^2} f(x) + V_0^{\mathsf{T}} V_0 + V_1^{\mathsf{T}} V_1 + \delta I_{N \times N} \geq$$

$$\mu \frac{\partial^2}{\partial x^2} f(x) + \delta I_{N \times N} \geq \delta \left(1 + \frac{\mu}{\delta} \lambda^- \right) I_{N \times N} > 0 \forall \delta_n > 0$$

$$\lambda^- := \min_{x \in X_{adm}} \lambda_{\min} \left(\frac{\partial^2}{\partial x^2} f(x) \right) \frac{\partial^2}{\partial u^2} \mathsf{F}_{\mu,\delta}(x,u) = I_{M_1 \times M_1} > 0$$

By the Schur lemma

$$\frac{\partial^2}{\partial x^2} \mathsf{F}_{\mu,\delta}(x,u) = \mu \frac{\partial^2}{\partial x^2} f(x) + V_0^{\mathsf{T}} V_0 + V_1^{\mathsf{T}} V_1 + \delta I_{N \times N} > \frac{\partial^2}{\partial u \partial x} \mathsf{F}_{\mu,\delta}(x,u) \cdot$$

$$\left[\frac{\partial^2}{\partial u^2} \mathsf{F}_{\mu,\delta}(x,u) \right]^{-1} \frac{\partial^2}{\partial x \partial u} \mathsf{F}_{\mu,\delta}(x,u) = (1+\delta)^{-1} V_1^{\mathsf{T}} V_1$$

implying, $\mu \frac{\partial^2}{\partial x^2} f(x) + V_0^{\mathsf{T}} V_0 + \frac{\delta}{1+\delta} V_1^{\mathsf{T}} V_1 + \delta I_{N \times N} > 0$ which holds for any $\delta > 0$ by the condition (13) since

$$(\mu \lambda^- + \delta) I_{N \times N} + V_0^{\mathsf{T}} V_0 + \frac{\delta}{1+\delta} V_1^{\mathsf{T}} V_1 \geq \delta \left(1 + \frac{\mu}{\delta} \lambda^- \right) I_{N \times N} = \delta (1 + o(1)) I_{N \times N} > 0$$

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So, $H>0$ which means that the penalty function (10) is strongly convex and, hence, has a unique minimal point defined below as $x^*(\mu,\delta)$ and $u^*(\mu,\delta)$.By the strictly convexity property (30) for any $z:=\begin{pmatrix} x \\ u \end{pmatrix}$ and any vector

$z_n^*:=\begin{pmatrix} x_n^*=x^*(\mu_n,\delta_n) \\ u_n^*=u^*(\mu_n,\delta_n) \end{pmatrix}$ for the function $\mathsf{F}_{\mu,\delta}(x,u)=\mathsf{F}_{\mu,\delta}(z)$ we have

$$0\geq (z_n^*-z)^{\mathsf{T}}\frac{\partial}{\partial z}\mathsf{F}_{\mu_n,\delta_n}(z_n^*)=\mu_n(x_n^*-x)^{\mathsf{T}}\frac{\partial}{\partial x}f(x_n^*)$$
$$+[V_0(x_n^*-x)]^{\mathsf{T}}[V_0x_n^*-b_0]+[V_1(x_n^*-x)]^{\mathsf{T}}\cdot[V_1x_n^*-b_1+u_n^*]+\delta_n(x_n^*-x)^{\mathsf{T}}x_n^*+$$
$$(u_n^*-u)^{\mathsf{T}}[V_1x_n^*-b_1+(1+\delta)u_n^*] \tag{A.3}$$

Selecting in (A.3) $x:=x^*\in X^*$ (x^* is one of admissible solutions such that $V_0x^*=b_0$) and $u:=b_1-V_1x_n^*$ we obtain

$$0\geq \mu_n(x_n^*-x^*)^{\mathsf{T}}\frac{\partial}{\partial x}f(x_n^*)+\|V_0(x_n^*-x^*)\|^2+\|V_1(x_n^*-x^*)\|^2+\delta_n(x_n^*-x^*)^{\mathsf{T}}x_n^*+$$
$$(1+\delta)^{-1}\|V_1x_n^*-b_1+(1+\delta)u_n^*\|^2+\delta_n(u_n^*-b_1-V_1x_n^*)^{\mathsf{T}}u_n^*$$

Dividing both sides of this inequality by δ_n we get

$$0\geq \frac{\mu_n}{\delta_n}(x_n^*-x^*)^{\mathsf{T}}\frac{\partial}{\partial x}f(x_n^*)+\frac{1}{\delta_n}\Big(\|V_0x_n^*-b_0\|^2+$$
$$\|V_1(x_n^*-x^*)\|^2+\|V_1x_n^*-b_1+(1+\delta)u_n^*\|^2\Big)+(x_n^*-x^*)^{\mathsf{T}}x_n^*+(u_n^*-b_1-V_1x_n^*)^{\mathsf{T}}u_n^*. \tag{A.4}$$

Notice also that from (A.3), taking $x=x_n^*$ and $u=0$, it follows

$$0\geq\left[\left\|\sqrt{1+\delta}u_n^*+\frac{(V_1x_n^*-b_1)}{2\sqrt{1+\delta}}\right\|^2-\left\|\frac{(V_1x_n^*-b_1)}{2\sqrt{1+\delta}}\right\|^2\right]$$

implying

$$1\geq\left\|e+2(1+\delta)u_n^*\|(V_1x_n^*-b_1)\|^{-1}\right\|^2,\ \|e\|=1$$

which means that the sequence $\{u_n^*\}$ is bounded. In view of this and taking into account that by the supposition (13) $\frac{\mu_n}{\delta_n}\underset{n\to\infty}{\to}0$, from (A.4) it follows

$$\text{Const}=\limsup_{n\to\infty}\begin{pmatrix}\left|(x_n^*-x^*)^{\mathsf{T}}x_n^*\right|+\\ \left|(u_n^*-b_1-V_1x_n^*)^{\mathsf{T}}u_n^*\right|\end{pmatrix}\geq \tag{A.5}$$
$$\limsup_{n\to\infty}\frac{1}{\delta_n}\Big(\|V_0x_n^*-b_0\|^2+\|V_1(x_n^*-x^*)\|^2+(1+\delta_n)^{-1}\|V_1x_n^*-b_1+(1+\delta_n)u_n^*\|^2\Big)$$

From (A.5) we may conclude that

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$$\left\|V_0 x_n^* - b_0\right\|^2 + \left\|V_1\left(x_n^* - x^*\right)\right\|^2 + (1+\delta_n)^{-1}\left\|V_1 x_n^* - b_1 + (1+\delta_n)u_n^*\right\|^2 = O(\delta_n) \qquad \text{(A.6)}$$

and

$$V_0 x_\infty^* - b_0 = 0, \qquad V_1 x_\infty^* - V_1 x^* = V_1 x_\infty^* - b_1 + u_\infty^* = 0$$

where $x_\infty^* \in X^*$ is a partial limit of the sequence $\{x_n^*\}$ which, obviousely, may be not unique. The vector u_∞^* is also a partial limit of the sequence $\{u_n^*\}$.Denote by $\hat{x}_n$ the projection of x_n^* to the set X_{adm}, namely,

$$\hat{x}_n = \Pr_{X_{adm}}\left(x_n^*\right) \qquad \text{(A.7)}$$

and show that

$$\left\|x_n^* - \hat{x}_n\right\| \le C\sqrt{\delta_n}, C = \text{const} > 0. \qquad \text{(A.8)}$$

From (A.6) we have

$$\left\|V_1 x_n^* - b_1 + u_n^*\right\| \le C_1\sqrt{\delta_n}, C_1 = \text{const} > 0$$

implying

$$V_1 x_n^* - b_1 \le C_1\sqrt{\delta_n}\, e - u_n^* \le C_1\sqrt{\delta_n}\, e, \|e\| = 1$$

where the vector inequality is treated in component-wise sense. Therefore

$$\left\|x_n^* - \hat{x}_n\right\|^2 \le \max_{V_1 x - b_1 \le C_1\sqrt{\delta_n} e, x \in X_{adm}} \min_{y \in X_{adm}} \|x - y\|^2 := d(\delta_n).$$

Introduce the new variable

$$\tilde{x} := (1-\nu_n)x + \nu_n x \in X_{adm} \qquad \text{(A.9)}$$

where by the Slater condition (12) $0 < \nu_n := \dfrac{C_1\sqrt{\delta_n}}{C_1\sqrt{\delta_n} + \min\limits_{j=1,\dots,M_1}\left|(V_1 x - b_1)_j\right|} < 1$. For new variable $x = \dfrac{\tilde{x} - \nu_n x}{1-\nu_n}$ we have

$$V_1\tilde{x} - b_1 \le \frac{C_1\sqrt{\delta_n}}{C_1\sqrt{\delta_n} + \min\limits_{j=1,\dots,M_1}\left|(V_1 x - b_1)_j\right|} \cdot \left(\min_{j=1,\dots,M_1}\left|(V_1 x - b_1)_j\right| e + (V_1 x - b_1)\right) \le 0$$

and therefore

$$d(\delta_n) \le \max_{V_1\tilde{x} - b_1 \le 0, \tilde{x} \in Xadm}\left\|\frac{\tilde{x} - \nu_n x}{1-\nu_n} - \tilde{x}\right\|^2 = \frac{\nu_n^2}{(1-\nu_n)^2}\max_{V_1\tilde{x} - b_1 \le 0, \tilde{x} \in Xadm}\|\tilde{x} - x\|^2 \le C_2\delta_n, \quad 0 < C_2 < \infty.$$

In view of that $\left\|x_n^* - \hat{x}_n\right\| \le \sqrt{d(\delta_n)} \le \sqrt{C_2}\sqrt{\delta_n}$ which proves (A.8).The last step is to prove the inequality

$$0 \ge \left(x_\infty^* - x^*\right)^{\mathsf{T}} x_\infty^* \text{ for any } x_\infty^* \le X^*. \qquad \text{(A.10)}$$

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From (A.4) we get

$$0 \geq \left(x_n^* - x^*\right)^{\mathsf{T}} \frac{\partial}{\partial x} f\left(x_n^*\right) + \frac{1}{\mu_n}\left(\left\|V_0 x_n^* - b_0\right\|^2 + \left\|V_1\left(x_n^* - x^*\right)\right\|^2\right) + \frac{\delta_n}{\mu_n}\left(x_n^* - x^*\right)^{\mathsf{T}} x_n^* + \frac{1}{\mu_n}\left\|V_1 x_n^* - b_1 + u_n^*\right\|^2 \geq \left(x_n^* - x^*\right)^{\mathsf{T}} \frac{\partial}{\partial x} f\left(x_n^*\right) + \frac{\delta_n}{\mu_n}\left(x_n^* - x^*\right)^{\mathsf{T}} x_n^*. \tag{A.11}$$

By the strong convexity property we have (see Corollary 21.4 in [23]) $(x-y)^{\mathsf{T}}\left(\frac{\partial}{\partial x} f(x) - \frac{\partial}{\partial x} f(y)\right) \geq 0$ for any $x, y \in \mathrm{R}^N$ which, in view of the property (A.8), implies

$$\left(x_n^* - \hat{x}_n\right)^{\mathsf{T}} \frac{\partial}{\partial x} f\left(x_n^*\right) = O\left(\sqrt{\delta_n}\right)\left(\hat{x}_n - x^*\right)^{\mathsf{T}} \cdot \frac{\partial}{\partial x} f\left(\hat{x}_n\right) \geq \left(\hat{x}_n - x^*\right)^{\mathsf{T}} \frac{\partial}{\partial x} f\left(x^*\right) \geq 0$$

and

$$\left(x_n^* - x^*\right)^{\mathsf{T}} \frac{\partial}{\partial x} f\left(x_n^*\right) = \left(x_n^* - \hat{x}_n\right)^{\mathsf{T}} \frac{\partial}{\partial x} f\left(x_n^*\right) +$$
$$\left(\hat{x}_n - x^*\right)^{\mathsf{T}} \frac{\partial}{\partial x} f\left(x_n^*\right) = O\left(\sqrt{\delta_n}\right) + \left(\hat{x}_n - x^*\right)^{\mathsf{T}}\left(\frac{\partial}{\partial x} f\left(x_n^*\right) - \frac{\partial}{\partial x} f\left(\hat{x}_n\right)\right)$$
$$\left(\hat{x}_n - x^*\right)^{\mathsf{T}} \frac{\partial}{\partial x} f\left(\hat{x}_n\right) \geq O\left(\sqrt{\delta_n}\right) - \left\|\hat{x}_n - x^*\right\|\left\|\frac{\partial}{\partial x} f\left(x_n^*\right) - \frac{\partial}{\partial x} f\left(\hat{x}_n\right)\right\|$$

Since any polinomial function is Lipschitz continuos on any bounded compact set, we can conclude that

$$\left\|\frac{\partial}{\partial x} f\left(x_n^*\right) - \frac{\partial}{\partial x} f\left(\hat{x}_n\right)\right\| \leq \mathrm{Const}\left\|x_n^* - \hat{x}_n\right\| = O\left(\sqrt{\delta_n}\right)$$

which gives $\left(x_n^* - \hat{x}^*\right)^{\mathsf{T}} \frac{\partial}{\partial x} f\left(x_n^*\right) = O\left(\sqrt{\delta_n}\right)$ which by (A.11) leads to

$$0 \geq \left(x_n^* - \hat{x}_n\right)^{\mathsf{T}} \frac{\partial}{\partial x} f\left(x_n^*\right) + \frac{\delta_n}{\mu_n}\left(x_n^* - x^*\right)^{\mathsf{T}} x_n^* = O\left(\sqrt{\delta_n}\right) + \frac{\delta_n}{\mu_n}\left(x_n^* - x^*\right)^{\mathsf{T}} x_n^* \tag{A.12}$$

Dividing both side of the inequality (A.12) by $\frac{\mu_n}{\delta_n}$ and in view (A.8) we finally obtain

$$0 \geq O\left(\frac{\mu_n}{\sqrt{\delta_n}}\right) + \left(x_n^* - x^*\right)^{\mathsf{T}} x_n^* = o(1)\sqrt{\delta_n} + \left(x_n^* - x^*\right)^{\mathsf{T}} x_n^* \tag{A.13}$$

This, by (13), for $n \to \infty$ leads to (A.10). Finally, for any $x^* \leq X^*$ it implies

$$0 \geq \left(x_\infty^* - x^*\right)^{\mathsf{T}} x_\infty^* = \left\|x_\infty^* - x^*\right\|^2 + \left(x_\infty^* - x^*\right)^{\mathsf{T}} x^* \geq \left(x_\infty^* - x^*\right)^{\mathsf{T}} x^*$$

This inequality exactly represents the necessary and sufficient condition that the point x^* is the minimum point of the function $\left\|x_\infty^*\right\|^2$ on the set X^*. Obliviously, this point is unique and has a minimal norm among all possible partial limits x_∞^*.

Theorem is proven.

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