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CONCAVITY AND BOUNDS INVOLVING GENERALIZED ELLIPTIC INTEGRAL OF THE FIRST KIND

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Abstract. In the article, we provide a sufficient condition for value range of the constant c such that the function $x \to \mathcal{K}_a(\sqrt{x})/\log(c/\sqrt{1-x})$ is strictly concave on $(0,1)$ for $a \in (0,1/2]$, which generalize a very recently obtained result that the function $x \to \mathcal{K}(\sqrt{x})/\log(c/\sqrt{1-x})$ is strictly concave on $(0,1)$ if and only if $c = e^{4/3}$. As applications, we present new bounds for $\mathcal{K}_a(x)$, $\mathcal{K}_a(\sqrt{1-x^2})/\mathcal{K}_a(\sqrt{x})$ and $\mathcal{K}_a(\sqrt{1-x^2})\mathcal{K}_a(\sqrt{x})$, where $\mathcal{K}_a(x)$ is the generalized elliptic integral of the first kind and $\mathcal{K}(x) = \mathcal{K}_{1/2}(x)$.

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