Reza Ebrahimi Atani; Shahabaddin Ebrahimi Atani

Comultiplication modules over a pullback of Dedekind domains

Czechoslovak Mathematical Journal, Vol. 59 (2009), No. 4, 1103–1114

Persistent URL: `http://dml.cz/dmlcz/140540`

Czechoslovak Mathematical Journal, 59 (134) (2009), 1103–1114

COMULTIPLICATION MODULES OVER A PULLBACK OF DEDEKIND DOMAINS

Reza Ebrahimi Atani, Shahabaddin Ebrahimi Atani, Rasht

(Received June 11, 2008)

Abstract. First, we give complete description of the comultiplication modules over a Dedekind domain. Second, if R is the pullback of two local Dedekind domains, then we classify all indecomposable comultiplication R-modules and establish a connection between the comultiplication modules and the pure-injective modules over such domains.

Keywords: pullback, separated modules and representations, non-separated modules, comultiplication modules, dedekind domain, pure-injective modules, Prüfer modules

MSC 2010: 13C05, 13C13, 16D70

1. Introduction

One of the aims of the modern representation theory is to solve classification problems for subcategories of modules over a unitary rings R. The reader is referred to [1] and [19, Chapters 1 and 14] for a detailed discussion of classification problems, their representation types (finite, tame, or wild), and useful computational reduction procedures, see also a recent paper [20] for a discussion of the notion of wild representation type for module classification problems.

Modules over pullback rings have been studied by several authors (see for example, [3], [16], [13], [20], [11], [23]). In the present paper we consider a new class of R-modules, called comultiplication modules, the dual notion of multiplication modules, (see Definition 1.2), and we study it in detail from the classification problem point of view. We are mainly interested in the case that either R is a Dedekind domain or R is a pullback of two local Dedekind domains. Let R be the pullback of two local Dedekind domains over a common factor field. The main purpose of this paper is to give a complete description of the indecomposable comultiplication modules over R. The classification is divided into two stages: the description of all indecomposable separated comultiplication R-modules and then, using this list of separated comultiplication modules we show that non-separated indecomposable comultiplication R-modules are factor modules of finite direct sums of separated comultiplication R-modules. Then we use the classification of separated comultiplication modules from Section 3, together with results of Levy [14], [15] on the possibilities for amalgamating finitely generated separated modules, to classify the non-separated indecomposable comultiplication modules M (see Theorem 4.8). We will see that the non-separated modules may be represented by certain amalgamation chains of separated indecomposable comultiplication modules (where infinite length comultiplication modules can occur only at the ends) where adjacency corresponds to amalgamation in the socles of these separated comultiplication modules.

For the sake of completeness, we state some definitions and notation used throughout. In this paper all rings are commutative with identity and all modules unitary. Let $v_1\colon R_1 \to \overline{R}$ and $v_2\colon R_2 \to \overline{R}$ be homomorphisms of two local Dedekind domains R_i onto a common field $\overline{R}$. Denote the *pullback* $R = \{(r_1, r_2) \in R_1 \oplus R_2\colon v_1(r_1) = v_2(r_2)\}$ by $(R_1 \xrightarrow{v_1} \overline{R} \xleftarrow{v_2} R_2)$, where $\overline{R} = R_1/J(R_1) = R_2/J(R_2)$. Then R is a ring under coordinate-wise multiplication. Denote the kernel of v_i, $i = 1, 2$, by P_i. Then $\mathrm{Ker}(R \to \overline{R}) = P = P_1 \times P_2$, $R/P \cong \overline{R} \cong R_1/P_1 \cong R_2/P_2$, and $P_1P_2 = P_2P_1 = 0$ (so R is not a domain). Furthermore, for $i \neq j$, the sequence $0 \to P_i \to R \to R_j \to 0$ is an exact sequence of R-modules (see [13]).

Definition 1.1. An R-module S is defined to be *separated* if there exist R_i-modules S_i, $i = 1, 2$, such that S is a submodule of $S_1 \oplus S_2$ (the latter is made into an R-module by setting $(r_1, r_2)(s_1, s_2) = (r_1s_1, r_2s_2)$).

Equivalently, S is separated if it is a pullback of an R_1-module and an R_2-module and then, using the same notation for pullbacks of modules as for rings, $S = (S/P_2S \to S/PS \leftarrow S/P_1S)$ [13, Corollary 3.3] and $S \leqslant (S/P_2S) \oplus (S/P_1S)$. Also, S is separated if and only if $P_1S \cap P_2S = 0$ [13, Lemma 2.9].

If R is a pullback ring, then every R-module is an epimorphic image of a separated R-module, indeed every R-module has a "minimal" such representation: a *separated representation* of an R-module M is an epimorphism $\varphi\colon S \to M$ of R-modules where S is separated and, if φ admits a factorization $\varphi\colon S \xrightarrow{f} S' \to M$ with S' separated, then f is one-to-one. The module $K = \mathrm{Ker}(\varphi)$ is then an $\overline{R}$-module, since $\overline{R} = R/P$ and $PK = 0$ [13, Proposition 2.3]. An exact sequence $0 \to K \to S \to M \to 0$ of R-modules with S separated and K an $\overline{R}$-module is a separated representation of M if and only if $P_iS \cap K = 0$ for each i and $K \subseteq PS$ [13, Proposition 2.3]. Every module M has a separated representation, which is unique up to isomorphism [13, Theorem 2.8]. Moreover, R-homomorphisms lift to separated representation, preserving epimorphisms and monomorphisms [13, Theorem 2.6].

If R is a ring and N is a submodule of an R-module M, the ideal $\{r \in R\colon\ rM \subseteq N\}$ is denoted by $(N : M)$. Then $(0 : M)$ is the annihilator of M. A proper submodule N of a module M over a ring R is said to be a prime submodule if whenever $rm \in N$ for some $r \in R$, $m \in M$, then $m \in N$ or $r \in (N : M)$, so $(N : M) = P$ is a prime ideal of R, and N is said to be a P-prime submodule. The set of all prime submodules in an R-module M is denoted $\mathrm{Spec}(M)$.

Definition 1.2. (a) An R-module M is a comultiplication module provided for each submodule N of M, $N = (0 :_M J)$ for some ideal J of R (see [2]).

(b) An R module M is defined to be a weak multiplication module if $\mathrm{Spec}(M) = \emptyset$ or for every prime submodule N of M, $N = IM$ for some ideal I of R. An R-module M is defined to be a multiplication module if for each submodule N, $N = IM$ for some ideal I of R [4].

(c) An R-submodule N of M is pure in M if any finite system of equations over N which is solvable in M is also solvable in N. A submodule N of an R-module M is called an RD-submodule if $rN = N \cap rM$ for all $r \in R$ (note that an important property of modules M, N over a Dedekind domains is that N is pure in M if and only if N is an RD-submodule of M [22], [17]).

(d) A module M is pure-injective if it has the injective property relative to all pure exact sequences [22], [12].

2. Comultiplication modules over a Dedekind domain

In this section we collect some basic properties concerning comultiplication modules. Our starting point is the following lemma.

Lemma 2.1. *Let M be a comultiplication module over a commutative ring R. If N is a direct summand of M, then M/N is a comultiplication R-module.*

Proof. There exists a submodule N' of M such that $M = N \oplus N'$. Suppose that K is a non-zero submodule of N', so $K = (0 :_M I)$ for some ideal I of R. Therefore, $K = (O :_M I) = (0 :_{N'} I)$. Thus N' is a comultiplication submodule, and the proof is complete. $\square$

Lemma 2.2. *Every comultiplication module over an integral domain R is a torsion R-module.*

Proof. Let M be a comultiplication R-module, $T(M)$ the torsion submodule of M, and N any R-submodule of M. Then $N = (0 :_M J)$ for some ideal J of R. Clearly, $N \subseteq T(M)$. Therefore, we have $M = \sum\limits_{m \in M} Rm = T(M)$. $\square$

Proposition 2.3. *Let M be a module over a Dedekind domain R. Then M is a comultiplication if and only if the R_P-module M_P is a comultiplication for every maximal ideal P of R.*

Proof. Assume that M is a comultiplication R-module and let G be a submodule of M_P, where P is a maximal ideal of R. There exists a submodule N of M such that $G = N_P$, so $N = (0 :_M J)$ for some ideal J of R. Therefore, $G = N_P = (0 :_M J)_P = (0 :_{M_P} J_P)$ by [21, Exercise 9.13]. Conversely, let K be a submodule of M. By assumption, there is an ideal Q of R such that $K_P = (0 :_{M_P} Q_P)$ for every maximal ideal P of R; we will show that $(K/(0 :_M Q))_P = 0$ for every maximal ideal. To see that, we have $K_P = (0 :_{M_P} Q_P) = ((0 :_M Q))_P$. Hence $K/(0 :_M Q))_P = 0$, so $K = (0 :_M Q)$, as required. $\square$

Reduction to the local case. Let R be a Dedekind domain. Our aim here is to classify the comultiplication R-modules. By Proposition 2.3, it suffices to consider the case where R is a local Dedekind domain (e.g. a discrete valuation domain) with a unique maximal ideal $P = Rp$.

Lemma 2.4. *Every non-zero comultiplication module over a discrete valuation domain R is indecomposable.*

Proof. Assume that $P = Rp$ is the unique maximal ideal of R and let M be a comultiplication R-module such that $M = N \oplus K$ with $N \neq 0$ and $K \neq 0$. There are positive integers m, n with $m < n$ such that $M = (0 :_M P^n) + (0 :_M P^m) = (0 :_M P^m)$ and this contradicts $N \cap K = 0$. Thus either $N = 0$ or $K = 0$, as required. $\square$

Theorem 2.5. *Let R be a discrete valuation domain with a unique maximal ideal $P = Rp$. Then the comultiplication modules over R are:*
(i) *R/P^n, $n \geqslant 1$;*
(ii) *$E(R/P)$, the injective hull of R/P.*

Proof. First we discuss the modules listed in (i)–(ii) and show that they are comultiplications. Next we show that there are no more comultiplication R-modules.

Since for each i, $1 \leqslant i \leqslant n$, we have $P^i/P^n = (0 :_{R/P^n} P^{n-i})$, so R/P^n ($n \geqslant 1$) is a comultiplication module. It remains to show that $E = E(R/P)$ is a comultiplication module. Set $A_n = (0 :_E P^n)$ for all positive integers n. If $0 \neq N$ is a proper submodule of E, then $N = A_m$ for some m by [10, Lemma 2.6]. Therefore, E is a comultiplication R-module.

Let M be a comultiplication R-module. Choose $0 \neq a$, $a \in M$. Define the height of a, $h(a) = \sup\{n \colon a \in P^n M\}$ (so $h(a)$ is either an integer $n \geqslant 0$ or "∞"). If $(0 : a) = P^{n+1} = p^{n+1}R$ with $n + 1 \geqslant 2$ then we have $p^n a \neq 0$ and $(0 : p^n a) = P$.

So, replacing a if necessary, we may suppose that $(0:a)$ is P since $a \neq 0$ and M is a torsion R-module by Lemma 2.2. We split the proof into two cases.

Case 1. $h(a) = n$, $(0:a) = P$.

Since $h(a) = n$, there is an element $b \in M$ such that $p^n b = a$. So $p^n b \neq 0$ and the maximal power of p dividing $p^n b$ is just p^n. Moreover, $(0:b) = p^{n+1}R$ gives $Rb \cong R/P^{n+1}$. By assumption, $Rb = (0 :_M P^s)$ for some ideal $P^s \neq R$, and so $p^s b = 0$; hence $R/P^{n+1} \cong Rb = (0:_M 0) = M$.

Case 2. $h(a) = \infty$, $(0:a) = P$.

Since $h(a) = \infty$, there is an element a_1 of M such that $a = a_0 = pa_1$ with $a \neq a_1$, since $a \neq 0$ and $pa = 0$. If $h(a_1) < \infty$, then by case (i), M is a module of finite length, and this contradicts the fact that the height of a is ∞. So $a_1 = pa_2$ for some $a_2 \in M$. By this process, one can show that $M \cong E(R/P)$ (see [9, Theorem 2.12]). $\square$

3. The separated case

Throughout this section we shall assume unless otherwise stated that

$$R = (R_1 \xrightarrow{v_1} \overline{R} \xleftarrow{v_2} R_2) \tag{3.1}$$

is the pullback of two local Dedekind domain R_1, R_2 with maximal ideals P_1, P_2 generated respectively by p_1, p_2, P denotes $P_1 \oplus P_2$ and $R_1/P_1 \cong R_2/P_2 \cong R/P \cong \overline{R}$ is a field. In particular, R is a commutative noetherian local ring with a unique maximal ideal P. The other prime ideals of R are easily seen to be P_1 (that is $P_1 \oplus 0$) and P_2 (that is $0 \oplus P_2$).

Proposition 3.1. *Let R be the pullback ring as described in* (3.1), *and let* $S = (S/P_2S = S_1 \xrightarrow{f_1} \overline{S} = S/PS \xleftarrow{f_2} S_2 = S/P_1S)$ *be any separated R-module. Then S is a comultiplication R-module if and only if S_i is a comultiplication R_i-module,* $i = 1, 2$.

Proof. Assume that S is a separated comultiplication R-module and let $0 \neq L$ be a non-zero submodule of S_1. Then there exists a separated submodule $T = (T/P_2S = T_1 \xrightarrow{g_1} \overline{T} \xleftarrow{g_2} T_2 = T/P_1T)$, where g_i is the restriction of f_i over T_i, $i = 1, 2$, such that $L = T_1$. We split the proof into two cases.

Case 1. $\overline{S} \neq 0$. By assumption, for each i, $S_i \neq 0$ and $T = (0 :_S P_1^n \oplus P_2^m)$ for some integers m, n; we will show that $T_1 = (0 :_{S_1} P_1^n)$. Let $s_1 \in (0 :_{S_1} P_1^n)$. Then $P_1^n s_1 = 0$, so $(P_1^n \oplus P_2^m)(s_1, 0) = 0$; hence $(s_1, 0) \in T$. Therefore, $(0 :_{S_1} P_1^n) \subseteq T_1$. Now suppose that $x \in T_1$. Then there is an element $y \in T_2$ such that $g_1(x) = g_2(y)$, so

$(x, y) \in T$; hence $P_1^n x = 0$, and so we have equality. Similarly, S_2 is a comultiplication R_2-module.

Case 2. $\overline{S} = 0$. Then by [5, Lemma 2.7], $S = S_1 \oplus S_2$; hence for each i, S_i is comultiplication by Lemma 2.1.

Conversely, assume that S_1, S_2 are comultiplication R_i-modules and let T be a non-zero submodule of S. If $\overline{T} \neq 0$, then for each i, $T_i \neq 0$ and there exist positive integers n, m such that $T_1 = (0 :_{S_1} P_1^n)$, $T_2 = (0 :_{S_2} P_2^m)$, and so $T = (0 :_S P_1^n \oplus P_2^m)$. If $\overline{T} = 0$, then $T = T_1 \oplus T_2 = (0 :_{S_1} P_1^n) \oplus (0 :_{S_2} P_2^m) = (0 :_S P_1^n + P_2^m)$. Therefore, for any case S is a comultiplication R-module. $\square$

Lemma 3.2. *Let R be the pullback ring as described in* (3.1)*. Then the indecomposable separated comultiplication modules over R are:*

(1) *$S = (E(R_1/P_1) \to 0 \leftarrow 0)$, $(0 \to 0 \leftarrow E(R_2/P_2)$ where $E(R_i/P_i$ is the R_i-injective hull of R_i/P_i for $i = 1, 2$;*

and, for all positive integers n, m,

(2) *$S = (R_1/P_1^n \to \overline{R} \leftarrow R_2/P_2^m)$.*

Proof. By [5, Lemma 2.8], these modules are indecomposable. By Proposition 3.1 and Theorem 2.5 they are comultiplication modules. $\square$

We refer to modules of type (1) in Lemma 3.2 as P_1-*Prüfer* and P_2-*Prüfer*, respectively.

Proposition 3.3. *Let R be the pullback ring as described in* (3.1)*, and let S be a separated comultiplication R-module. Then S is of the form $S = M \oplus N$, where M is one of the modules as described in* (1) *and N is one of the modules described in* (2) *of Lemma* 3.2*. In particular, every separated comultiplication R-module is pure-injective.*

Proof. Let T denote an indecomposable summand of S. Then we can write $T = (T_1 \to \overline{T} \leftarrow T_2)$, and T is a comultiplication R-module by Lemma 2.1. First suppose that $T = PT$. Then by [5, Lemma 2.7 (i)], $T = T_1$ or T_2 and so T is an indecomposable comultiplication R_i-module for some i and, since $T = PT$, is of type (1) in the list of Lemma 3.2, So we may assume that $T/PT \neq 0$.

By Theorem 2.5 and Proposition 3.1, T_i is an indecomposable comultiplication R_i-module, for each $i = 1, 2$. Hence, by the structure of comultiplication modules over a discrete valuation domain (see Theorem 2.5), we have $S_i = E(R_i/P_i)$ or R_i/P_i^n ($n \geqslant 1$). Since $T/PT \neq 0$ it follows that for each $i = 1, 2$, T_i is a torsion module and it is not a divisible R_i-module. Then there are positive integers m, n and k such that $P_1^m T_1 = 0$, $P_2^k T_2 = 0$ and $P^n T = 0$. For $t \in T$, let $o(t)$ denote the least positive integer m such that $P^m t = 0$. Now choose $t \in T_1 \cup T_2$ with $\bar{t} \neq 0$ and such that $o(t)$ is maximal (given that $\bar{t} \neq 0$). There exists a $t = (t_1, t_2)$ such that $o(t) = n$, $o(t_1) = m$ and $o(t_2) = k$. Then for each $i = 1, 2$, $R_i t_i$ is pure in T_i (see [5, Theorem 2.9]). Thus, $R_1 t_1 \cong R_1/(0 : t_1) \cong R_1/P_1^m$ is a direct summand of T_1 since $R_1 t_1$ is pure-injective; hence $T_1 = R_1 t_1$ since T_1 is indecomposable. Similarly, $T_2 = R_2 t_2 \cong R_2/P_2^k$. Let $\overline{M}$ be the $\overline{R}$-subspace of $\overline{T}$ generated by $\bar{t}$. Then $\overline{M} \cong \overline{R}$. Let $M = (R_1 t_1 \to \overline{M} \leftarrow R_2 t_2)$. Then $T = M$, and T satisfies case (2) (see [5, Theorem 2.9]). □

Theorem 3.4. *Let R be the pullback ring as described in* (3.1), *and let S be an indecomposable separated comultiplication R-module. Then S is isomorphic to one of the modules listed in Lemma* 3.2.

P r o o f. Apply Proposition 3.3 and Lemma 3.2. □

Theorem 3.5. *Let R be the pullback ring as described in* (3.1), *and let S be a separated comultiplication R-module. Then S has a finite-dimensional top.*

P r o o f. Apply Proposition 3.3 (note that $S = U \oplus X$, where $\dim_{\overline{R}}(U/PU) \leqslant 1$ and $X/PX = 0$). □

4. The non-separated case

We continue to use the notation already established, so R is a pullback ring as in (3.1).

In this section we find the indecomposable non-separated comultiplication module modules. We begin by describing one indecomposable non-separated comultiplication, namely the injective hull of the unique simple module.

For each $i = 1, 2$, let E_i be the R_i-injective hull of R_i/P_i regarded as an R-module (so E_1, E_2 are the modules listed under (1) in Lemma 3.3). Set $A_n = \operatorname{Ann}_{E_1}(P_1^n)$ and $B_n = \operatorname{Ann}_{E_2}(P_2^n)$ ($n \geqslant 1$). Then A_n is a cyclic R_1-module, say $A_n = R_1 a_n$, and we may choose the elements a_n so that $a_n = p_1 a_{n+1}$ for each $n \geqslant 0$. Also, $p_1 a_0 = 0$ and $R_1 a_0 \cong R/P$. Similarly, B_n is a cyclic R_2-module with $B_n = R_2 b_n$, where we may suppose that $b_n = p_2 b_{n+1}$, $p_2 b_0 = 0$ and $R_2 b_0 \cong R/P$. Then $F = (E_1 \oplus E_2)/ < a_0 - b_0 >$ is the injective hull of $Ra_0 = Rb_0 \cong R/P$ and it is a non-separated R-module (see [5, p. 4053]). Consider the R-module F with $a_0 = b_0$ and let $C_n = \operatorname{Ann}_F(P^n)$. Moreover, we identify A_n (B_n) with the submodule A'_n (B'_n) of F, consisting of all elements of the form $a + \langle a_0, b_0 \rangle$ ($b + \langle a_0, b_0 \rangle$), where $a \in A_n$ ($b \in B_n$). The above notation will be kept in the first two results.

Proposition 4.1 [8, Proposition 3.1]. *Let R be the pullback ring as described in* (3.1)*. Then the following assertions hold:*

(i) *For each n, $C_n = A_n + B_n$, $C_0 = R/P = Ra_0 = Rb_0$, $C_n \subseteq C_{n+1}$ and $F = \bigcup C_n$.*

(ii) *The non-zero proper R-submodules of F are E_1, E_2, A_n, B_m, $E_1 + B_n$, $A_m + E_2$ and $A_m + B_n$ for all $n \geqslant 1$, $m \geqslant 1$.*

Proposition 4.2. *Let R be the pullback ring as described in* (3.1)*. Then F, the injective hull of R/P, is a non-separated comultiplication R-module.*

Proof. Let L be a non-zero submodule of F, say $A_n + B_m$; we will show that $A_n + B_m = (0 :_F P_1^n + P_2^m)$. If $x \in (0 :_F P_1^n \oplus P_2^m)$, then $(P_1^n + P_2^m)x = 0$ and $x = x_1 + x_2$, where $x_i \in E_i$, $i = 1, 2$. It follows that $P_1^n x = P_2^m x = 0$ and $0 = P_1^n(x_1 + x_2) = P_1^n x_1$. Similarly, $P_2^m x_2 = 0$, so $x \in A_n + B_m$; hence $(0 :_F P_1^n \oplus P_2^m) \subseteq A_n + B_m$. The proof of the other inclusion is similar. $\square$

Proposition 4.3. *Let R be the pullback ring as described in* (3.1) *and let M be any R-module. Let $0 \to K \xrightarrow{i} S \xrightarrow{\varphi} M \to 0$ be a separated representation of M. Then the following assertions hold:*

(i) *for every positive integer n, $0 \to K \to P^n S \to P^n M \to 0$ is a separated representation of $P^n M$. In particular, $K \subseteq P^n S$.*

(ii) *If T is a non-zero submodule of M, then $K \subseteq T$.*

Proof. (i) Since $\varphi^{-1}(P^n M) = P^n S$, the result follows from [6, Lemma 3.1].

(ii) If $(T : S) = P$, then [13, Proposition 2.3] gives $K \subseteq PS \subseteq T$. So suppose that $(T : S) = P_1 \oplus 0$ and $x \in K$. Then $(P_1 \oplus 0)^2 S \subseteq (P_1 \oplus 0)T$ and $P_i^2 S \cap K = 0$ for every i and $K \subseteq P^2 S$ by (i). Then $K \subseteq (P_1 \oplus 0)T + (0 \oplus P_2)^2 S$; hence $x = (x_1, x_2) = (p_1 t_1, p_2^2 s_1)$ for some $t_1 \in T_1$ and $s_1 \in S_1$. Therefore, $x_2 = 0$ and $K \subseteq T$. Likewise, if $(T : S) = 0 \oplus P_2$, then $K \subseteq T$. $\square$

Theorem 4.4. *Let R be the pullback ring as described in* (3.1) *and let M be any R-module. Let $0 \to K \xrightarrow{i} S \xrightarrow{\varphi} M \to 0$ be a separated representation of M. Then S is a comultiplication module if and only if M is a comultiplication module.*

Proof. Suppose that M is a comultiplication R-module and let T be a non-zero submodule of S. Then by Proposition 4.3, $K \subseteq T$ and T/K is a submodule of S/K. Since $M \cong S/K$ is comultiplication, we have $T/K = (0 :_{S/K} P_1^n \oplus P_2^m)$ for some m, n; we show that $T = (0 :_S P_1^n \oplus P_2^m)$. Let $t \in T$. Then $(P_1^n \oplus P_2^m)(t + K) = 0$, so $(P_1^n \oplus P_2^m)t = 0$ since $P_i S \cap K = 0$; hence $T \subseteq (0 :_S P_1^n \oplus P_2^m)$. For the reverse inclusion, assume that $s \in (0 :_S P_1^n \oplus P_2^m)$, so $(P_1^n \oplus P_2^m)s = 0$; hence $(P_1^n \oplus P_2^m)(s + K) = 0$. Therefore $s \in T$, so we have equality. Thus S is comultiplication. Conversely, assume that S is a comultiplication and let N be a non-separated submodule of M. Then

$\varphi^{-1}(N) = U$ is a submodule of S, so $U = (0 :_S P_1^n \oplus P_2^m)$ for some integers m, n. By [6, Lemma 3.1], $U/K \cong N$ is a submodule of $S/K \cong M$, so an inspection shows that $N = U/K = (0 :_{S/K} P_1^n \oplus P_2^m)$, as required. □

Proposition 4.5. *Let R be the pullback ring as described in* (3.1) *and let M be an indecomposable comultiplication non-separated R-module. Let $0 \to K \to S \to M \to 0$ be a separated representation of M. Then S is pure-injective.*

Proof. Apply Proposition 3.3 and Theorem 4.4. □

Let R be the pullback ring as described in (3.1) and let M be an indecomposable comultiplication non-separated R-module. Consider the separated representation $0 \to K \to S \to M \to 0$. By Proposition 4.5, S is pure-injective. Moreover, M has finite-dimensional top by [5, Proposition 2.6 (i)] and Theorem 3.5. So in the proofs of [5, Proposition 3.2 and Proposition 3.4] (here the pure-injectivity of M implies the pure-injectivity of S by [5, Proposition 2.6 (ii)]) we can replace the statement "M is an indecomposable pure-injective non-separated R-module" by "M is an indecomposable comultiplication non-separated R-module", because the main key in those results are the pure-injectivity of S, and indecomposability and non-separability of M. So we have the following results:

Corollary 4.6. *Let R be the pullback ring as described in* (3.1)*, let M be an indecomposable comultiplication non-separated R-module and let $0 \to K \to S \to M \to 0$ be a separated representation of M. Then S is a direct sum of finitely many indecomposable comultiplication modules.*

Corollary 4.7. *Let R be the pullback ring as described in* (3.1)*, let M be an indecomposable comultiplication non-separated R-module and let $0 \to K \to S \to M \to 0$ be a separated representation of M. Then at most two copies of modules of infinite length can occur among the indecomposable summands of S.*

Before we state the main theorem of this section let us explain the idea of proof. Let M be an indecomposable comultiplication non-separated R-module, and let $0 \to K \to S \to M \to 0$ be a separated representation of M. Then by Corollary 4.6, S is a direct sum of just finitely many indecomposable separated comultiplication modules and these are known by Theorem 3.4. In any separated representation $0 \to K \xrightarrow{i} S \xrightarrow{\varphi} M \to 0$ the kernel of the map φ to M is annihilated by P, hence it is contained in the socle of the separated module S. Thus M is obtained by amalgamation in the socle of the various direct summands of S. So the questions are: does this provide any further condition on the possible direct summands of S? How can these summands be amalgamated in order to form M?

In [15], Levy shows that the indecomposable finitely generated R-modules are of two non-overlapping types which he calls deleted cycle and block cycle types. It is the modules of deleted cycle type which are most relevant to us. Such a module is obtained from a direct sum S of indecomposable separated modules by amalgamating the direct summands of S in pairs to form a chain but leaving the two ends unamalgamated [15], see also [14, section 11].

Recall that, by Lemma 3.2 and Theorem 4.4, every indecomposable R-module of finite length is a comultiplication one. So by Corollary 4.7, the infinite length non-separated indecomposable comultiplication modules are obtained in just the same way as the deleted cycle type indecomposable ones are except that at least one of the two "end" modules must be a separated indecomposable comultiplication module of infinite length (that is, P_1-Prüfer and P_2-Prüfer). Note that one cannot have, for instance, a P_1-Prüfer module at each end (consider the alternation of primes P_1, P_2 along the amalgamation chain). So, apart from any finite length modules, we have amalgamations involving two Prüfer modules as well as modules of finite length (the injective hull $E(R/P)$ is the simplest module of this type), a P_1-Prüfer module and a P_2-Prüfer module. If the P_1-Prüfer and the P_2-Prüfer modules are direct summands of S then we will describe these modules as *doubly infinite*. Those where S has just one infinite length summand we will call *singly infinite* (see [4, Section 3]). It remains to show that the modules obtained by these amalgamation are, indeed, indecomposable comultiplication modules.

Theorem 4.8. *Let $R = (R_1 \to \overline{R} \leftarrow R_2)$ be the pullback of two discrete valuation domains R_1, R_2 with a common factor field $\overline{R}$. Then the indecomposable non-separated comultiplication modules are the following ones:*

(i) *the indecomposable modules of finite length (apart from R/P which is separated);*

(ii) *the doubly infinite comultiplication modules as described above;*

(iii) *the singly infinite comultiplication modules as described above, apart from the two Prüfer modules* (1) *in Lemma* 3.3.

P r o o f. We know already that every indecomposable comultiplication non-separated module has one of these forms so it remains to show that the modules obtained by these amalgamation are, indeed, indecomposable comultiplication modules. Let M be an indecomposable non-separated comultiplication R-module and let $0 \to K \xrightarrow{i} S \xrightarrow{\varphi} M \to 0$ be a separated representation of M.

(i) Since M is of finite length, then M is a comultiplication R-module. Indecomposability follows from [15, 1.9].

(ii) and (iii) (involving one or two Prüfer modules): M is a comultiplication module (see Proposition 3.3 and Proposition 4.2) and indecomposability follows from [5, Theorem 3.5]. $\square$

Corollary 4.9. *Let R be the pullback ring as described in Theorem* 4.8. *Then every indecomposable comultiplication R-module is pure-injective.*

Proof. Apply [5, Theorem 3.5] and Theorem 4.8. $\square$

Remark 4.10. For a given field k, the infinite-dimensional k-algebra $T = k[x, y\colon xy = 0]_{(x,y)}$ is the pullback $(k[x]_{(x)} \to k \leftarrow k[y]_{(y)})$ of the local Dedekind domains $k[x]_{(x)}, k[y]_{(y)}$. This paper includes the classification of indecomposable comultiplication modules over T.

Acknowledgments. The second author expresses his thanks to Professor Mike Prest for several useful suggestions on the first draft of the manuscript. The authors thank the referee for his careful reading of the manuscript and suggesting improvements in the presentation.

References

[1] *I. Assen, D. Simson and A. Skowroński*: Representation Theory of Associative Algebra. Vol. 1. Techniques of Representation Theory. London Math. Soc. Student Texts 65, Cambridge Univ. Press, Cambridge-New York, 2007.
[2] *H. Ansari-Toroghy and F. Farshadifar*: On endomorphisms of multiplication and comultiplication modules. Archivum Mathematicum *44* (2008), 9–15.
[3] *H. Bass*: On the ubiquity of Gorenstein rings. Math. Z. *82* (1963), 8–29.
[4] *Z. A. El-Bast and P. F. Smith*: Multiplication modules. Comm. Algebra. *16* (1988), 755–779.
[5] *S. Ebrahimi Atani*: On pure-injective modules over pullback rings. Comm. Algebra *28* (2000), 4037–4069.
[6] *S. Ebrahimi Atani*: On secondary modules over pullback rings. Comm. Algebra *30* (2002), 2675–2685.
[7] *S. Ebrahimi Atani*: On prime modules over pullback rings. Czech. Math. J. *54* (2004), 781–789.
[8] *S. Ebrahimi Atani and F. Farzalipour*: Weak multiplication modules over a pullback of Dedekind domains. To appear in Colloq. Math.
[9] *S. Ebrahimi Atani*: Indecomposable weak multiplication modules over Dedekind domains. Demonstratio Mathematica XLI *41* (2008), 33–43.
[10] *S. Ebrahimi Atani*: On secondary modules over Dedekind domains. Southeast Asian Bull. Math. *25* (2001), 1–6.
[11] *A. Facchini and P. Vamos*: Injective modules over pullbacks. J. London Math. Soc. *31* (1985), 125–138.
[12] *R. Kietpiński*: On Γ-pure injective modules. Bull. Polon. Acad. Sci. Math. *15* (1967), 127–131.
[13] *L. Levy*: Modules over pullbacks and subdirect sums. J. Algebra *71* (1981), 50–61.
[14] *L. Levy*: Modules over Dedekind-like rings. J. Algebra *93* (1985), 1–116.

[15] *L. Levy*: Mixed modules over ZG, G cyclic of prime order, and over related Dedekind pullbacks. J. Algebra *70* (1981), 62–114.
[16] *L. A. Nazarova and A. V. Roiter*: Finitely generated modules over a dyad of local Dedekind rings and finite group having an abelian normal subgroup of index p. Izv. Acad. Nauk. SSSR *33* (1969), 65–89. (In Russian.)
[17] *M. Prest*: Model Theory and Modules. London Mathematical Society, Cambridge University Press, Cambridge, 1988.
[18] *M. Prest*: Ziegler spectra of tame hereditary algebras. J. Algebra *207* (1998), 146–164.
[19] *D. Simson*: Linear Representations of Partially Ordered Sets and Vector Space Categories. Algebra, Logic and Applications. Vol. 4, Gordon and Breach Science Publishers, Switzerland-Australia, 1992.
[20] *D. Simson*: On Corner type Endo-Wild algebras. J. Pure Appl. Algebra *202* (2005), 118–132.
[21] *R. Y. Sharp*: Steps in Commutative Algebra. London Mathematical Society, Student Texts, Vol. 19, Cambridge University Press, Cambridge, 1990.
[22] *R. B. Warfield*: Purity and algebraic compactness for modules. Pacific J. Math. *28* (1969), 699–719.
[23] *A. N. Wiseman*: Projective modules over pullback rings. Math. Proc. Cambridge Philos. Soc. *97* (1985), 399–106.

Authors' addresses: Shahabaddin Ebrahimi Atani, Department of Mathematics, University of Guilan, P.O. Box 1914, Rasht, Iran, e-mail: ebrahimi@guilan.ac.ir; Reza Ebrahimi Atani, Faculty of Engineering, University of Guilan, P.O. Box 3756, Rasht, Iran, e-mail: rebrahimi@guilan.ac.ir.