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Computing Tropical Varieties Over Fields with Valuation

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Abstract

We show how the tropical variety of an ideal $I \unlhd K[x_1, \ldots, x_n]$ over a field K with non-trivial discrete valuation can always be traced back to the tropical variety of an ideal $\pi^{-1}I \unlhd R[\![t]\!][x_1, \ldots, x_n]$ over some dense subring R in its ring of integers. We show that this connection is compatible with the Gröbner polyhedra covering them. Combined with previous works, we thus obtain a framework for computing tropical varieties over general fields with valuations, which relies on the existing theory of standard bases if $\pi^{-1}I$ is generated by elements in $R[t, x_1, \ldots, x_n]$.

Keywords Tropical geometry · Tropical varieties · Gröbner fans · Standard bases · Valued fields

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1 Introduction

Given a polynomial ideal I over a field K with a non-trivial valuation $\nu : K \to \mathbb{R} \cup \{\infty\}$, its tropical variety $\mathcal{T}(I)$ is commonly described as the combinatorial shadow of its vanishing set over the algebraic closure of K. Tropical varieties arise naturally in many contexts in mathematics [1,22] and beyond, such as phylogenetic trees in biology [24, §4], product-mix auctions in economics [2,32] or finiteness of central configurations in the 5-body problem in physics [12].

However, computing tropical varieties is an algorithmically highly challenging task, requiring sophisticated techniques from computer algebra and convex geometry. The first techniques were developed by Bogart, Jensen, Speyer, Sturmfels and Thomas [5] for the rational function field over the complex numbers $\mathbb{C}(t)$ using classical Gröbner basis methods. More recently, Chan and Maclagan [7] generalised the notion of Gröbner bases to general fields with valuation in order to compute tropical varieties thereover. The linchpin of both works is the ability to compute initial ideals. Moreover, significant advances have been made in specific parts of the computations: Chan [6], Hofmann and Ren [13], Sommars and Verschelde [29] all worked on improving the main bottleneck that is the computation of tropical links. The first two works developed new algorithms based on projections and intersections, respectively, whereas the latter improved the computation of so-called tropical prevarieties which was essential for the original algorithm. At the same time, Vaccon [34] showed that Matrix-F5 ideas can be applied to improve the performance of the generalised Gröbner bases computation.

In contrast, this article revisits the problem on a more fundamental level. As in [7], the overall goal is to develop a framework for general fields with valuation in which the original algorithms in [5] work almost ad verbum. However, instead of introducing a new notion of Gröbner bases, we aim to base it on the existing theory of standard bases [25]. The key idea is to use Cohen's Structure Theorem and replace the valued field K with a power series ring $R[\![t]\!]$ with its natural valuation. This replaces the original ideal $I \trianglelefteq K[x]$ with an ideal in $R[\![t]\!][x]$, which is generated by polynomials in both t and x under mild assumptions on I. Our approach is to a certain extent equivalent to that of Chan and Maclagan, which can be seen from the fact that we naturally obtain an algorithm for computing their Gröbner bases. However, we can leverage existing implementations, such as in the computer algebra system SINGULAR [8], for a better performance (see Timings 1).

Our framework relies heavily on two previous works: In [19], we introduced standard bases for ideals in $R[\![t]\!][x]$, whose elements are multivariate polynomials in x and univariate power series in t over a coefficient ring R. In [20], we introduced Gröbner fans for ideals in $R[\![t]\!][x]$, which are a natural amalgamation of the existing notions of Gröbner fans for power series rings [3,26,33] and Gröbner fans for polynomial rings over coefficient rings [23]. In both works, special emphasis is put on ideals in $R[\![t]\!][x]$ generated by polynomials in t and x. For those, our standard bases coincide with the existing notion of standard bases for polynomials over coefficient rings and our algorithms consist of a finite sequence of basic polynomial arithmetic.

This article is organised as follows: First, in Sect. 2, we recall Cohen's Structure Theorem and use it to establish a bijection between the tropical variety of an ideal in $K[x]$ and the tropical variety of a corresponding ideal in $R[\![t]\!][x]$ (see Theorem 4).

Next, in Sect. 3, we show that this bijection is compatible with the polyhedral structures covering the respective tropical varieties (see Corollary 3). Finally, in Sect. 4, we explain how the corresponding ideal in $R[\![t]\!][x]$ can be computed from the original ideal in $K[x]$.

Furthermore, modified versions of the algorithms in [5] in our framework have been implemented in the SINGULAR library `tropical.lib` [15], relying on GFANLIB [14,16] for computations in convex geometry (see Example 5). They are publicly available as part of the official SINGULAR distribution, and a detailed account on the modified algorithms can be found in [27].

2 Tracing Tropical Varieties to a Trivial Valuation

The aim of this section is to show how tropical varieties over a valued field K can be traced back to tropical varieties over a power series ring $R[\![t]\!]$ as in Convention 2. The linchpin is to show how initial ideals over one can be described through initial ideals over the other, and the remaining results then follow naturally from this. Let us start by recalling Cohen's Structure Theorem.stop

Theorem 1 (Cohen's Structure Theorem, [21, §23 Corollary 5]) *Let R be a Noetherian ring, $P = \langle p \rangle \trianglelefteq R$ an ideal and $\widehat{R}$ the P-adic completion of R. Then*

$$\widehat{R} \cong R[\![t]\!] \big/ \langle p - t \rangle,$$

where $R[\![t]\!]$ denotes the ring of formal power series in t with coefficients in R.

Convention 2 *For the remainder of the article, fix a field K with a non-trivial discrete valuation $v : K \to \mathbb{R} \cup \{\infty\}$. Without loss of generality, we may assume that K is complete in the topology induced by the valuation, as in our context we can always pass to its completion if it is not. Let $\mathcal{O}_K$ be its ring of integers, and let $\mathfrak{K}$ denote its residue field. Fix a uniformising parameter $p \in \mathcal{O}_K$, and $R \leq \mathcal{O}_K$ a subring that is dense in the topology induced by the valuation, i.e. $\widehat{R} = \mathcal{O}_K$. By Theorem 1, we have two exact sequences*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \langle p - t \rangle \cdot R[\![t]\!]_{\langle p-t \rangle} & \longrightarrow & R[\![t]\!]_{\langle p-t \rangle} & \longrightarrow & K & \longrightarrow & 0, \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \langle p - t \rangle \cdot R[\![t]\!] & \longrightarrow & R[\![t]\!] & \xrightarrow[t \,\mapsto\, p]{\pi} & \mathcal{O}_K & \longrightarrow & 0.
\end{array}
$$

Moreover, fix a multivariate polynomial ring $K[x] = K[x_1, \ldots, x_n]$. By abuse of notation, we also use π to refer to both the map $R[\![t]\!][x] \to \mathcal{O}_K[x]$ as well as the composition $R[\![t]\!][x] \to \mathcal{O}_K[x] \hookrightarrow K[x]$, where $R[\![t]\!][x]$ and $\mathcal{O}_K[x]$ denote the rings of polynomials in $x_1, \ldots, x_n$ with coefficients in $R[\![t]\!]$ and $\mathcal{O}_K$, respectively.

Example 1 (*p-adic numbers*) The most interesting example is the field $K := \mathbb{Q}_p$ of p-adic numbers with $\mathcal{O}_K := \mathbb{Z}_p$ the ring of p-adic integers. Then $R := \mathbb{Z} \leq \mathbb{Z}_p$ is a natural dense subring, which is computationally easy to work with. The exact sequences in Convention 2 merely reflect the presentation of p-adic integers as power series in p:

$$0 \longrightarrow \langle p - t \rangle \cdot \mathbb{Z}[\![t]\!]_{\langle p-t \rangle} \longrightarrow \mathbb{Z}[\![t]\!]_{\langle p-t \rangle} \longrightarrow \mathbb{Q}_p \longrightarrow 0,$$

$$0 \longrightarrow \langle p - t \rangle \cdot \mathbb{Z}[\![t]\!] \longrightarrow \mathbb{Z}[\![t]\!] \xrightarrow[t \mapsto p]{\pi} \mathbb{Z}_p \longrightarrow 0.$$

Example 2 Given the choice of $R \leq \mathcal{O}_K$ in Convention 2, choosing $R := \mathcal{O}_K$ is always possible. However, in many examples there are natural choices for R, which are computationally much easier to handle than $\mathcal{O}_K$ itself:

1. $K = k(\!(t)\!)$ the field of Laurent series over a field k with $\mathcal{O}_K = k[\![t]\!]$ the ring of power series over k, $R = k[t]$ and $p = t$; e.g. $k = \mathbb{F}_q$ with q a prime power, as used in [30, Sect. 7] or [17], or $k = \mathbb{Q}$ as considered in [5] (see Example 4).
2. Finite extensions K of $\mathbb{Q}_p$ and $\mathbb{F}_q(\!(t)\!)$, i.e. all local fields with non-trivial valuation, and also all higher-dimensional local fields.
3. $\mathcal{O}_K$ any completion of a localisation of a Dedekind domain R at a prime ideal $P \trianglelefteq R$, $p \in P$ a suitable element. Note that p does not need to generate P and hence $\mathcal{O}_K$ need not be the completion with respect to the ideal generated by p, e.g. $R = \mathbb{Z}[\sqrt{-5}]$, $P = \langle 2, 1 + \sqrt{-5} \rangle$ and $p = 2$.
4. For an odd choice of R, consider $K := \mathbb{Q}(s)(\!(t)\!)$ so that $\mathcal{O}_K = \mathbb{Q}(s)[\![t]\!]$. Set $R := S^{-1}\mathbb{Q}[s, t]$, where $S := \mathbb{Q}[s, t] \setminus (\langle t - 1, s \rangle \cup \langle t \rangle)$ is multiplicatively closed as the complement of two prime ideals. Then R is a non-catenarian, dense subring of $\mathcal{O}_K$.

To fix the notation, we briefly recall some basic notions in tropical geometry that are of immediate relevance to us. For an in-depth introduction to tropical geometry, we refer to the reader to [18]. For a brief survey with a view towards algebraic geometry, we recommend [9].

Definition 1 (*Initial forms, initial ideals, tropical varieties over valued fields*) For a polynomial $0 \neq f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha \cdot x^\alpha \in K[x]$ and a weight vector $w \in \mathbb{R}^n$, we define the *valued weighted degree* and *initial form* of f with respect to w to be:

$$\deg_{v,w}(f) := \max\{w \cdot \alpha - v(c_\alpha) \mid c_\alpha \neq 0\} \in \mathbb{R},$$

$$\text{in}_{v,w}(f) := \sum_{w \cdot \alpha - v(c_\alpha) \text{ maximal}} \overline{c_\alpha \cdot p^{-v(c_\alpha)}} \cdot x^\alpha \in \mathfrak{K}[x].$$

For an ideal $I \trianglelefteq K[x]$ and a weight vector $w \in \mathbb{R}^n$, we define the *initial ideal* of I with respect to w to be:

$$\text{in}_{v,w}(I) := \langle \text{in}_{v,w}(f) \mid 0 \neq f \in I \rangle \trianglelefteq \mathfrak{K}[x].$$

We refer to the set of weight vectors for which the initial ideal contains no monomial as the *tropical variety* of I,

$$\mathcal{T}_v(I) := \left\{ w \in \mathbb{R}^n \mid \text{in}_{v,w}(I) \text{ monomial free} \right\}.$$

Theorem 3 (Structure Theorem for Tropical Varieties, [18, Theorem 3.3.5]) *Let $I \trianglelefteq K[x]$ define an irreducible subvariety in $(K^*)^n$ of dimension d. Then $\mathcal{T}_v(I)$ is the*

support of a pure polyhedral complex of the same dimension that is connected through codimension 1.

Next, we introduce initial forms and initial ideals in for elements and ideals in $R[\![t]\!][x]$ and show how initial ideals of ideals in $R[\![t]\!][x]$ can be used to compute the initial ideals of ideals in $K[x]$.

Definition 2 (*Initial forms, initial ideals*) Given an element $0 \neq f = \sum_{\beta,\alpha} c_{\alpha,\beta} \cdot t^\beta x^\alpha \trianglelefteq R[\![t]\!][x]$ and a weight vector $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$, we define the *weighted degree* and *initial form* of f with respect to w to be

$$\deg_w(f) := \max\{w \cdot (\beta, \alpha) \mid c_{\beta,\alpha} \neq 0\} \in \mathbb{R},$$
$$\operatorname{in}_w(f) := \sum_{w \cdot (\beta,\alpha) \text{ maximal}} c_{\alpha,\beta} \cdot t^\beta x^\alpha \in R[t, x],$$

where $R[t, x]$ denotes the ring of polynomials in $t, x_1, \ldots, x_n$ with coefficients in R.

Given an ideal $I \trianglelefteq R[\![t]\!][x]$ and a weight vector $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$, we define the *initial ideal* of I with respect to w to be:

$$\operatorname{in}_w(J) := \langle \operatorname{in}_w(f) \mid 0 \neq f \in J \rangle \trianglelefteq R[t, x].$$

This can be thought of as a natural extension of Definition 1 with trivial valuation on the coefficients. Note that we only allow weight vectors with negative weight in t, so that the maximum of all $w \cdot (\beta, \alpha)$ exists.

Example 3 ([6, §3.6]) Consider, over the field of 3-adic numbers $\mathbb{Q}_3$, the ideal

$$I = \langle 2x_1^2 + 3x_1x_2 + 24x_3x_4, 8x_1^3 + x_2x_3x_4 + 18x_3^2x_4 \rangle \trianglelefteq \mathbb{Q}_3[x_1, \ldots, x_4] = \mathbb{Q}_3[x],$$

and the weight vector $w := (1, 11, 3, 19)$. The initial ideal of I under the 3-adic valuation is then

$$\operatorname{in}_{v_3, w}(I) = \langle x_1^2, x_1x_3x_4, x_1x_2^2x_3, x_1x_2^4, x_3^4x_4^2 \rangle \trianglelefteq \mathbb{F}_3[x].$$

Moreover, we have

$$\pi^{-1}I = \langle 3 - t, 2x_1^2 + 3x_1x_2 + 24x_3x_4, 8x_1^3 + x_2x_3x_4 + 18x_3^2x_4 \rangle \trianglelefteq \mathbb{Z}[\![t]\!][x],$$

and for the weight vector $(-1, w) \in \mathbb{R}_{<0} \times \mathbb{R}^4$ a short standard bases computation (see Proposition 3) yields

$$\operatorname{in}_{(-1,w)}(\pi^{-1}I) = \langle 3, x_1^2, tx_1x_3x_4, t^3x_1x_2^2x_3, t^4x_1x_2^4, t^3x_3^4x_4^2 \rangle \trianglelefteq \mathbb{Z}[t, x].$$

The similarity to the initial ideal of I under the 3-adic valuation is no coincidence.

Proposition 1 *For any ideal $I \trianglelefteq \mathcal{O}_K[x]$ and any weight vector $w \in \mathbb{R}^n$, we have:*

$$\overline{\operatorname{in}_{(-1,w)}(\pi^{-1}I)}|_{t=1} = \operatorname{in}_{v,w}(I),$$

where $\overline{(\cdot)}$ denotes the canonical projection $\overline{(\cdot)} : R[x] \to \mathfrak{K}[x]$.

Proof $\supseteq$ Any term $s \in \mathcal{O}_K[x]$ is of the form $s = (\sum_\beta c_\beta p^\beta) \cdot x^\alpha$ with $p \nmid c_\beta$ for all $\beta \in \mathbb{N}$. Then the element $s' := (\sum_\beta c_\beta t^\beta) \cdot x^\alpha \in R[\![t]\!][x]$ is a natural preimage of s under π for which we have

$$\mathrm{in}_{v,w}(s) = \bar{c}_{\beta_0} \cdot x^\alpha = \overline{\mathrm{in}_{(-1,w)}(s')}\big|_{t=1}, \text{ where } \beta_0 = \min\{\beta \in \mathbb{N} \mid c_\beta \neq 0\}.$$

And because the valued weighted degree of s in $\mathcal{O}_K[x]$, i.e. the left-hand side in the following equation, and the weighted degree of s' in $R[\![t]\!][x]$, i.e. the right-hand side of the following equation, coincide,

$$\deg_w(x^\alpha) - \nu\left(\sum_\beta c_\beta p^\beta\right) = \max\{w \cdot \alpha - \beta \mid c_\beta \neq 0\} = \deg_{(-1,w)}\left(\sum_\beta c_\beta \cdot t^\beta x^\alpha\right),$$

this implies that any $f \in \mathcal{O}_K[x]$ has a preimage $f' \in R[\![t]\!][x]$ under π such that

$$\mathrm{in}_{v,w}(f) = \overline{\mathrm{in}_{(-1,w)}(f')}\big|_{t=1},$$

simply by applying the above argument to each of its terms.

$\subseteq$ Once again consider a term $s = \sum_\beta c_\beta p^\beta \cdot x^\alpha \in \mathcal{O}_K[x]$ with $p \nmid c_\beta$ for all $\beta \in \mathbb{N}$. Then any preimage of it under π is of the form $s' = \sum_\beta c_\beta t^\beta x^\alpha + r$ for some $r \in \langle p - t \rangle$.

If $\deg_{(-1,w)}(r) > \deg_{(-1,w)}(\sum_\beta c_\beta t^\beta x^\alpha)$, we have

$$\overline{\mathrm{in}_{(-1,w)}(s')}\big|_{t=1} = \overline{\mathrm{in}_{(-1,w)}(r)}\big|_{t=1} = 0,$$

since $\mathrm{in}_{(-1,w)}(r) \in \mathrm{in}_{(-1,w)}\langle p - t \rangle = \langle p \rangle$.

And if $\deg_{(-1,w)}(r) < \deg_{(-1,w)}(\sum_\beta c_\beta t^\beta x^\alpha)$, we have

$$\overline{\mathrm{in}_{(-1,w)}(s')}\big|_{t=1} = \overline{\mathrm{in}_{(-1,w)}\left(\sum_\beta c_\beta t^\beta x^\alpha\right)}\big|_{t=1} = \bar{c}_{\beta_0} \cdot x^\alpha$$
$$= \mathrm{in}_{v,w}\left(\sum_\beta c_\beta p^\beta \cdot x^\alpha\right) = \mathrm{in}_{v,w}(s),$$

where $\beta_0 := \min\{\beta \in \mathbb{N} \mid c_\beta \neq 0\}$.

Now suppose $\deg_{(-1,w)}(r) = \deg_{(-1,w)}(\sum_\beta c_\beta t^\beta x^\alpha)$. First observe that because t is weighted negatively, there can be no cancellation amongst the highest weighted terms of r and the terms of $\sum_\beta c_\beta t^\beta x^\alpha$, as the terms of $\sum_\beta c_\beta t^\beta x^\alpha$ are not divisible by p, unlike the terms of the highest weighted terms of r. Therefore, we have

$$\overline{\mathrm{in}_{(-1,w)}(s')}\big|_{t=1} = \underbrace{\overline{\mathrm{in}_{(-1,w)}\left(\sum_\beta c_\beta t^\beta x^\alpha\right)}\big|_{t=1}}_{=\mathrm{in}_{v,w}(\sum_\beta c_\beta p^\beta \cdot x^\alpha)} + \underbrace{\overline{\mathrm{in}_{(-1,w)}(r)}\big|_{t=1}}_{=\bar{0}} = \mathrm{in}_{v,w}(s).$$

Either way, we always have $\overline{\mathrm{in}_{(-1,w)}(s')}|_{t=1} \in \langle \mathrm{in}_{v,w}(s) \rangle$ for any arbitrary preimage $s' \in \pi^{-1}(s)$, and, as before, the same hence holds true for any arbitrary element $f \in \mathcal{O}_K[x]$. □

Corollary 1 *For any ideal $I \trianglelefteq K[x]$ and any weight vector $w \in \mathbb{R}^n$, we have:*

$$\overline{\mathrm{in}_{(-1,w)}(\pi^{-1}I)}|_{t=1} = \mathrm{in}_{v,w}(I).$$

Proof The statement follows from $\mathrm{in}_{v,w}(I) = \mathrm{in}_{v,w}(I \cap \mathcal{O}_K[x])$ and Proposition 1. □

Finally, we can introduce tropical varieties in $R[\![t]\!][x]$ and show how they relate to tropical varieties in $K[x]$. In particular, we note how the tropical varieties in $R[\![t]\!][x]$ that are of interest to us are pure and connected through codimension 1. This is not a given for tropical varieties over coefficient rings [18, §1.6] and very important algorithmically, as it allows us to run over it via a fan-traversal through the facets of the maximal cones.

Definition 3 *(tropical variety)* For an ideal $I \trianglelefteq R[\![t]\!][x]$, we define its *tropical variety* to be

$$\mathcal{T}(I) = \mathrm{cl}\left(\{w \in \mathbb{R}_{<0} \times \mathbb{R}^n \mid \mathrm{in}_w(I) \text{ monomial free}\}\right) \subseteq \mathbb{R}_{\leq 0} \times \mathbb{R}^n,$$

where $\mathrm{cl}(\cdot)$ denotes the closure in the Euclidean topology.

Theorem 4 *Let $I \trianglelefteq K[x]$ be an ideal. The projection $\mathbb{R}_{\leq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ induces a bijection*

$$\mathcal{T}(\pi^{-1}I) \cap (\{-1\} \times \mathbb{R}^n) \xrightarrow{\sim} \mathcal{T}_v(I)$$
$$(-1, w_1, \ldots, w_n) \longmapsto (w_1, \ldots, w_n),$$

where v is the valuation on K in Convention 2.

Proof For the bijection, we show that

$$\mathrm{in}_{(-1,w)}(\pi^{-1}I) \text{ is not monomial free} \iff \mathrm{in}_{v,w}(I) \text{ is not monomial free}.$$

$\implies$ Suppose $\mathrm{in}_{(-1,w)}(\pi^{-1}I) \trianglelefteq R[\![t]\!][x]$ contains a monomial $t^\beta x^\alpha$. By Corollary 1, we have $\mathrm{in}_{v,w}(I) = \overline{\mathrm{in}_{(-1,w)}(\pi^{-1}I)}|_{t=1}$, which means $\mathrm{in}_{v,w}(I)$ must contain the monomial $x^\alpha \in \mathfrak{K}[x]$.

$\impliedby$ Suppose $\mathrm{in}_{v,w}(I) \trianglelefteq \mathfrak{K}[x]$ contains a monomial x^α. For the remainder of the proof, we abbreviate $(-1, w)$-weighted degree and $(-1, w)$-weighted homogeneous with weighted degree and weighted homogeneous, respectively. Consider all $r \in R[t, x]$ such that

$$f := t^\beta \cdot \left(x^\alpha + (t-1) \cdot r\right) \in \mathrm{in}_{(-1,w)}(\pi^{-1}I), \text{ for some } \beta \in \mathbb{N}.$$

Such f exist, because, by Corollary 1, $\mathrm{in}_{(-1,w)}(\pi^{-1}I)$ must contain an element of the form $x^\alpha + (t-1) \cdot r + p \cdot s$ for some $r, s \in R[t, x]$, and p lies in $\mathrm{in}_{(-1,w)}(\pi^{-1}I)$. We may decompose each r into its weighted homogeneous layers,

$$r = r_{q_1} + \ldots + r_{q_l}.$$

with $\deg_{(-1,w)}(r_{q_i}) = q_i$ and $q_i < q_{i+1}$, and we may choose $\beta \in \mathbb{N}$ and $r \in R[t, x]$ such that l is minimal. We now use the weighted homogeneity of $\mathrm{in}_{(-1,w)}(\pi^{-1}I)$ to show that $l = 0$, which means that $\mathrm{in}_{(-1,w)}(\pi^{-1}I)$ contains the monomial $t^\beta x^\alpha$ for some $\beta \in \mathbb{N}$.

Assume $l > 0$. Setting $d := \deg_{(-1,w)}(x^\alpha)$ and $r_q = 0$ for $q \notin \{q_1, \ldots, q_l\}$, we obtain the following weighted homogeneous layers of f in weighted degree $q - \beta$:

$$f_{q-\beta} := \begin{cases} t^\beta \cdot (t \cdot r_{q+1} - r_q), & \text{if } q \neq d \\ t^\beta \cdot (x^\alpha + t \cdot r_{d+1} - r_d), & \text{if } q = d. \end{cases}$$

Since $\mathrm{in}_{(-1,w)}(\pi^{-1}I)$ is weighted homogeneous, all $f_{q-\beta}$ are contained in this ideal.

Now if q_l was strictly bigger than d, we would get $f_{q_l-\beta} = -t^\beta \cdot r_{q_l} \in \mathrm{in}_{(-1,w)}(\pi^{-1}I)$, and thus

$$t^\beta \cdot (x^\alpha + (t-1) \cdot (r_{q_1} + \ldots + r_{q_{l-1}})) = f - t^\beta (t-1) \cdot r_{q_l} \in \mathrm{in}_{(-1,w)}(\pi^{-1}I),$$

contradicting our choice of β and r with minimal l.

Similarly, if q_1 was less than or equal to d, we would get $f_{q_1-1-\beta} = t^\beta \cdot t \cdot r_{q_1} \in \mathrm{in}_{(-1,w)}(\pi^{-1}I)$, and thus

$$t^{\beta+1} \cdot (x^\alpha + (t-1) \cdot (r_{q_2} + \ldots + r_{q_l})) = t \cdot f - t^{\beta+1}(t-1) \cdot r_{q_1} \in \mathrm{in}_{(-1,w)}(\pi^{-1}I),$$

again contradicting our choice of β and r with minimal l.

Hence, $q_l \leq d < q_1$, which, however, contradicts $q_i < q_{i+1}$. $\qquad\square$

Corollary 2 *If $I \trianglelefteq K[x]$ defines an irreducible subvariety of $(K^*)^n$ of dimension d, then $\mathcal{T}(\pi^{-1}I)$ is the support of a pure polyhedral fan of dimension $d+1$ connected through codimension one.*

Proof Follows immediately from Definition 3 and Theorem 4, which imply that $\mathcal{T}(\pi^{-1}I)$ is the polyhedral fan over $\mathcal{T}_\nu(I)$. And by Theorem 3, the latter is pure of dimension d and connected through codimension one. $\qquad\square$

We close the section with a couple of examples of tropical varieties over $K[x]$, their counterparts in $R[\![t]\!][x]$ and how they can be computed in SINGULAR.

Example 4 Let $K := \mathbb{Q}(\!(\varepsilon)\!)$ be the field of Laurent series, equipped with is natural valuation ν, and let $I \trianglelefteq K[x, y]$ be the principal ideal generated by $(x + y + 1) \cdot (\varepsilon^2 x + y + \varepsilon)$. Then $\mathcal{T}_\nu(I)$ is the union of two tropical lines, one with vertex at $(0, 0)$ and one with vertex at $(1, -1)$. Setting $R := \mathbb{Q}[t] \subseteq \mathcal{O}_K = \mathbb{Q}[\![t]\!]$, Theorem 4 implies that for
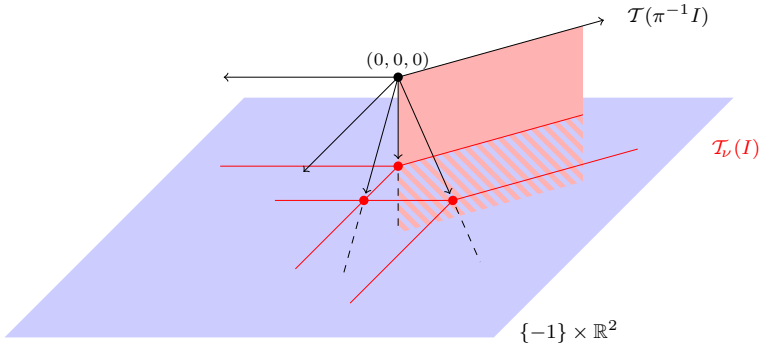
Fig. 1 $\mathcal{T}(\pi^{-1}I)$ as polyhedral fan over $\mathcal{T}_\nu(I)$

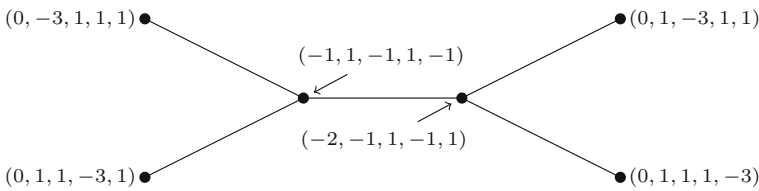**Fig. 2** $\mathcal{T}(\langle x_1 - 2x_2 + 3x_3, 3x_2 - 4x_3 + 5x_4, 2 - t\rangle)$

any weight vector $w = (w_t, w_x, w_y) \in \mathbb{R}_{<0} \times \mathbb{R}^2$ in the lower open half-space we have

$$w \in \mathcal{T}(\pi^{-1}I) \quad \Longleftrightarrow \quad \left(\frac{w_x}{|w_t|}, \frac{w_y}{|w_t|}\right) \in \mathcal{T}_\nu(I).$$

Hence $\mathcal{T}(\pi^{-1}I)$ is the polyhedral fan over $\mathcal{T}_\nu(I)$ as shown in Fig. 1. It consists of 6 rays and 8 two-dimensional cones in a way that the intersection with the affine hyperplane yields $\mathcal{T}_\nu(I)$.

Example 5 ([6, §3.6]) Consider $I := \langle x_1 - 2x_2 + 3x_3, 3x_2 - 4x_3 + 5x_4\rangle \trianglelefteq \mathbb{Q}[x_1, \ldots, x_4] \subseteq \mathbb{Q}_2[x_1, \ldots, x_4]$, whose preimage is given by

$$\pi^{-1}I = \langle x_1 - 2x_2 + 3x_3, 3x_2 - 4x_3 + 5x_4, 2 - t\rangle \trianglelefteq \mathbb{Z}[\![t]\!][x_1, \ldots, x_4].$$

The tropical variety of the preimage is combinatorially of the form shown in Fig. 2 and invariant by translation under the one-dimensional subspace generated by $(0, 1, 1, 1, 1)$. Hence, each of the six drawn vertices represents a two-dimensional cone and each of the five edges represents a three-dimensional cone.

Intersecting with the affine hyperplane $\{-1\} \times \mathbb{R}^4$, we obtain a polyhedral complex as shown in the top left of Fig. 3, the vertices of Fig. 2 in $\{0\} \times \mathbb{R}^4$ becoming points at infinity.
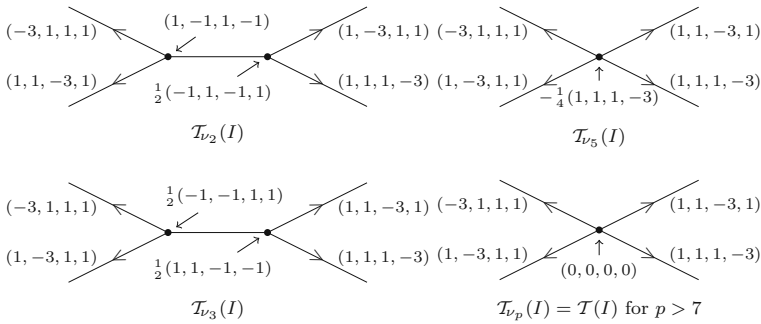
Fig. 3 $\mathcal{T}_\nu(I)$ for all p-adic valuations and the trivial valuation

```
                    SINGULAR                              /
A Computer Algebra System for Polynomial Computations  /   Version 4.1.1
                                                      0<
by: W. Decker, G.-M. Greuel, G. Pfister, H. Schoenemann \   Feb 2018
FB Mathematik der Universitaet, D-67653 Kaiserslautern  \


> LIB "tropical.lib";                 LINEALITY_SPACE
> ring r = 0,x(1..4),dp;              0 -1 -1 -1 -1    # 0
> ideal I =
.   x(1)+2*x(2)-3*x(3),               F_VECTOR
.   3*x(2)-4*x(3)+5*x(4);             1 6 5
> number p = 2;
> tropicalVariety(I,p);               MAXIMAL_CONES
RAYS                                  {0 1}    # Dimension 3
-2 -1 1 -1 1     # 0                  {0 2}
-1 1 -1 1 -1     # 1                  {0 4}
0 -3 1 1 1       # 2                  {1 3}
0 1 -3 1 1       # 3                  {1 5}
0 1 1 -3 1       # 4
0 1 1 1 -3       # 5
```

Fig. 4 Computing tropical varieties over fields with valuation in SINGULAR

Figure 3 further shows the tropical varieties of $I \trianglelefteq \mathbb{Q}[x_1, \ldots, x_4]$ for other possible valuations on $\mathbb{Q}$. We see that regardless of the valuation, all tropical varieties share the same recession fan, as was proven by Gubler [11]. The latter is also necessarily the tropical variety under the trivial valuation. Note that for p sufficiently large, the tropical varieties over $\mathbb{Q}_p$ coincide with the tropical variety under the trivial valuation. This is because p is simply too large for $p - t$ to matter in any of our standard basis computations that arise in the computation of $\mathcal{T}(\pi^{-1}I)$. In the theory of modular Gröbner bases [4], these p are referred to as *good primes* while the other p are referred to as *bad primes*.

Figure 4 shows the input and output of the SINGULAR computation of $\mathcal{T}_{\nu_2}(I)$. Currently, the computation of tropical varieties is limited to the fields $\mathbb{C}\{\{t\}\}$ and $\mathbb{Q}_p$ and to ideals defined over $\mathbb{Q}$.

3 Tracing Gröbner Complexes to a Trivial Valuation

In this section, we show how the Gröbner complexes of ideals in $K[x]$ can be traced back to the Gröbner fans of ideals in $R[\![t]\!][x]$. We show how the latter induces a refinement of the former and how to determine whether two Gröbner cones map to the same Gröbner polyhedron. We close this section with a remark on p-adic Gröbner bases as introduced by Chan and Maclagan [7].

Definition 4 (*Gröbner polyhedra, Gröbner complexes over valued fields*) For a homogeneous ideal $I \trianglelefteq K[x]$ and a weight vector $w \in \mathbb{R}^n$, we define its *Gröbner polyhedra* to be

$$C_{v,w}(I) := \mathrm{cl}\left(\{v \in \mathbb{R}^n \mid \mathrm{in}_{v,v}(I) = \mathrm{in}_{v,w}(I)\}\right) \subseteq \mathbb{R}^n.$$

We refer to the collection $\Sigma_v(I) := \{C_{v,w}(I) \mid w \in \mathbb{R}^n\}$ as the *Gröbner complex* of I.

Theorem 5 ([18, Theorem 2.5.3]) *Let $I \trianglelefteq K[x]$ be a homogeneous ideal. Then all $C_{v,w}(I)$ are convex polytopes and $\Sigma_v(I)$ is a finite polyhedral complex.*

Definition 5 For an x-homogeneous ideal $I \trianglelefteq R[\![t]\!][x]$, i.e. an ideal generated by elements which are homogeneous if considered as polynomials in x with coefficients in $R[\![t]\!]$, and a weight vector $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$ we define its Gröbner cone to be

$$C_w(I) := \mathrm{cl}\left(\{v \in \mathbb{R}_{<0} \times \mathbb{R}^n \mid \mathrm{in}_v(I) = \mathrm{in}_w(I)\}\right).$$

We refer to the collection $\Sigma(I) := \{C_w(I), C_w(I) \cap \{0\} \times \mathbb{R}^n \mid w \in \mathbb{R}_{<0} \times \mathbb{R}^n\}$ as the *Gröbner fan* of I.

Proposition 2 ([20, Theorem 3.19]) *Let $I \trianglelefteq R[\![t]\!][x]$ be an x-homogeneous ideal. Then all $C_w(I)$ are polyhedral cones and $\Sigma(I)$ is a finite polyhedral fan.*

Corollary 3 *The map $\{-1\} \times \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n, (-1, w) \longmapsto w$ is compatible with the Gröbner fan $\Sigma(\pi^{-1}I)$ and the Gröbner complex $\Sigma_v(I)$, i.e. it maps the restriction of a Gröbner cone $C_{(-1,w)}(\pi^{-1}I) \cap \left(\{-1\} \times \mathbb{R}^n\right)$ into the Gröbner polytope $C_{v,w}(I)$.*

Proof Follows directly from Proposition 1, as two weight vectors with the same initial ideal of $\pi^{-1}I \trianglelefteq R[\![t]\!][x]$ yield the same initial ideal of $I \trianglelefteq K[x]$. □

Note that it may happen that several cones are mapped into the same Gröbner polytope, i.e. that the image of the restricted Gröbner fan is a refinement of the Gröbner complex (see Example 6).

We now recall the notion of initially reduced standard bases of ideals in $R[\![t]\!][x]$ from [19] and how they determine the inequalities and equations of Gröbner cones as shown in [20]. We then use these to decide whether two Gröbner cones are mapped to the same Gröbner polytope and, by doing so, show that no separate standard basis computation is required for this.

Definition 6 (*initially reduced standard bases*) Fix the lexicographical ordering $>$ such that $x_1 > \ldots > x_n > 1 > t$. Given a weight vector $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$, we define the *weighted ordering* $>_w$ to be

$$t^\beta x^\alpha >_w t^\delta x^\gamma \quad :\Longleftrightarrow \quad w \cdot (\beta, \alpha) > w \cdot (\delta, \gamma) \text{ or}$$
$$w \cdot (\beta, \alpha) = w \cdot (\delta, \gamma) \text{ and } t^\beta x^\alpha > t^\delta x^\gamma.$$

For $g \in R[\![t]\!][x]$, the *leading term* $\mathrm{LT}_{>_w}(g)$ is the unique term of g with maximal monomial under $>_w$ and for $I \trianglelefteq R[\![t]\!][x]$, the *leading ideal* $\mathrm{LT}_{>_w}(I)$ is the ideal generated by the leading terms of all its elements. A finite subset $G \subseteq I$ is called a *standard basis* of I with respect to $>_w$, if the leading terms of its elements generate $\mathrm{LT}_{>_w}(I)$.

Suppose $G = \{g_1, \ldots, g_k\}$ with $g_i = \sum_{\alpha \in \mathbb{N}^n} g_{i,\alpha} \cdot x^\alpha$, $g_{i,\alpha} \in R[\![t]\!]$. We call G *initially reduced*, if the set

$$G' := \left\{ \sum_{\alpha \in \mathbb{N}} \mathrm{LT}_>(g_{i,\alpha}) \cdot x^\alpha \ \middle| \ i = 1, \ldots, k \right\} \subseteq R[t, x],$$

is reduced in the classical sense, i.e. no term in the tail of an element of G' is in the ideal generated by the leading terms of the elements in G'.

Proposition 3 ([20, Algorithm 4.6]) *Let $I \trianglelefteq R[\![t]\!][x]$ be an x-homogeneous ideal and $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$ a weight vector. Then an initially reduced standard basis G of I with respect to $>_w$ can be computed using a finite sequence of arithmetic operations in $R[\![t]\!][x]$. Moreover, if I is generated by elements in $R[t, x]$, then it can be computed using a finite sequence of arithmetic operations in $R[t, x]$.*

Proposition 4 ([20, Proposition 3.8, 3.11]) *Let $I \trianglelefteq R[\![t]\!][x]$ be an x-homogeneous ideal, let $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$ be a weight vector and let G an initially reduced standard basis of I with respect to $>_w$. Then $\{\mathrm{in}_w(g) \mid g \in G\}$ is an initially reduced standard basis of $\mathrm{in}_w(I)$ with respect to $>_w$, and the Gröbner cone of I around w is given by*

$$C_w(I) = \mathrm{cl}\left(\{v \in \mathbb{R}_{<0} \times \mathbb{R}^n \mid \mathrm{in}_v(g) = \mathrm{in}_w(g) \text{ for all } g \in G\}\right).$$

We now show that our standard bases of $\pi^{-1}I \trianglelefteq R[\![t]\!][x]$ yield Gröbner bases of initial ideals of $I \trianglelefteq K[x]$, allowing us to immediately decide whether two Gröbner cones of the former are mapped to the same Gröbner polytope of the latter.

Corollary 4 *Let $I \trianglelefteq K[x]$ be a homogeneous ideal, let $w \in \mathbb{R}^n$ be a weight vector and let G be an initially reduced standard basis of $\pi^{-1}I$ with respect to the weighted ordering $>_{(-1,w)}$. Then*

$$\left\{ \overline{\mathrm{in}_{(-1,w)}(g)|_{t=1}} \ \middle| \ g \in G \right\}$$

is a standard basis of $\mathrm{in}_{v,w}(I)$ with respect to the fixed lexicographical ordering $>$ restricted to monomials in x.

Proof By Proposition 4, the set $\text{in}_{(-1,w)}(G) := \{\text{in}_{(-1,w)}(g) \mid g \in G\}$ is an initially reduced standard basis of $\text{in}_{(-1,w)}(\pi^{-1}I)$ with respect to $>_{(-1,w)}$. And because it is homogeneous with respect to the weight vector $(-1, w)$, it is also an initially reduced standard basis with respect to $>$. By choice of $>$, the set $\text{in}_{(-1,w)}(G)|_{t=1}$ remains a standard basis of $\text{in}_{(-1,w)}(\pi^{-1}I)|_{t=1}$ with respect to the restriction of $>$ to monomials in x. And since $p \in \text{in}_{(-1,w)}(G)|_{t=1}$, $\overline{\text{in}_{(-1,w)}(G)}|_{t=1}$ is a standard basis of $\overline{\text{in}_{(-1,w)}(\pi^{-1}I)}|_{t=1}$ with respect to the restriction of $>$. □

Example 6 Consider the preimage $\pi^{-1}I \trianglelefteq \mathbb{Z}[\![t]\!][x, y, z]$ of the ideal $I = \langle 2y + x, z^2 + y^2 \rangle \trianglelefteq \mathbb{Q}_2[x, y, z]$ and the two weight vectors $w = (1, 3, 7)$, $v = (1, 10, 5) \in \mathbb{R}^3$. Fix a lexicographical tiebreaker $>$ with $x > y > z > 1 > t$.

The initially reduced standard basis of $\pi^{-1}I$ with respect to $>_{(-1,w)}$ and $>_{(-1,v)}$ is the following two sets, respectively (initial forms underlined):

$$G_{(-1,w)} = \{\underline{2} - t, \underline{ty} + x, \underline{z^2} + y^2\}, \ G_{(-1,v)}$$
$$= \{\underline{2} - t, \underline{ty} + x, \underline{xy} - tz^2, \underline{t^2z^2} + x^2, \underline{y^2} + z^2\},$$

yielding the following Gröbner basis of $\text{in}_{v,w}(I)$ and $\text{in}_{v,v}(I)$ under $>$:

$$G_{v,w} = \{y, z^2\}, \quad G_{v,v} = \{y, xy, z^2, y^2\}.$$

One immediately sees that both initial ideals coincide, meaning that the two Gröbner cones $C_{(-1,w)}(\pi^{-1}I)$ and $C_{(-1,v)}(\pi^{-1}I)$ are mapped to the same Gröbner polytope $C_{v_2,w}(I) = C_{v_2,v}(I)$.

Remark 1 (*p-adic Gröbner bases*) By [18, Sect. 2.4], a Gröbner basis of an ideal $I \trianglelefteq K[x]$ over valued fields with respect to a weight vector $w \in \mathbb{R}^n$ is a finite generating set whose initial forms generate the initial ideal $\text{in}_{v,w}(I)$. By Corollary 4, $\pi(G)$ is such a Gröbner basis if $G \subseteq \pi^{-1}I \trianglelefteq R[\![t]\!][x]$ is a standard basis under the monomial ordering $>_w$.

Lines 1 to 6 in Fig. 5 illustrate the computation of a Gröbner bases over the 2-adic numbers in SINGULAR: Line 3 creates the Katsura(4) ideal in $x_1, \ldots, x_4$, Line 5 homogenises it on x_0 and adds the generator $2 - t$, and Line 6 computes its standard basis. Note that ds is a weighted ordering with weight vector $(-1, \ldots, -1, -1)$ which

```
1   LIB "poly.lib"; LIB "gfan.lib";  // for katsura() and initial() commands
2   ring r = 0,(x1,x2,x3,x4),dp;
3   ideal I = katsura(4);
4   ring s = integer,(t,x0,x1,x2,x3,x4),ds;
5   ideal I = homog(imap(r0,I),x0), 2-t;
6   ideal stdI = std(I);
7   option(infRedTail);
8   stdI = reduce(stdI,2-t);
9   ideal inI = initial(stdI,intvec(-1,0,0,0,0,0));
10  inI = subst(inI,t,1);
11  ring r2 = 2,(x0,x1,x2,x3,x4,x5,x6),dp;
12  ideal inI = imap(r1,inI);
```

Fig. 5 Computing Gröbner bases over valued fields in SINGULAR

Table 1 Timings in seconds unless aborted after 1 CPU day

Examples	GFAN	MACAULAY2	SAGE	SINGULAR
Cyclic(4)	–	1	10	1
Cyclic(5)	–	–	–	1
Cyclic(6)	–	–	–	2
Katsura(3)	1	1	1	1
Katsura(4)	–	–	10	1
Katsura(5)	–	–	190	1
Katsura(6)	–	–	2900	–
Chan	1	1	4	–

is equivalent to a weighted ordering with weight vector $(0, \ldots, 0, -1)$ since the ideal is homogeneous in $x_0, \ldots, x_4$. Substituting t with 2 in `stdI` yields a Gröbner basis, however the monomials will be out of order since the ordering ignores the 2-adic valuation.

Lines 7 to 12 in Fig. 5 construct the initial ideal: Line 7 forces SINGULAR to do tail reductions even though this might cause infinite loops in non-global orderings. Line 8 reduces `stdI` with respect to $2 - t$, so that the minimal degrees in t reflect the 2-valuations. Line 9 computes the desired initial form, and Line 10 replaces all t with 1 so that Line 12 can safely pass to a polynomial ring without t over the residue field.

Table 1 shows timings of the GFAN command `gfan_padic -groebnerBasis` by Anders Jensen [14], the MACAULAY2 Package `GroebnerValuations` by Andrew Chan [7,10], an implementation of a p-adic Matrix-F5 algorithm by Tristan Vaccon in SAGE [28,34], and the standard basis engine of SINGULAR over integers under mixed orderings [8]. We consider the following examples:

Cyclic(n)   In $\mathbb{Q}_2[x_0, \ldots, x_n]$, the cyclic ideal in the variables $x_1, \ldots, x_n$, homogenised using the variable x_0, and weight vector $(0, \ldots, 0)$.

Katsura(n)   In $\mathbb{Q}_2[x_0, \ldots, x_n]$, the Katsura ideal in the variables $x_1, \ldots, x_n$, homogenised using the variable x_0, and weight vector $(0, \ldots, 0)$.

Chan   In $\mathbb{Q}_3[x_0, \ldots, x_4]$, the ideal $\langle 2x_1^2 + 3x_1x_2 + 24x_3x_4, 8x_1^3 + x_2x_3x_4 + 18x_3^2x_4 \rangle$ and weight vector $(-1, -11, -3, -19)$ taken from [6, §3.6].

All computations were done on a server running Gentoo-3.16.5 with Intel Xeon E5-2690 processors. Computations exceeding 1 CPU day were aborted. Note that the computations in SAGE were done up to a finite precision of p^{50} and that the correctness of the result could only be verified for the examples for which either GFAN, MACAULAY2 or SINGULAR finished.

4 Computing the Preimage

This article was dedicated to show how $\mathcal{T}_v(I)$ can be computed via $\mathcal{T}(\pi^{-1}I)$, however until now we have not addressed how to determine the preimage $\pi^{-1}I$ in the first place.

We therefore end the article with two results: The first shows that $\pi^{-1}I$ can be obtained by a saturation, and the second describes how to compute it.

Lemma 1 *Let $I \trianglelefteq K[x]$ be an ideal, and let $\{f_1, \ldots, f_k\} \subseteq I \cap \mathcal{O}_K[x]$ be a generating set over the valuation ring. Since $\pi : R[\![t]\!][x] \to \mathcal{O}_K[x]$ is surjective, there are $f_1', \ldots, f_k' \in R[\![t]\!][x]$ such that $\pi(f_i') = f_i \in R[x]$. Then*

$$\pi^{-1}I = \left(\langle f_1', \ldots, f_k' \rangle + \langle p - t \rangle \right) : p^\infty \trianglelefteq R[\![t]\!][x],$$

where $(\cdot) : p^\infty$ denotes the saturation of the ideal with respect to p.

Proof The $\supseteq$ inclusion is obvious, as $p - t$ is mapped to 0 and p is invertible in K.

For the converse inclusion, let $g \in \pi^{-1}I$. Then there are $q_1, \ldots, q_k \in K[x]$ such that

$$\pi(g) = q_1 \cdot f_1 + \ldots + q_k \cdot f_k \in K[x],$$

which means that for a sufficiently high power $l \in \mathbb{N}$ we have

$$p^l \cdot \pi(g) = \underbrace{p^l q_1}_{\in \mathcal{O}_K[x]} \cdot f_1 + \ldots + \underbrace{p^l q_k}_{\in \mathcal{O}_K[x]} \cdot f_k \in \mathcal{O}_K[x].$$

Since the map $\pi : R[\![t]\!][x] \to \mathcal{O}_K[x]$ is surjective, there exist $q_1', \ldots, q_k' \in R[\![t]\!][x]$ such that

$$p^l \cdot \pi(g) = \pi(q_1' \cdot f_1' + \ldots + q_k' \cdot f_k'),$$

or rather

$$p^l \cdot g - q_1' \cdot f_1' + \ldots + q_k' \cdot f_k' \in \ker(\pi) = \langle p - t \rangle.$$

Thus $p^l \cdot g \in \langle f_1', \ldots, f_k' \rangle + \langle p - t \rangle$, and hence

$$g \in (\langle f_1', \ldots, f_k' \rangle + \langle p - t \rangle) : p^\infty. \qquad \square$$

The next example shows that $\langle f_1, \ldots, f_k' \rangle$ in Lemma 1 is not necessarily saturated with respect to p, which is why Proposition 5 shows how to compute it.

Example 7 Consider $I = \langle f_1, f_2 \rangle \trianglelefteq \mathbb{Q}_2[x, y]$, where $f_1 = x^2 + \frac{1}{2}y$, $f_2 = y^2 + \frac{1}{2}y$. Then $g = x^2 - y^2 \in I \cap \mathbb{Z}_2[x, y]$ and $2g \in \langle f_1, f_2 \rangle \trianglelefteq \mathbb{Z}_2[x, y]$, but $g \notin \langle f_1, f_2 \rangle \trianglelefteq \mathbb{Z}_2[x, y]$.

Lemma 2 *Given the same conditions as in Lemma 1, we have*

$$\pi^{-1}I = \left(\langle f_1', \ldots, f_k' \rangle + \langle p - t \rangle \right) : t^\infty \trianglelefteq R[\![t]\!][x].$$

Proof Follows directly from Lemma 1, since $p - t \in \langle f'_1, \ldots, f'_k \rangle + \langle p - t \rangle$ implies that its saturation with respect to p coincides with the saturation at t. ☐

The following proposition shows how to compute the a standard basis of the preimage. It requires the notion of strong standard bases as in [25, Definition A.1.1.8]. The result and its proof is a straightforward generalisation of [31, Lemma 12.1].

Proposition 5 *Let $I \trianglelefteq K[x]$ be an ideal, and let $\{f_1, \ldots, f_k\} \subseteq I \cap \mathcal{O}_K[x]$ be a generating set of I in the valuation ring. Since $\pi : R[\![t]\!][x] \to \mathcal{O}_K[x]$ is surjective, there are $f'_1, \ldots, f'_k \in R[\![t]\!][x]$ such that $\pi(f'_i) = f_i \in \mathcal{O}_K[x]$.*

Let $G \subseteq \langle f'_1, \ldots, f'_k \rangle + \langle p - t \rangle$ be a strong standard basis with respect to some weighted ordering $>_w$ with weight vector $w := (-1, 0, \ldots, 0)$. Then

$$\left\{ \frac{g}{t^k} \ \middle| \ g \in G, \ t^k \text{ highest power of } t \text{ dividing } g \right\}$$

is a strong standard basis of $\pi^{-1} I$ with respect to $>_w$.

Proof By Lemma 2, we have $G \subseteq \pi^{-1} I$. It remains to show that for any $f \in \pi^{-1} I$ there exists a $g \in G$ such that $\mathrm{LM}_{>_w}(g)$ divides $\mathrm{LM}_{>_w}(f)$. For that, observe that for any $f \in R[\![t]\!][x]$ our ordering satisfies

$$t \text{ divides } \mathrm{LM}_{>_w}(f) \quad \Longleftrightarrow \quad t \text{ divides } f.$$

Let $f \in \pi^{-1} I$. By Lemma 2, $t^l f \in \langle f'_1, \ldots, f'_k \rangle + \langle p - t \rangle$ for $l \in \mathbb{N}$ sufficiently big. Since G is strong, there is a $g \in G$ such that $\mathrm{LM}_{>_w}(g)$ divides $\mathrm{LM}_{>_w}(t^l f)$. Let t^k be the highest power of t dividing g, so that $g/t^k \in G$. Then $\mathrm{LM}_{>_w}(g/t^k)$ has no common divisor with t^l, as by our first observation that would contradict the maximality of k. Hence it divides $\mathrm{LM}_{>_w}(f)$. ☐

Example 8 Let I be the ideal Katsura(4) from Remark 1. Figure 6 shows the initially reduced standard basis of the computation in Fig. 5. By Proposition 5, dividing `stdI[6]` and `stdI[7]` by t yields a standard basis of the preimage. This shows that `stdI` does not generate the entire preimage in $\mathbb{Z}[\![t]\!][x_1, \ldots, x_4]$.

```
> stdI;
stdI[1]=2-t;
stdI[2]=x0-x1-tx2-tx3-tx4
stdI[3]=x1x2-tx2^2-tx2x4+tx3x4
stdI[4]=x2^2-x0x3+tx1x3+tx2x4
stdI[5]=x1^2x3+tx0x2x3-3tx1x2x3+tx2^2x3-tx0x3^2+tx1x3^2
          +tx0x2x4-tx0x3x4+tx1x3x4-tx2x3x4-t^2x2^2x4
stdI[6]=tx3^2+tx1x4-tx4^2+t^2x2x3
stdI[7]=tx1^3x4-tx1^2x4^2+t^2x1^2x2x3-t^2x0x2x3^2+3t^2x1x2x3^2
          -t^2x2^2x3^2+t^2x0x3^3-t^2x1x3^3-t^2x0x2x3x4+t^2x0x3^2x4
          -t^2x1x3^2x4+t^2x2x3^2x4+t^3x2^2x3x4
```

Fig. 6 Standard basis for computing a preimage SINGULAR

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