# CONCENTRATIONS FOR NONLINEAR SCHRÖDINGER EQUATION WITH MAGNETIC POTENTIAL AND CONSTANT ELECTRIC POTENTIAL 

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#### Abstract

This paper studies the concentration phenomena to nonlinear Schrödinger equations with magnetic potential and constant electric potential. The existing results show that the magnetic field has no effect on the location of point concentrations, as long as the electric potential is not constant. This paper finds out what the role of the magnetic field plays in the location of concentrations when the electric potential is constant.


## 1. Introduction

In this paper, we investigate the magnetic Schrödinger equation in $\mathbb{R}^{N}$

$$
\begin{equation*}
\mathrm{i} \varepsilon \frac{\partial \psi}{\partial t}=(\mathrm{i} \varepsilon \nabla+\boldsymbol{A}(x))^{2} \psi+Q(x) \psi-|\psi|^{p-1} \psi \tag{1}
\end{equation*}
$$

where $\varepsilon>0$ is a small parameter, $1<p<\frac{N+2}{N-2}$ if $N \geq 3, p>1$ if $N=2$ and i is the imaginary unit. The function $\psi$ is complex-valued. Vector $\boldsymbol{A}=\left(A_{1}, A_{2}, \ldots, A_{N}\right): \mathbb{R}^{N} \rightarrow \mathbb{R}^{N}$ is the magnetic potential and assumed to be smooth, bounded and $Q: \mathbb{R}^{N} \rightarrow \mathbb{R}$ represents the electric potential. The magnetic Laplacian is defined by

$$
(\mathrm{i} \varepsilon \nabla+\boldsymbol{A})^{2} \psi:=-\varepsilon^{2} \Delta \psi+2 \mathrm{i} \varepsilon \boldsymbol{A} \cdot \nabla \psi+|\boldsymbol{A}|^{2} \psi+\mathrm{i} \varepsilon \psi \nabla \cdot \boldsymbol{A} .
$$

Terms involving $\boldsymbol{A}$ model the presence in some quantum model of the magnetic field $\boldsymbol{B}$ given by

$$
\boldsymbol{B}= \begin{cases}\partial_{1} A_{2}-\partial_{2} A_{1}, & \text { for } N=2 \\ \operatorname{curl} \boldsymbol{A}, & \text { for } N=3 \\ \left(\partial_{j} A_{k}-\partial_{k} A_{j}\right)_{N \times N}, & \text { for } N>3\end{cases}
$$

For the discussion of this operator, one may refer to (22] and 25].
The equation (1) arises in various physical contexts such as Bose-Einstein condensates and nonlinear optics, see [5] or plasma physics where one can simulate the interaction effect among many particles by introducing some nonlinear term, see 27. Concerning nonlinear Schrödinger equation with the magnetic field, the pioneer work is by Esteban-Lions 15 where they studied some minimization problems under suitable assumptions on the magnetic field by concentrations and compactness arguments. For more results, one can refer to ,1, 19, 18, 21] and the references therein.

From now on we consider standing wave solutions to problem (11), namely $\psi(t, x)=\mathrm{e}^{\mathrm{i} \lambda \varepsilon^{-1} t} u(x)$ for some complex-valued function $u(x)$. Substituting this ansatz into problem (1), $u(x)$ should satisfy

$$
\begin{equation*}
(\mathrm{i} \varepsilon \nabla+\boldsymbol{A})^{2} u+V(x) u-|u|^{p-1} u=0 \quad \text { in } \mathbb{R}^{N} \tag{2}
\end{equation*}
$$

where $V(x)=Q(x)+\lambda$. The potential $V(x)$ is usually assumed to be smooth and

$$
\inf _{\mathbb{R}^{N}} V(x)>0
$$

Problem (2) seems a very interesting problem since the Correspondence's Principle establishes that classical mechanics is, roughly speaking, contained in quantum mechanics. The mathematical transition from quantum mechanics to classical mechanics can be formally described by letting the Planck constant $\varepsilon \rightarrow 0$, and thus the existence of solutions for $\varepsilon$ small has physical interest. Standing waves for $\varepsilon$ small are usually referred as semi-classical bound states, see e.g. 19. In the linear case, Helffer et al. in 18, 20 have studied the asymptotic behavior of the eigenfunctions of the Schrödinger operators with magnetic fields in the semiclassical limit. Note that in these results, one can not see the magnetic field in the definition of the well.

Concentration plays the important role in the problem (2). The pioneer work of concentration to problem (2) with $\boldsymbol{A} \equiv 0$ in one dimension is given by Floer and Weistein 16]. Really they proved that if the electric potential $V(x)$ has a non-degenerate critical point, then $u(x)$ concentrates near this critical point as $\varepsilon \rightarrow 0$. After that, it

[^0]appears many results concerning concentration phenomena about problem (2). Different approaches are used to cover different cases, see [13, 24, 26, 33. Cingolani and Secchi 10] proved that for bounded vector potentials, concentration can happen at any non-degenerate critical point, not necessarily a minimum of $V$, as $\varepsilon \rightarrow 0$ using a perturbation approach given by Ambrosetti, Malchiodi and Secchi in [3]. Later they [11] extended the result to the case of a possibly unbounded vector potential by a penalization procedure (see 12]). Semiclassical multi-peak solutions were found for bounded vector potentials in 11, 9, 6, 17]. Li-Peng-Wang [23] constructed infinitely many non-radial solutions if the potentials $\boldsymbol{A}$ and $V$ are radially symmetric and suitably decay at infinity by using the number of bumps as parameter motivated by [34, see also e.g. 30] and the references therein for the application of such a very novel idea. For more recent results, we can refer to [7, 8, 28, 29] and the references therein. Also readers can refer to 2, 14, 31, 32 for high dimensional concentration. Till now, all these concentrations, especially point concentrations, are very dependent on critical points of $V(x)$ while the effect of the magnetic vector potential $\boldsymbol{A}$ is always ignored as higher order. In other words, $\boldsymbol{A}$ has no contribution to decide the location of point concentrations when $V(x)$ has critical points. On the other hand, for the case of the constant $V(x)$, there is no result on the concentration as far as we know.

Our aim in this paper is to exhibit how the magnetic potential $\boldsymbol{A}$ plays the role in the concentration. For this purpose, we study the following nonlinear magnetic Schrödinger equation

$$
\begin{equation*}
(\mathrm{i} \varepsilon \nabla+\boldsymbol{A}(x))^{2} u+u-|u|^{p-1} u=0 \quad \text { in } \mathbb{R}^{N} \tag{3}
\end{equation*}
$$

Here $p>1$ and $\boldsymbol{A}=\left(A_{1}, \ldots, A_{N}\right): \mathbb{R}^{N} \rightarrow \mathbb{R}^{N}$ is assumed in $W^{1, \infty}\left(\mathbb{R}^{N}, \mathbb{R}^{N}\right)$ and smooth everywhere for simplicity. To state our main result, we introduce the Frobenius norm of a matrix $\boldsymbol{\mathcal { M }}=\left(m_{i j}\right)_{I \times J}$

$$
\|\boldsymbol{\mathcal { M }}\|_{F}=\left(\sum_{i=1}^{I} \sum_{j=1}^{J}\left|m_{i j}\right|^{2}\right)^{\frac{1}{2}}
$$

And denote by $w(y)=w(|y|)$ the unique radial real-valued solution of

$$
\begin{equation*}
\Delta w-w+w^{p}=0 \quad \text { in } \mathbb{R}^{N}, \quad w(0)=\max _{\mathbb{R}^{N}} w>0, \quad w( \pm \infty)=0 \tag{4}
\end{equation*}
$$

Now our main results are the following.
Theorem 1.1. Assume that the Forbenius norm $\|\boldsymbol{B}\|_{F}$ of the magnetic field $\boldsymbol{B}$ admits $K$ local maximum (minimum) points $\left\{P_{m}\right\}_{m=1}^{K}$ (which may be degenerate) and $K$ disjoint, closed and bounded regions $\left\{\Omega_{m}\right\}_{m=1}^{K}$ of $\mathbb{R}^{N}$ such that

$$
\left\|\boldsymbol{B}\left(P_{m}\right)\right\|_{F}=\max _{\Omega_{m}}\|\boldsymbol{B}\|_{F}>\max _{\partial \Omega_{m}}\|\boldsymbol{B}\|_{F} . \quad\left(\left\|\boldsymbol{B}\left(P_{m}\right)\right\|_{F}=\min _{\Omega_{m}}\|\boldsymbol{B}\|_{F}<\min _{\partial \Omega_{m}}\|\boldsymbol{B}\|_{F} .\right)
$$

Then there exists an $\varepsilon_{0}>0$, such that for every $0<\varepsilon<\varepsilon_{0}$, the problem (3) admits a solution $u_{\varepsilon}$ with the form

$$
u_{\varepsilon}(x)=\sum_{m=1}^{K}\left[w\left(\frac{\left|x-\zeta_{m}\right|}{\varepsilon}\right)+\varepsilon \Psi_{m}\left(\frac{x}{\varepsilon}\right)\right] \mathrm{e}^{\mathrm{i} \sigma_{m}+\mathrm{i} \varepsilon^{-1} \boldsymbol{A}\left(\zeta_{m}\right) \cdot x}+O\left(\varepsilon^{2}\right)
$$

for some $\left(\sigma_{1}, \ldots, \sigma_{K}\right) \in[0,2 \pi)^{K}, \zeta_{m} \in \Omega_{m}$. The definition of $\Psi_{m}$ is given in (8).
For general critical points, i.e. not local extremum points, we also have the following result.
Theorem 1.2. Assume that $p>\frac{3}{2}$ and $P_{1}, P_{2}, \ldots, P_{K}, K \geq 1$ are all non-degenerate critical points of $\|\boldsymbol{B}\|_{F}^{2}$. Then there exists an $\varepsilon_{0}>0$ such that for any $0<\varepsilon<\varepsilon_{0}$, the problem (3) admits a solution $u_{\varepsilon}$ with the form

$$
u_{\varepsilon}(x)=\sum_{m=1}^{K}\left[w\left(\frac{\left|x-\zeta_{m}\right|}{\varepsilon}\right)+\varepsilon \Psi_{m}\left(\frac{x}{\varepsilon}\right)\right] \mathrm{e}^{\mathrm{i} \sigma_{m}+\mathrm{i} \varepsilon^{-1} \boldsymbol{A}\left(\zeta_{m}\right) \cdot x}+O\left(\varepsilon^{2}\right)
$$

for some $\left(\sigma_{1}, \ldots, \sigma_{K}\right) \in[0,2 \pi)^{K}$ and $\zeta_{m}=P_{m}+o(1)$. The definition of $\Psi_{m}$ is given in (乌).
Remark 1.3. Theorem 1.1 and Theorem 1.7 clearly show the principle of the magnetic vector $\boldsymbol{A}$, or precisely the magnetic field $\boldsymbol{B}$ driving the location of concentration if the electric potential is constant. This is a completely new phenomenon up to now.
Remark 1.4. Theorem 1.1 holds for any $p>1$. And Theorem 1.2 holds only for $p>\frac{3}{2}$. The reason is that the critical point of the function deduced by the energy functional is the local extremum point in Theorem 1.1, which may be gotten by direct comparison. While in Theorem 1.8, $P_{1}, P_{2}, \ldots, P_{K}$ may not be extremum points any more. Thus we have to study the derivatives of the energy functional. Naturally, the corresponding estimates should be higher accuracy for which the better regularity of the nonlinear term $|u|^{p-1} u$ is required.

From the point view of physics, the magnetic field $\boldsymbol{B}$ is essential, not the particular choice of magnetic potential $\boldsymbol{A}$. At the same time, there is a gauge invariance for the magnetic Laplacian correspondingly, that is, the magnetic field $\boldsymbol{B}$ is invariant under the transform of the potential $\boldsymbol{A} \rightarrow \boldsymbol{A}+\nabla f$. Also it is easy to see that the energy of the problem (3) is unchanged under the gauge invariance, with which our result coincides (see Proposition 4.3).

When the electric potential $V(x)$ has some non-degeneracy, we just need to make the first approximation by the limit equation, which is enough to deduce the role of the function $V(x)$. But in order to see the role of magnetic vector potential $\boldsymbol{A}$ in our case, we need the approximation up to order $\varepsilon$ of the problem (3). Hence it is necessary to get a more accurate expression. On the other hand, higher accuracy allows us to deal with extremum points not necessary minimum points, which is different from the case of the non-constant $V(x)$, see e.g. 17. Finally we note that the solutions given in Theorem 1.1 and Theorem 1.2 both show simple concentrations. In the forthcoming paper, we will deal with multi-bump solutions for constant electric potential.

The paper is organized as follows. In section 2, the ansatz and the estimate of the error are given. The corresponding non-linear problem is solved in section 3. In section 4, the original problem is reduced to the finite dimension problem using variational reduction process and the expansion of the energy functional is shown. Finally, in section 5, Theorem 1.1 and Theorem 1.2 are proved.

## Notations.

1. Constant $\beta=\min \{p-1,1\}$.
2. The real part of $z \in \mathbb{C}$ will be denoted by $\operatorname{Re} z$.
3. The complex conjugate of $z \in \mathbb{C}$ will be denoted by $\bar{z}$.
4. $C$ denotes a generic positive constant, which may be different from lines to lines.
5. Landau symbols $O(\varepsilon)$ is a generic function such that $|O(\varepsilon)| \leq C \varepsilon$ and $o(\varepsilon)$ means that $\lim _{\varepsilon \rightarrow 0^{+}} \frac{|o(\varepsilon)|}{\varepsilon}=0$.

## 2. Ansatz

In this section we will present the approximation of the problem and give the corresponding error estimate. Recall in 21 that the problem

$$
\Delta \tilde{w}-\tilde{w}+|\tilde{w}|^{p-1} \tilde{w}=0, \quad \tilde{w} \in H^{1}\left(\mathbb{R}^{N}, \mathbb{C}\right)
$$

possesses a unique ground state solution $\tilde{w}(y)=w(y) \mathrm{e}^{\mathrm{i} \sigma}, \forall \sigma \in[0,2 \pi]$ where $w(y)$ is the radial solution of problem (4). Thus by the gauge invariance,

$$
\widetilde{U}(x)=w\left(\frac{|x-\zeta|}{\varepsilon}\right) \mathrm{e}^{\mathrm{i} \sigma+\mathrm{i} \varepsilon^{-1} \boldsymbol{A}(\zeta) \cdot x}, \quad \forall \sigma \in[0,2 \pi],
$$

is also the ground state solution to the constant magnetic potential problem

$$
(\mathrm{i} \varepsilon \nabla+\boldsymbol{A}(\zeta))^{2} \widetilde{U}+\widetilde{U}-|\widetilde{U}|^{p-1} \widetilde{U}=0, \quad \widetilde{U} \in H^{1}\left(\mathbb{R}^{N}, \mathbb{C}\right)
$$

In the frame of large variable $y=x / \varepsilon$, the original problem (3) is equivalent to

$$
\begin{equation*}
(\mathrm{i} \nabla+\boldsymbol{A}(\varepsilon y))^{2} u+u-|u|^{p-1} u=0 \quad \text { in } \mathbb{R}^{N} \tag{5}
\end{equation*}
$$

Therefore, the function

$$
U(y)=w\left(\left|y-\zeta^{\prime}\right|\right) \mathrm{e}^{\mathrm{i} \sigma+\mathrm{i} \boldsymbol{A}(\zeta) \cdot y}, \quad \zeta^{\prime}=\zeta / \varepsilon
$$

formally approximates the solution at least near $y=\zeta^{\prime}$. Now it's time to give the first approximation

$$
\begin{equation*}
W(y)=\sum_{m=1}^{K} U_{m}(y), \quad U_{m}(y)=w\left(\left|y-\zeta_{m}^{\prime}\right|\right) \mathrm{e}^{\mathrm{i} \sigma_{m}+\mathrm{i} \boldsymbol{A}\left(\zeta_{m}\right) \cdot y}, \quad \sigma_{m} \in[0,2 \pi] \tag{6}
\end{equation*}
$$

where $\zeta_{m} \in \Omega_{m}, m=1,2, \ldots, K$. Denote

$$
\rho:=\min _{1 \leq i \neq j \leq K} \operatorname{dist}\left(\Omega_{i}, \Omega_{j}\right)>0
$$

and take a positive number $\delta$ such that $\delta \leq \frac{1}{4} \rho$. Let $\widehat{R}(y)$ be the error caused by the first approximation $W$, which is

$$
\widehat{R}(y)=(\mathrm{i} \nabla+\boldsymbol{A}(\varepsilon y))^{2} W+W-|W|^{p-1} W .
$$

It is checked that for $\left|y-\zeta_{1}^{\prime}\right| \leq \frac{\delta}{\sqrt{\varepsilon}}, \quad\left|U_{m}\right| \leq\left|U_{1}\right| e^{-\frac{\left|\zeta_{m}^{\prime}-\zeta_{1}^{\prime}\right|}{3}}, m=2,3, \ldots, K$ and

$$
\begin{aligned}
\widehat{R}(y) & =\widetilde{R}(y)+O\left(w\left(\left|y-\zeta_{1}^{\prime}\right|\right) \max _{2 \leq m \leq K} e^{-\frac{\left|\zeta_{m}-\zeta_{1}\right|}{3 \varepsilon}}+w^{p-1}\left(\left|y-\zeta_{1}^{\prime}\right|\right) \max _{2 \leq m \leq K} e^{-\frac{\left|\zeta_{m}-\zeta_{1}\right|}{2 \varepsilon}}\right) \\
& =\widetilde{R}(y)+O\left(w\left(\left|y-\zeta_{1}^{\prime}\right|\right)+w^{p-1}\left(\left|y-\zeta_{1}^{\prime}\right|\right)\right) e^{-\frac{\delta}{\varepsilon}}
\end{aligned}
$$

where

$$
\begin{align*}
\widetilde{R}(y)= & (\mathrm{i} \nabla+\boldsymbol{A}(\varepsilon y))^{2} U_{1}+U_{1}-\left|U_{1}\right|^{p-1} U_{1}=(\mathrm{i} \nabla+\boldsymbol{A}(\varepsilon y))^{2} U_{1}-\left(\mathrm{i} \nabla+\boldsymbol{A}\left(\zeta_{1}\right)\right)^{2} U_{1} \\
= & 2 \mathrm{i}\left(\boldsymbol{A}(\varepsilon y)-\boldsymbol{A}\left(\zeta_{1}\right)\right) \cdot \nabla U_{1}+\mathrm{i} \varepsilon\left(\nabla_{x} \cdot A\right) U_{1}+\left(|\boldsymbol{A}(\varepsilon y)|^{2}-\left|\boldsymbol{A}\left(\zeta_{1}\right)\right|^{2}\right) U_{1} \\
= & 2 \mathrm{i}\left(\boldsymbol{A}(\varepsilon y)-\boldsymbol{A}\left(\zeta_{1}\right)\right) \cdot\left[\nabla w\left(y-\zeta_{1}^{\prime}\right)+\mathrm{i} \boldsymbol{A}\left(\zeta_{1}\right) w\left(y-\zeta_{1}^{\prime}\right)\right] \mathrm{e}^{\mathrm{i} \sigma_{1}+\mathrm{i} \boldsymbol{A}\left(\zeta_{1}\right) \cdot y}+\mathrm{i} \varepsilon\left(\nabla_{x} \cdot \boldsymbol{A}\right) U_{1} \\
& +\left(|\boldsymbol{A}(\varepsilon y)|^{2}-\left|\boldsymbol{A}\left(\zeta_{1}\right)\right|^{2}\right) w\left(y-\zeta_{1}^{\prime}\right) \mathrm{e}^{\mathrm{i} \sigma_{1}+\mathrm{i} \boldsymbol{A}\left(\zeta_{1}\right) \cdot y} \\
= & {\left[-2 \boldsymbol{A}\left(\zeta_{1}\right) \cdot\left(\boldsymbol{A}(\varepsilon y)-\boldsymbol{A}\left(\zeta_{1}\right)\right)+\left(|\boldsymbol{A}(\varepsilon y)|^{2}-\left|\boldsymbol{A}\left(\zeta_{1}\right)\right|^{2}\right)\right] U_{1} } \\
& +\mathrm{i}\left[2\left(\boldsymbol{A}(\varepsilon y)-\boldsymbol{A}\left(\zeta_{1}\right)\right) \cdot \nabla w\left(y-\zeta_{1}^{\prime}\right)+\varepsilon\left(\nabla_{x} \cdot A\right) w\left(y-\zeta_{1}^{\prime}\right)\right] \mathrm{e}^{\mathrm{i} \sigma_{1}+\mathrm{i} \boldsymbol{A}\left(\zeta_{1}\right) \cdot y} \\
= & \left(\left|\boldsymbol{A}(\varepsilon y)-\boldsymbol{A}\left(\zeta_{1}\right)\right|\right)^{2} w\left(y-\zeta_{1}^{\prime}\right) \mathrm{e}^{\mathrm{i} \sigma_{1}+\mathrm{i} \boldsymbol{A}\left(\zeta_{1}\right) \cdot y} \\
& +\mathrm{i}\left[2\left(\boldsymbol{A}(\varepsilon y)-\boldsymbol{A}\left(\zeta_{1}\right)\right) \cdot \nabla w\left(y-\zeta_{1}^{\prime}\right)+\varepsilon\left(\nabla_{x} \cdot \boldsymbol{A}\right) w\left(y-\zeta_{1}^{\prime}\right)\right] \mathrm{e}^{\mathrm{i} \sigma_{1}+\mathrm{i} \boldsymbol{A}\left(\zeta_{1}\right) \cdot y} . \tag{7}
\end{align*}
$$

As usual one writes $\zeta_{m}^{\prime}=\left(\zeta_{m, 1}^{\prime}, \zeta_{m, 2}^{\prime}, \ldots, \zeta_{m, N}^{\prime}\right)$. Direct calculation shows that

$$
\begin{aligned}
& \left|\boldsymbol{A}(\varepsilon y)-\boldsymbol{A}\left(\zeta_{1}\right)\right|^{2}=\sum_{i=1}^{N}\left(A_{i}(\varepsilon y)-A_{i}\left(\zeta_{1}\right)\right)^{2} \\
= & \sum_{i=1}^{N}\left(\varepsilon \nabla A_{i}\left(\zeta_{1}\right) \cdot\left(y-\zeta_{1}^{\prime}\right)+\frac{\varepsilon^{2}}{2}\left(y-\zeta_{1}^{\prime}\right)^{\perp} \cdot \nabla^{2} A_{i}\left(\zeta_{1}\right) \cdot\left(y-\zeta_{1}^{\prime}\right)+O\left(\varepsilon^{3}\left|y-\zeta_{1}^{\prime}\right|^{3}\right)\right)^{2} \\
= & \varepsilon^{2} \sum_{i, j, k=1}^{N} \partial_{j} A_{i}\left(\zeta_{1}\right) \partial_{k} A_{i}\left(\zeta_{1}\right)\left(y_{j}-\zeta_{1, j}^{\prime}\right)\left(y_{k}-\zeta_{1, k}^{\prime}\right) \\
& +\varepsilon^{3} \sum_{i, j, k, \ell=1}^{N} \partial_{j} A_{i}\left(\zeta_{1}\right) \partial_{k \ell} A_{i}\left(\zeta_{1}\right)\left(y_{j}-\zeta_{1, j}^{\prime}\right)\left(y_{k}-\zeta_{1, k}^{\prime}\right)\left(y_{\ell}-\zeta_{1, \ell}^{\prime}\right)+O\left(\varepsilon^{4}\left|y-\zeta_{1}^{\prime}\right|^{4}\right)
\end{aligned}
$$

Similarly, it is verified that

$$
\begin{aligned}
& \left(\boldsymbol{A}(\varepsilon y)-\boldsymbol{A}\left(\zeta_{1}\right)\right) \cdot \nabla w\left(\left|y-\zeta_{1}^{\prime}\right|\right)=\sum_{i=1}^{N}\left(A_{i}(\varepsilon y)-A_{i}\left(\zeta_{1}\right)\right) \partial_{i} w\left(\left|y-\zeta_{1}^{\prime}\right|\right) \\
= & \varepsilon \sum_{i, j=1}^{N} \partial_{j} A_{i}\left(\zeta_{1}\right)\left(y_{j}-\zeta_{1, j}^{\prime}\right) \partial_{i} w\left(\left|y-\zeta_{1}^{\prime}\right|\right)+\frac{\varepsilon^{2}}{2} \sum_{i, j, k=1}^{N} \partial_{j k} A_{i}\left(\zeta_{1}\right)\left(y_{j}-\zeta_{1, j}^{\prime}\right)\left(y_{k}-\zeta_{1, k}^{\prime}\right) \partial_{i} w\left(\left|y-\zeta_{1}^{\prime}\right|\right) \\
& +\frac{\varepsilon^{3}}{6} \sum_{i, j, k, \ell=1}^{N} \partial_{j k \ell} A_{i}\left(\zeta_{1}\right)\left(y_{j}-\zeta_{1, j}^{\prime}\right)\left(y_{k}-\zeta_{1, k}^{\prime}\right)\left(y_{\ell}-\zeta_{1, \ell}^{\prime}\right) \partial_{i} w\left(\left|y-\zeta_{1}^{\prime}\right|\right)+O\left(\varepsilon^{4}\left|y-\zeta_{1}^{\prime}\right|^{4}\left|\nabla w\left(\left|y-\zeta_{1}^{\prime}\right|\right)\right|\right) .
\end{aligned}
$$

As for the term $\nabla_{x} \cdot \boldsymbol{A}(\varepsilon y) w\left(\left|y-\zeta_{1}^{\prime}\right|\right)$, we expand it similarly to

$$
\begin{aligned}
& \nabla_{x} \cdot \boldsymbol{A}(\varepsilon y) w\left(\left|y-\zeta_{1}^{\prime}\right|\right) \\
= & \nabla_{x} \cdot \boldsymbol{A}\left(\zeta_{1}\right) w\left(\left|y-\zeta_{1}^{\prime}\right|\right)+\varepsilon \nabla\left(\nabla_{x} \cdot \boldsymbol{A}\right)\left(\zeta_{1}\right) \cdot\left(y-\zeta_{1}^{\prime}\right) w\left(\left|y-\zeta_{1}^{\prime}\right|\right) \\
& +\frac{1}{2} \varepsilon^{2}\left(y-\zeta_{1}^{\prime}\right)^{\perp} \cdot \nabla^{2}\left(\nabla_{x} \cdot \boldsymbol{A}\right)\left(\zeta_{1}\right) \cdot\left(y-\zeta_{1}^{\prime}\right)+O\left(\varepsilon^{3}\left|y-\zeta_{1}^{\prime}\right|^{3} w\left(\left|y-\zeta_{1}^{\prime}\right|\right)\right) \\
= & \nabla_{x} \cdot \boldsymbol{A}\left(\zeta_{1}\right) w\left(\left|y-\zeta_{1}^{\prime}\right|\right)+\varepsilon \sum_{i, j=1}^{N} \partial_{i j} A_{i}\left(\zeta_{1}\right)\left(y_{j}-\zeta_{1, j}^{\prime}\right) w\left(\left|y-\zeta_{1}^{\prime}\right|\right) \\
& +\frac{\varepsilon^{2}}{2} \sum_{i, j, k=1}^{N} \partial_{j k i} A_{i}\left(\zeta_{1}\right)\left(y_{j}-\zeta_{1, j}^{\prime}\right)\left(y_{k}-\zeta_{1, k}^{\prime}\right) w\left(\left|y-\zeta_{1}^{\prime}\right|\right)+O\left(\varepsilon^{3}\left|y-\zeta_{1}^{\prime}\right|^{3} w\left(\left|y-\zeta_{1}^{\prime}\right|\right)\right)
\end{aligned}
$$

Therefore, in the region $\left|y-\zeta_{1}^{\prime}\right|<\frac{\delta}{\sqrt{\varepsilon}}$, we get that

$$
\begin{aligned}
& \widetilde{R}(y) \mathrm{e}^{-\mathrm{i} \sigma_{1}-\mathrm{i} \boldsymbol{A}\left(\zeta_{1}\right) \cdot y} \\
= & \varepsilon \mathrm{i}\left(2 \sum_{i, j=1}^{N} \partial_{j} A_{i}\left(\zeta_{1}\right)\left(y_{j}-\zeta_{1, j}^{\prime}\right) \partial_{i} w\left(\left|y-\zeta_{1}^{\prime}\right|\right)+\nabla_{x} \cdot \boldsymbol{A}\left(\zeta_{1}\right) w\left(\left|y-\zeta_{1}^{\prime}\right|\right)\right) \\
& +\varepsilon^{2} \sum_{i, j, k=1}^{N} \partial_{j} A_{i}\left(\zeta_{1}\right) \partial_{k} A_{i}\left(\zeta_{1}\right)\left(y_{j}-\zeta_{1, j}^{\prime}\right)\left(y_{k}-\zeta_{1, k}^{\prime}\right) w\left(\left|y-\zeta_{1}^{\prime}\right|\right)
\end{aligned}
$$

$$
\begin{aligned}
& +\varepsilon^{2} \mathrm{i} \sum_{i, j, k=1}^{N} \partial_{j k} A_{i}\left(\zeta_{1}\right)\left(y_{j}-\zeta_{1, j}^{\prime}\right)\left(y_{k}-\zeta_{1, k}^{\prime}\right) \partial_{i} w\left(\left|y-\zeta_{1}^{\prime}\right|\right)+\varepsilon^{2} \mathrm{i} \sum_{i, j=1}^{N} \partial_{i j} A_{i}\left(\zeta_{1}\right)\left(y_{j}-\zeta_{1, j}^{\prime}\right) w\left(\left|y-\zeta_{1}^{\prime}\right|\right) \\
& +\varepsilon^{3} \sum_{i, j, k, \ell=1}^{N} \partial_{j} A_{i}\left(\zeta_{1}\right) \partial_{k \ell} A_{i}\left(\zeta_{1}\right)\left(y_{j}-\zeta_{1, j}^{\prime}\right)\left(y_{k}-\zeta_{1, k}^{\prime}\right)\left(y_{\ell}-\zeta_{1, \ell}^{\prime}\right) w\left(\left|y-\zeta_{1}^{\prime}\right|\right) \\
& +\frac{\varepsilon^{3}}{3} \mathrm{i} \sum_{i, j, k, \ell=1}^{N} \partial_{j k \ell} A_{i}\left(\zeta_{1}\right)\left(y_{j}-\zeta_{1, j}^{\prime}\right)\left(y_{k}-\zeta_{1, k}^{\prime}\right)\left(y-\zeta_{1, \ell}^{\prime}\right) \partial_{i} w\left(\left|y-\zeta_{1}^{\prime}\right|\right) \\
& +\frac{\varepsilon^{3}}{2} \mathrm{i} \sum_{i, j, k=1}^{N} \partial_{j k i} A_{i}\left(\zeta_{1}\right)\left(y_{j}-\zeta_{1, j}^{\prime}\right)\left(y_{k}-\zeta_{1, k}^{\prime}\right) w\left(\left|y-\zeta_{1}^{\prime}\right|\right) \\
& +O\left(\varepsilon^{4}\left|y-\zeta_{1}^{\prime}\right|^{4}\left|w\left(\left|y-\zeta_{1}^{\prime}\right|\right)\right|\right)+\mathrm{i} O\left(\varepsilon^{4}\left|y-\zeta_{1}^{\prime}\right|^{4}\left|\nabla w\left(\left|y-\zeta_{1}^{\prime}\right|\right)\right|+\varepsilon^{4}\left|y-\zeta_{1}^{\prime}\right|^{3}\left|w\left(\left|y-\zeta_{1}^{\prime}\right|\right)\right|\right)
\end{aligned}
$$

Note that the imaginary part of $\widetilde{R}(y) \mathrm{e}^{-\mathrm{i} \sigma_{1}-\mathrm{i} \boldsymbol{A}\left(\zeta_{1}\right) \cdot y}$ is $O(\varepsilon)$. It is of less accuracy for later application. Complying with the guideline that the better approximation, the more possibility to get a solution, we should improve the accuracy of the approximation. To this purpose, we find that the real-valued function $\Psi_{1, i j}(y)=$ $\frac{1}{2}\left(y_{i}-\zeta_{1, i}^{\prime}\right)\left(y_{j}-\zeta_{1, j}^{\prime}\right) w\left(\left|y-\zeta_{1}^{\prime}\right|\right)(i \neq j)$ is the solution to

$$
-\Delta \Psi_{1, i j}+\Psi_{1, i j}-w^{p-1}\left(\left|y-\zeta_{1}^{\prime}\right|\right) \Psi_{1, i j}=-2\left(y_{j}-\zeta_{1, j}^{\prime}\right) \partial_{i} w\left(\left|y-\zeta_{1}^{\prime}\right|\right)=-2\left(y_{i}-\zeta_{1, i}^{\prime}\right) \partial_{j} w\left(\left|y-\zeta_{1}^{\prime}\right|\right)
$$

Besides, $\Psi_{1, i i}(y)=\frac{1}{2}\left(y_{i}-\zeta_{1, i}^{\prime}\right)^{2} w\left(\left|y-\zeta_{1}^{\prime}\right|\right)$ satisfies the equation

$$
-\Delta \Psi_{1, i i}+\Psi_{1, i i}-w^{p-1}\left(\left|y-\zeta_{1}^{\prime}\right|\right) \Psi_{1, i i}=-2\left(y_{i}-\zeta_{1, i}^{\prime}\right) \partial_{i} w\left(\left|y-\zeta_{1}^{\prime}\right|\right)-w
$$

Then obviously the function

$$
\begin{aligned}
\Psi_{1}(y) & =\mathrm{i}\left(\sum_{i=1}^{N} \sum_{j \neq i} \partial_{j} A_{i}\left(\zeta_{1}\right) \Psi_{1, i j}(y)+\sum_{i=1}^{N} \partial_{i} A_{i}\left(\zeta_{1}\right) \Psi_{1, i i}(y)\right) \mathrm{e}^{\mathrm{i} \sigma_{1}+\mathrm{i} \boldsymbol{A}\left(\zeta_{1}\right) \cdot y} \\
& =\mathrm{i}\left(\sum_{i, j=1}^{N} \partial_{j} A_{i}\left(\zeta_{1}\right) \Psi_{1, i j}(y)\right) \mathrm{e}^{\mathrm{i} \sigma_{1}+\mathrm{i} \boldsymbol{A}\left(\zeta_{1}\right) \cdot y}:=\mathrm{i} \psi_{1}(y) \mathrm{e}^{\mathrm{i} \sigma_{1}+\mathrm{i} \boldsymbol{A}\left(\zeta_{1}\right) \cdot y}
\end{aligned}
$$

satisfies

$$
\begin{gathered}
\left(\mathrm{i} \nabla+\boldsymbol{A}\left(\zeta_{1}\right)\right)^{2} \Psi_{1}+\Psi_{1}-(p-1)\left|U_{1}\right|^{p-3} \operatorname{Re}\left(\bar{U}_{1} \Psi_{1}\right) U_{1}-\left|U_{1}\right|^{p-1} \Psi_{1} \\
=\quad \mathrm{i}\left[-2 \sum_{i, j=1}^{N} \partial_{j} A_{i}\left(\zeta_{1}\right)\left(y_{j}-\zeta_{1, j}^{\prime}\right) \partial_{i} w\left(\left|y-\zeta_{1}^{\prime}\right|\right)-\nabla_{x} \cdot \boldsymbol{A}\left(\zeta_{1}\right) w\left(\left|y-\zeta_{1}^{\prime}\right|\right)\right] \mathrm{e}^{\mathrm{i} \sigma_{1}+\mathrm{i} \boldsymbol{A}\left(\zeta_{1}\right) \cdot y},
\end{gathered}
$$

since $\operatorname{Re}\left(\bar{U}_{1} \Psi_{1}\right)=0$. Moreover, the same computation may also be carried out in the region $\left|y-\zeta_{m}^{\prime}\right|<\frac{\delta}{\sqrt{\varepsilon}}$ for $m=2, \cdots, K$. The ultimate approximation is then selected as

$$
\begin{equation*}
\mathcal{W}(y)=W(y)+\varepsilon \Psi(y) \tag{8}
\end{equation*}
$$

where $\Psi(y)=\sum_{m=1}^{K} \Psi_{m}(y)$. Here

$$
\begin{equation*}
\Psi_{m}(y)=\mathrm{i}\left(\sum_{i, j=1}^{N} \partial_{j} A_{i}\left(\zeta_{m}\right) \Psi_{m, i j}(y)\right) \mathrm{e}^{\mathrm{i} \sigma_{m}+\mathrm{i} \boldsymbol{A}\left(\zeta_{m}\right) \cdot y}:=\mathrm{i} \psi_{m}(y) \mathrm{e}^{\mathrm{i} \sigma_{m}+\mathrm{i} \boldsymbol{A}\left(\zeta_{m}\right) \cdot y} \tag{9}
\end{equation*}
$$

and

$$
\Psi_{m, i j}(y)=\frac{1}{2}\left(y_{i}-\zeta_{m, i}^{\prime}\right)\left(y_{j}-\zeta_{m, j}^{\prime}\right) w\left(\left|y-\zeta_{m}^{\prime}\right|\right)
$$

Now the approximation $\mathcal{W}$ is good enough to help us find a solution. Precisely our aim is to find a solution with the form $\mathcal{W}(y)+\phi(y)$ of problem (3), where $\phi$ is a small perturbation and satisfies the equation

$$
\begin{equation*}
L \phi:=(\mathrm{i} \nabla+\boldsymbol{A}(\varepsilon y))^{2} \phi+\phi-(p-1)|\mathcal{W}|^{p-3} \operatorname{Re}(\overline{\mathcal{W}} \phi) \mathcal{W}-|\mathcal{W}|^{p-1} \phi=-R(y)+N(\phi) \tag{10}
\end{equation*}
$$

Here $R(y)$ denotes the error caused by $\mathcal{W}$, which is

$$
R(y)=(\mathrm{i} \nabla+\boldsymbol{A}(\varepsilon y))^{2} \mathcal{W}+\mathcal{W}-|\mathcal{W}|^{p-1} \mathcal{W}
$$

and the nonlinear term

$$
N(\phi)=|\mathcal{W}+\phi|^{p-1}(\mathcal{W}+\phi)-|\mathcal{W}|^{p-1} \mathcal{W}-(p-1)|\mathcal{W}|^{p-3} \operatorname{Re}(\overline{\mathcal{W}} \phi) \mathcal{W}-|\mathcal{W}|^{p-1} \phi
$$

Obviously, with the notation $\beta=\min \{p-1,1\}$,

$$
|N(\phi)| \leq C|\phi|^{1+\beta}
$$

To solve problem (10), it is important to estimate the error $R(y)$.
Proposition 2.1. We have that

$$
\|R(y)\|_{L^{2}} \leq C \varepsilon^{2}
$$

and

$$
\left\|\partial_{\zeta_{m, k}^{\prime}} R(y)\right\|_{L^{2}} \leq C \varepsilon^{2}, \quad\left\|\partial_{\sigma_{m}} R(y)\right\|_{L^{2}} \leq C \varepsilon^{2}, \quad \forall m=1, \ldots, K, k=1, \ldots, N
$$

Proof. First we consider the domain $\left|y-\zeta_{m}^{\prime}\right| \leq \frac{\delta}{\sqrt{\varepsilon}}$, without loss of generality, say $m=1$. Then

$$
\begin{aligned}
R(y)= & \widehat{R}(y)+\varepsilon\left[(\mathrm{i} \nabla+\boldsymbol{A}(\varepsilon y))^{2} \Psi+\Psi-(p-1)|W|^{p-3} \operatorname{Re}(\bar{W} \Psi) W-|W|^{p-1} \Psi\right] \\
& -\left[|W+\varepsilon \Psi|^{p-1}(W+\varepsilon \Psi)-|W|^{p-1} W-\varepsilon(p-1)|W|^{p-3} \operatorname{Re}(\bar{W} \Psi) W-\varepsilon|W|^{p-1} \Psi\right] \\
= & \widetilde{R}(y)-\varepsilon \mathrm{i}\left(2 \sum_{i, j=1}^{N} \partial_{j} A_{i}\left(\zeta_{1}\right)\left(y_{j}-\zeta_{1, j}^{\prime}\right) \partial_{i} w\left(y-\zeta_{1}^{\prime}\right)+\left(\nabla_{x} \cdot \boldsymbol{A}\right)\left(\zeta_{1}\right) w\left(\left|y-\zeta_{1}^{\prime}\right|\right)\right) \mathrm{e}^{\mathrm{i} \sigma_{1}+\mathrm{i} \boldsymbol{A}\left(\zeta_{1}\right) \cdot y} \\
& +\varepsilon\left[(\mathrm{i} \nabla+\boldsymbol{A}(\varepsilon y))^{2} \Psi_{1}-\left(\mathrm{i} \nabla+\boldsymbol{A}\left(\zeta_{1}\right)\right)^{2} \Psi_{1}\right]+\varepsilon\left[(\mathrm{i} \nabla+\boldsymbol{A}(\varepsilon y))^{2} \sum_{m=2}^{K} \Psi_{m}+\sum_{m=2}^{K} \Psi_{m}\right] \\
& -\frac{p-1}{2} \varepsilon^{2} \psi_{1}^{2}(y) w^{p-2}\left(\left|y-\zeta_{1}^{\prime}\right|\right) \mathrm{e}^{\mathrm{i} \sigma_{1}+\mathrm{i} \boldsymbol{A}\left(\zeta_{1}\right) \cdot y}-\mathrm{i} \frac{p-1}{2} \varepsilon^{3} \psi_{1}^{3}(y) w^{p-3}\left(\left|y-\zeta_{1}^{\prime}\right|\right) \mathrm{e}^{\mathrm{i} \sigma_{1}+\mathrm{i} \boldsymbol{A}\left(\zeta_{1}\right) \cdot y} \\
& +O\left(\varepsilon^{4}\left|y-\zeta_{1}^{\prime}\right|^{8} \left\lvert\, w^{p}\left(\left|y-\zeta_{1}^{\prime}\right|\right)+\mathrm{e}^{-\delta / \varepsilon} w\left(\left|y-\zeta_{1}^{\prime}\right|\right)+\mathrm{e}^{-\frac{\delta}{\varepsilon}} w^{p-1}\left(\left|y-\zeta_{1}^{\prime}\right|\right)\right.\right)
\end{aligned}
$$

Obviously we have that

$$
\begin{aligned}
& \varepsilon^{2} \psi_{1}(y)^{2} w^{p-2}\left(y-\zeta_{1}^{\prime}\right)=\varepsilon^{2} w^{p-2}\left(\left|y-\zeta_{1}^{\prime}\right|\right) \sum_{i, j=1}^{N} \partial_{j} A_{i}\left(\zeta_{1}\right) \Psi_{1, i j}(y) \sum_{k, \ell=1}^{N} \partial_{\ell} A_{k}\left(\zeta_{1}\right) \Psi_{1, k \ell}(y) \\
= & \frac{\varepsilon^{2}}{4} w^{p}\left(\left|y-\zeta_{1}^{\prime}\right|\right) \sum_{i, j, k, \ell=1}^{N} \partial_{j} A_{i}\left(\zeta_{1}\right) \partial_{\ell} A_{k}\left(\zeta_{1}\right)\left(y_{i}-\zeta_{1, i}^{\prime}\right)\left(y_{j}-\zeta_{1, j}^{\prime}\right)\left(y_{k}-\zeta_{1, k}^{\prime}\right)\left(y_{\ell}-\zeta_{1, \ell}^{\prime}\right)
\end{aligned}
$$

and

$$
\varepsilon\left|\left[(\mathrm{i} \nabla+\boldsymbol{A}(\varepsilon y))^{2} \sum_{m=2}^{K} \Psi_{m}+\sum_{m=2}^{K} \Psi_{m}\right]\right|=O\left(\sum_{m=2}^{K} w\left(y-\zeta_{m}^{\prime}\right)\right)=O\left(\mathrm{e}^{-\frac{\delta}{\varepsilon}} w\left(y-\zeta_{1}^{\prime}\right)\right)
$$

Moreover, it is checked as in (7) that

$$
\begin{align*}
& (\mathrm{i} \nabla+\boldsymbol{A}(\varepsilon y))^{2} \Psi_{1}-\left(\mathrm{i} \nabla+\boldsymbol{A}\left(\zeta_{1}\right)\right)^{2} \Psi_{1} \\
= & 2 \mathrm{i}\left(\boldsymbol{A}(\varepsilon y)-\boldsymbol{A}\left(\zeta_{1}\right)\right) \cdot \nabla \Psi_{1}+\mathrm{i} \varepsilon\left(\nabla_{x} \cdot \boldsymbol{A}(\varepsilon y)\right) \Psi_{1}+\left(|\boldsymbol{A}(\varepsilon y)|^{2}-\left|\boldsymbol{A}\left(\zeta_{1}\right)\right|^{2}\right) \Psi_{1} \\
= & -\left[2\left(\boldsymbol{A}(\varepsilon y)-\boldsymbol{A}\left(\zeta_{1}\right)\right) \cdot \nabla \psi_{1}+\varepsilon\left(\nabla_{x} \cdot \boldsymbol{A}(\varepsilon y)\right) \psi_{1}\right] e^{\mathrm{i} \sigma_{1}+\mathrm{i} \boldsymbol{A}\left(\zeta_{1}\right) \cdot y}+\mathrm{i}\left|\boldsymbol{A}(\varepsilon y)-\boldsymbol{A}\left(\zeta_{1}\right)\right|^{2} \psi_{1} e^{\mathrm{i} \sigma_{1}+\mathrm{i} \boldsymbol{A}\left(\zeta_{1}\right) \cdot y} . \tag{11}
\end{align*}
$$

Let us estimate them one by one in (11) for $y \neq \zeta_{1}^{\prime}$. Note that

$$
\begin{aligned}
& \left(\boldsymbol{A}(\varepsilon y)-\boldsymbol{A}\left(\zeta_{1}\right)\right) \cdot \nabla \psi_{1}=\sum_{k=1}^{N}\left(A_{k}(\varepsilon y)-A_{k}\left(\zeta_{1}\right)\right) \partial_{k} \psi_{1} \\
= & \varepsilon \sum_{k=1}^{N} \nabla A_{k}\left(\zeta_{1}\right) \cdot\left(y-\zeta_{1}^{\prime}\right) \partial_{k} \psi_{1}+\frac{\varepsilon^{2}}{2} \sum_{k=1}^{N}\left(y-\zeta_{1}^{\prime}\right)^{\perp} \cdot \nabla^{2} A_{k}\left(\zeta_{1}\right) \cdot\left(y-\zeta_{1}^{\prime}\right) \partial_{k} \psi_{1}+O\left(\varepsilon^{3}\left|y-\zeta_{1}^{\prime}\right|^{3}\left|\nabla \psi_{1}\right|\right) \\
= & \varepsilon \sum_{k, \ell=1}^{N} \partial_{\ell} A_{k}\left(\zeta_{1}\right)\left(y_{\ell}-\zeta_{1, \ell}^{\prime}\right)\left[\sum_{i, j=1}^{N} \partial_{j} A_{i}\left(\zeta_{1}\right) \partial_{k} \Psi_{1, i j}\right] \\
& +\frac{\varepsilon^{2}}{2} \sum_{k, \ell, s=1}^{N} \partial_{\ell s} A_{k}\left(\zeta_{1}\right)\left(y \ell-\zeta_{1, \ell}^{\prime}\right)\left(y_{s}-\zeta_{1, s}^{\prime}\right)\left[\sum_{i, j=1}^{N} \partial_{j} A_{i}\left(\zeta_{1}\right) \partial_{k} \Psi_{1, i j}\right]+O\left(\varepsilon^{3}\left|y-\zeta_{1}^{\prime}\right|^{5} w\left(\left|y-\zeta_{1}^{\prime}\right|\right)\right) .
\end{aligned}
$$

It is easy to see that ( $\delta_{i k}$ is the Kronecker symbol) for $y \neq \zeta_{1}^{\prime}$,

$$
\partial_{k} \Psi_{1, i j}(y)=\frac{1}{2}\left[\left(\delta_{i k}\left(y_{j}-\zeta_{1, j}^{\prime}\right)+\delta_{j k}\left(y_{i}-\zeta_{1, i}^{\prime}\right)\right) w\left(\left|y-\zeta_{1}^{\prime}\right|\right)+\left(y_{i}-\zeta_{1, i}^{\prime}\right)\left(y_{j}-\zeta_{1, j}^{\prime}\right)\left(y_{k}-\zeta_{1, k}^{\prime}\right) \frac{w^{\prime}\left(\left|y-\zeta_{1}^{\prime}\right|\right)}{\left|y-\zeta_{1}^{\prime}\right|}\right]
$$

Thus we conclude that

$$
\begin{align*}
& \left(\boldsymbol{A}(\varepsilon y)-\boldsymbol{A}\left(\zeta_{1}\right)\right) \cdot \nabla \psi_{1} \\
= & \frac{\varepsilon}{2} \sum_{j, k, \ell=1}^{N} \partial_{\ell} A_{k}\left(\zeta_{1}\right)\left[\partial_{j} A_{k}\left(\zeta_{1}\right)+\partial_{k} A_{j}\left(\zeta_{1}\right)\right]\left(y_{j}-\zeta_{1, j}^{\prime}\right)\left(y_{\ell}-\zeta_{1, \ell}^{\prime}\right) w\left(\left|y-\zeta_{1}^{\prime}\right|\right) \\
& +\frac{\varepsilon}{2} \sum_{i, j, k, \ell=1}^{N} \partial_{\ell} A_{k}\left(\zeta_{1}\right) \partial_{j} A_{i}\left(\zeta_{1}\right)\left(y_{i}-\zeta_{1, i}^{\prime}\right)\left(y_{j}-\zeta_{1, j}^{\prime}\right)\left(y_{k}-\zeta_{1, k}^{\prime}\right)\left(y_{\ell}-\zeta_{1, \ell}^{\prime}\right) \frac{w^{\prime}\left(\left|y-\zeta_{1}^{\prime}\right|\right)}{\left|y-\zeta_{1}^{\prime}\right|} \\
& +\frac{\varepsilon^{2}}{4} \sum_{j, k, \ell, s=1}^{N} \partial_{\ell s} A_{k}\left(\zeta_{1}\right)\left[\partial_{j} A_{k}\left(\zeta_{1}\right)+\partial_{k} A_{j}\left(\zeta_{1}\right)\right]\left(y_{j}-\zeta_{1, j}^{\prime}\right)\left(y_{\ell}-\zeta_{1, \ell}^{\prime}\right)\left(y_{s}-\zeta_{1, s}^{\prime}\right) w\left(y-\zeta_{1}^{\prime}\right) \\
& +\frac{\varepsilon^{2}}{4} \sum_{i, j, k, \ell, s=1}^{N} \partial_{\ell s} A_{k}\left(\zeta_{1}\right) \partial_{j} A_{i}\left(\zeta_{1}\right)\left(y_{i}-\zeta_{1, i}^{\prime}\right)\left(y_{j}-\zeta_{1, j}^{\prime}\right)\left(y_{k}-\zeta_{1, k}^{\prime}\right)\left(y_{\ell}-\zeta_{1, \ell}^{\prime}\right)\left(y_{s}-\zeta_{1, s}^{\prime}\right) \frac{w^{\prime}}{\left|y-\zeta_{1}^{\prime}\right|} \\
& +O\left(\varepsilon^{3}\left|y-\zeta_{1}^{\prime}\right|^{5} w\left(\left|y-\zeta_{1}^{\prime}\right|\right)\right) . \tag{12}
\end{align*}
$$

Also it can be obtained that

$$
\begin{aligned}
\left(\nabla_{x} \cdot \boldsymbol{A}(\varepsilon y)\right) \psi_{1}= & \frac{1}{2}\left(\nabla_{x} \cdot \boldsymbol{A}\right)\left(\zeta_{1}\right) \sum_{i, j=1}^{N} \partial_{j} A_{i}\left(\zeta_{1}\right)\left(y_{i}-\zeta_{1, i}^{\prime}\right)\left(y_{j}-\zeta_{1, j}^{\prime}\right) w\left(\left|y-\zeta_{1}^{\prime}\right|\right) \\
& +\frac{\varepsilon}{2} \sum_{i, j, k, \ell=1}^{N} \partial_{j} A_{i}\left(\zeta_{1}\right) \partial_{\ell k} A_{\ell}\left(\zeta_{1}\right)\left(y_{i}-\zeta_{1, i}^{\prime}\right)\left(y_{j}-\zeta_{1, j}^{\prime}\right)\left(y_{k}-\zeta_{1, k}^{\prime}\right) w\left(y-\zeta_{1}^{\prime}\right) \\
& +O\left(\varepsilon^{2}\left|y-\zeta_{1}^{\prime}\right|^{4} w\left(y-\zeta_{1}^{\prime}\right)\right)
\end{aligned}
$$

and

$$
\left|\boldsymbol{A}(\varepsilon y)-\boldsymbol{A}\left(\zeta_{1}\right)\right|^{2} \psi_{1}=O\left(\varepsilon^{2}\left|y-\zeta_{1}^{\prime}\right|^{4} w\left(y-\zeta_{1}^{\prime}\right)\right)
$$

Now (11) can be expanded as

$$
\begin{aligned}
& (\mathrm{i} \nabla+\boldsymbol{A}(\varepsilon y))^{2} \Psi_{1}-\left(\mathrm{i} \nabla+\boldsymbol{A}\left(\zeta_{1}\right)\right)^{2} \Psi_{1} \\
= & -\varepsilon \sum_{j, k, \ell=1}^{N} \partial_{\ell} A_{k}\left(\zeta_{1}\right)\left[\partial_{j} A_{k}\left(\zeta_{1}\right)+\partial_{k} A_{j}\left(\zeta_{1}\right)\right]\left(y_{j}-\zeta_{1, j}^{\prime}\right)\left(y_{\ell}-\zeta_{1, \ell}^{\prime}\right) w\left(\left|y-\zeta_{1}^{\prime}\right|\right) e^{\mathrm{i} \sigma_{1}+\mathrm{i} \boldsymbol{A}\left(\zeta_{1}\right) \cdot y} \\
& -\varepsilon \sum_{i, j, k, \ell=1}^{N} \partial_{\ell} A_{k}\left(\zeta_{1}\right) \partial_{j} A_{i}\left(\zeta_{1}\right)\left(y_{i}-\zeta_{1, i}^{\prime}\right)\left(y_{j}-\zeta_{1, j}^{\prime}\right)\left(y_{k}-\zeta_{1, k}^{\prime}\right)\left(y_{\ell}-\zeta_{1, \ell}^{\prime}\right) \frac{w^{\prime}\left(\left|y-\zeta_{1}^{\prime}\right|\right)}{\left|y-\zeta_{1}^{\prime}\right|} \mathrm{e}^{\mathrm{i} \sigma_{1}+\mathrm{i} \boldsymbol{A}\left(\zeta_{1}\right) \cdot y} \\
& -\frac{\varepsilon}{2}\left(\nabla_{x} \cdot \boldsymbol{A}\left(\zeta_{1}\right)\right) \sum_{i, j=1}^{N} \partial_{j} A_{i}\left(\zeta_{1}\right)\left(y_{i}-\zeta_{1, i}^{\prime}\right)\left(y_{j}-\zeta_{1, j}^{\prime}\right) w\left(\left|y-\zeta_{1}^{\prime}\right|\right) e^{\mathrm{i} \sigma_{1}+\mathrm{i} \boldsymbol{A}\left(\zeta_{1}\right) \cdot y} \\
& -\frac{\varepsilon^{2}}{2} \sum_{j, k, \ell, s=1}^{N} \partial_{\ell s} A_{k}\left(\zeta_{1}\right)\left[\partial_{j} A_{k}\left(\zeta_{1}\right)+\partial_{k} A_{j}\left(\zeta_{1}\right)\right]\left(y_{j}-\zeta_{1, j}^{\prime}\right)\left(y_{\ell}-\zeta_{1, \ell}^{\prime}\right)\left(y_{s}-\zeta_{1, s}^{\prime}\right) w\left(y-\zeta_{1}^{\prime}\right) \mathrm{e}^{\mathrm{i} \sigma_{1}+\mathrm{i} \boldsymbol{A}\left(\zeta_{1}\right) \cdot y} \\
& -\frac{\varepsilon^{2}}{2} \sum_{i, j, k, \ell, s=1}^{N} \partial_{\ell s} A_{k}\left(\zeta_{1}\right) \partial_{j} A_{i}\left(\zeta_{1}\right)\left(y_{i}-\zeta_{1, i}^{\prime}\right)\left(y_{j}-\zeta_{1, j}^{\prime}\right)\left(y_{k}-\zeta_{1, k}^{\prime}\right)\left(y_{\ell}-\zeta_{1, \ell}^{\prime}\right)\left(y_{s}-\zeta_{1, s}^{\prime}\right) \frac{w^{\prime}}{\left|y-\zeta_{1}^{\prime}\right|} \mathrm{e}^{\mathrm{i} \sigma_{1}+\mathrm{i} \boldsymbol{A}\left(\zeta_{1}\right) \cdot y} \\
& -\frac{\varepsilon^{2}}{2} \sum_{i, j, k, \ell=1}^{N} \partial_{j} A_{i}\left(\zeta_{1}\right) \partial_{\ell k} A_{\ell}\left(\zeta_{1}\right)\left(y_{i}-\zeta_{1, i}^{\prime}\right)\left(y_{j}-\zeta_{1, j}^{\prime}\right)\left(y_{k}-\zeta_{1, k}^{\prime}\right) w\left(\left|y-\zeta_{1}^{\prime}\right|\right) \mathrm{e}^{\mathrm{i} \sigma_{1}+\mathrm{i} \boldsymbol{A}\left(\zeta_{1}\right) \cdot y} \\
& \left.+\left[O\left(\varepsilon^{3}\left|y-\zeta_{1}^{\prime}\right|^{5}+\varepsilon^{3}\left|y-\zeta_{1}^{\prime}\right|^{4}\right)+\mathrm{i} O\left(\varepsilon^{2}\left|y-\zeta_{1}^{\prime}\right|^{4}\right)\right] w\left(\left|y-\zeta_{1}^{\prime}\right|\right)\right) \mathrm{e}^{\mathrm{i} \sigma_{1}+\mathrm{i} \boldsymbol{A}\left(\zeta_{1}\right) \cdot y} .
\end{aligned}
$$

Therefore, in $\left|y-\zeta_{m}^{\prime}\right| \leq \frac{\delta}{\sqrt{\varepsilon}}$

$$
R(y) \mathrm{e}^{-\mathrm{i} \sigma_{m}-\mathrm{i} \boldsymbol{A}\left(\zeta_{m}\right) \cdot y}:=R_{m, 1}(y)+\mathrm{i} R_{m, 2}(y),
$$

where

$$
R_{m, 1}(y)
$$

$$
\begin{align*}
= & \varepsilon^{2} \sum_{i, j, k=1}^{N} \partial_{j} A_{i}\left(\zeta_{m}\right) \partial_{k} A_{i}\left(\zeta_{m}\right)\left(y_{j}-\zeta_{m, j}^{\prime}\right)\left(y_{k}-\zeta_{m, k}^{\prime}\right) w\left(y-\zeta_{m}^{\prime}\right) \\
& -\varepsilon^{2} \sum_{j, k, \ell=1}^{N} \partial_{\ell} A_{k}\left(\zeta_{m}\right)\left[\partial_{j} A_{k}\left(\zeta_{m}\right)+\partial_{k} A_{j}\left(\zeta_{m}\right)\right]\left(y_{j}-\zeta_{m, j}^{\prime}\right)\left(y_{\ell}-\zeta_{m, \ell}^{\prime}\right) w\left(\left|y-\zeta_{m}^{\prime}\right|\right) \\
& -\varepsilon^{2} \sum_{i, j, k, \ell=1}^{N} \partial_{\ell} A_{k}\left(\zeta_{m}\right) \partial_{j} A_{i}\left(\zeta_{m}\right)\left(y_{i}-\zeta_{m, i}^{\prime}\right)\left(y_{j}-\zeta_{m, j}^{\prime}\right)\left(y_{k}-\zeta_{m, k}^{\prime}\right)\left(y \ell-\zeta_{m, \ell}^{\prime}\right) \frac{w^{\prime}\left(\left|y-\zeta_{m}^{\prime}\right|\right)}{\left|y-\zeta_{m}^{\prime}\right|} \\
& -\frac{\varepsilon^{2}}{2}\left(\nabla_{x} \cdot \boldsymbol{A}\left(\zeta_{m}\right)\right) \sum_{i, j=1}^{N} \partial_{j} A_{i}\left(\zeta_{m}\right)\left(y_{i}-\zeta_{m, i}^{\prime}\right)\left(y_{j}-\zeta_{m, j}^{\prime}\right) w\left(\left|y-\zeta_{m}^{\prime}\right|\right) \\
& -\frac{(p-1) \varepsilon^{2}}{8} \sum_{i, j, k, \ell=1}^{N} \partial_{j} A_{i}\left(\zeta_{m}\right) \partial_{\ell} A_{k}\left(\zeta_{m}\right)\left(y_{i}-\zeta_{m, i}^{\prime}\right)\left(y_{j}-\zeta_{m, j}^{\prime}\right)\left(y_{k}-\zeta_{m, k}^{\prime}\right)\left(y_{\ell}-\zeta_{m, \ell}^{\prime}\right) w^{p}\left(\left|y-\zeta_{m}^{\prime}\right|\right) \\
& +\varepsilon^{3} \sum_{i, j, k, \ell=1}^{N} \partial_{j} A_{i}\left(\zeta_{m}\right) \partial_{k \ell} A_{i}\left(\zeta_{m}\right)\left(y_{j}-\zeta_{m, j}^{\prime}\right)\left(y_{k}-\zeta_{m, k}^{\prime}\right)\left(y_{\ell}-\zeta_{m, \ell}^{\prime}\right) w\left(\left|y-\zeta_{m}^{\prime}\right|\right) \\
& -\frac{\varepsilon^{3}}{2} \sum_{j, k, \ell, s=1}^{N} \partial_{\ell s} A_{k}\left(\zeta_{m}\right)\left[\partial_{j} A_{k}\left(\zeta_{m}\right)+\partial_{k} A_{i}\left(\zeta_{m}\right)\right]\left(y_{j}-\zeta_{m, j}^{\prime}\right)\left(y_{\ell}-\zeta_{m, \ell}^{\prime}\right)\left(y_{s}-\zeta_{m, s}^{\prime}\right) w\left(\left|y-\zeta_{m}^{\prime}\right|\right) \\
& -\frac{\varepsilon^{3}}{2} \sum_{i, j, k, \ell, s=1}^{N} \partial_{\ell s} A_{k}\left(\zeta_{m}\right) \partial_{j} A_{i}\left(\zeta_{m}\right)\left(y_{i}-\zeta_{m, i}^{\prime}\right)\left(y_{j}-\zeta_{m, j}^{\prime}\right)\left(y_{k}-\zeta_{m, k}^{\prime}\right)\left(y_{\ell}-\zeta_{m, \ell}^{\prime}\right)\left(y_{s}-\zeta_{m, s}^{\prime}\right) \frac{w^{\prime}\left(\left|y-\zeta_{m}^{\prime}\right|\right)}{\left|y-\zeta_{m}^{\prime}\right|} \\
& -\frac{\varepsilon^{3}}{2} \sum_{i, j, k, \ell=1}^{N} \partial_{j} A_{i}\left(\zeta_{m}\right) \partial_{\ell k} A_{\ell}\left(\zeta_{m}\right)\left(y_{i}-\zeta_{m, i}^{\prime}\right)\left(y_{j}-\zeta_{m, j}^{\prime}\right)\left(y_{k}-\zeta_{m, k}^{\prime}\right) w\left(\left|y-\zeta_{m}^{\prime}\right|\right) \\
& +O\left(\varepsilon^{4}\left|y-\zeta_{m}^{\prime}\right|^{4}+\varepsilon^{4}\left|y-\zeta_{m}^{\prime}\right|^{5}\right) w\left(\left|y-\zeta_{m}^{\prime}\right|\right)+O\left(\varepsilon^{4}\left|y-\zeta_{m}^{\prime}\right|^{8} w^{p}\left(\left|y-\zeta_{m}^{\prime}\right|\right)\right) \tag{13}
\end{align*}
$$

and

$$
\begin{aligned}
R_{m, 2}(y)= & \varepsilon^{2} \sum_{i, j, k=1}^{N} \partial_{j k} A_{i}\left(\zeta_{m}\right)\left(y_{j}-\zeta_{m, j}^{\prime}\right)\left(y_{k}-\zeta_{m, k}^{\prime}\right) \partial_{i} w\left(\left|y-\zeta_{m}^{\prime}\right|\right) \\
& +\varepsilon^{2} \sum_{i, j=1}^{N} \partial_{i j} A_{i}\left(\zeta_{m}\right)\left(y_{j}-\zeta_{m, j}^{\prime}\right) w\left(\left|y-\zeta_{m}^{\prime}\right|\right) \\
& +O\left(\varepsilon^{3}\left|y-\zeta_{m}^{\prime}\right|^{4}+\varepsilon^{3}\left|y-\zeta_{m}^{\prime}\right|^{2}\right) w\left(\left|y-\zeta_{m}^{\prime}\right|\right)+O\left(\varepsilon^{4}\left|y-\zeta_{m}^{\prime}\right|^{8}\right) w^{p}\left(\left|y-\zeta_{m}^{\prime}\right|\right) .
\end{aligned}
$$

Hence we get the estimate

$$
\sum_{m=1}^{K} \int_{B\left(\zeta_{m}^{\prime} ; \frac{\delta}{\sqrt{\varepsilon}}\right)}|R(y)|^{2} d y \leq C \varepsilon^{2}
$$

As for the domain $\left|y-\zeta_{m}^{\prime}\right| \geq \frac{\delta}{\sqrt{\varepsilon}}, \forall m=1,2, \ldots, K$, using the asymptotic behaviour of $w\left(\left|y-\zeta_{m}^{\prime}\right|\right)$, it is easy to see that

$$
\int_{\mathbb{R}^{N} \backslash \cup_{m=1}^{K} B\left(\zeta_{m}^{\prime} ; \frac{\delta}{\sqrt{\varepsilon}}\right)}|R(y)|^{2} d y \leq C \mathrm{e}^{-\frac{\delta}{\sqrt{\varepsilon}}}
$$

The result for $R(y)$ is concluded.
As for the estimates of $\partial_{\zeta_{m, k}^{\prime}} R$ and $\partial_{\sigma_{m}} R$, one may check it similarly.

## 3. The linear problem and the nonlinear problem

This section is devoted to the invertibility of the linear operator $L$ in order to solve problem (10):

$$
L \phi=(\mathrm{i} \nabla+\boldsymbol{A}(\varepsilon y))^{2} \phi+\phi-(p-1)|\mathcal{W}|^{p-3} \operatorname{Re}(\overline{\mathcal{W}} \phi) \mathcal{W}-|\mathcal{W}|^{p-1} \phi=-R(y)+N(\phi) .
$$

Let $H$ be the Hilbert space as the closure of $C_{0}^{\infty}\left(\mathbb{R}^{N}, \mathbb{C}\right)$ under the scalar product

$$
(u, v)=\operatorname{Re} \int_{\mathbb{R}^{N}}(\mathrm{i} \nabla u+\boldsymbol{A}(\varepsilon y) u) \overline{(\mathrm{i} \nabla v+\boldsymbol{A}(\varepsilon y) v)}+u \bar{v}
$$

The norm deduced by the above scalar product is equivalent to the usual norm of $H^{1}\left(\mathbb{R}^{N}, \mathbb{C}\right)$ due to the boundness of $|\boldsymbol{A}(x)|$, see 10. In $\left|y-\zeta_{m}^{\prime}\right| \leq \frac{\delta}{\sqrt{\varepsilon}}$, the operator $L$ formally looks like

$$
\left(\mathrm{i} \nabla+\boldsymbol{A}\left(\zeta_{m}\right)\right)^{2} \phi+\phi-(p-1)\left|U_{m}\right|^{p-3} \operatorname{Re}\left(\bar{U}_{m} \phi\right) U_{m}-\left|U_{m}\right|^{p-1} \phi,
$$

which is not invertible. Precisely, the null space of this limit operator is

$$
\operatorname{span}_{\mathbb{R}}\left\{Z_{m, 0}, Z_{m, 1}, \cdots, Z_{m, N}\right\}
$$

where

$$
Z_{m, 0}=\mathrm{i} w\left(\left|y-\zeta_{m}^{\prime}\right|\right) \mathrm{e}^{\mathrm{i} \sigma_{m}+\mathrm{i} \boldsymbol{A}\left(\zeta_{m}\right) \cdot y}=\mathrm{i} U_{m} \quad \text { and } \quad Z_{m, i}=\frac{\partial w\left(\left|y-\zeta_{m}^{\prime}\right|\right)}{\partial \zeta_{m, i}^{\prime}} \mathrm{e}^{\mathrm{i} \sigma_{m}+\mathrm{i} \boldsymbol{A}\left(\zeta_{m}\right) \cdot y}, \quad 1 \leq i \leq N
$$

The symbol $\operatorname{span}_{\mathbb{R}}$ means the linear combinations on real numbers, see for instance 10, 11. Therefore, we study the following linear problem with $h \in L^{2}\left(\mathbb{R}^{N}, \mathbb{C}\right)$

$$
\left\{\begin{array}{l}
L \phi=h+\sum_{i=0}^{N} \sum_{m=1}^{K} c_{m, i} \chi_{m} Z_{m, i},  \tag{14}\\
\operatorname{Re} \int_{\mathbb{R}^{N}} \chi_{m} \bar{Z}_{m, i} \phi=0, \quad i=0,1, \cdots, N, \quad m=1, \ldots, K
\end{array}\right.
$$

where $\chi_{m}(y)=\chi\left(\left|y-\zeta_{m}^{\prime}\right|\right)$ is a smooth cut-off function on the large ball $B_{R}\left(\zeta_{m}^{\prime}\right)$, satisfying $\chi(s)=1$ for $|s| \leq R$ and $\chi(s)=0$ for $|s| \geq R+1$.

Next, we will prove the following invertibility proposition which is the main result in this section.
Proposition 3.1. The linear problem (14) admits a unique solution $\left(\phi, c_{m, i}\right)=\left(\widetilde{T}(h), c_{m, i}\right), i=0,1, \cdots, N, m=$ $1, \ldots, K$ satisfying

$$
\|\phi\|_{H^{2}}=\|\widetilde{T}(h)\| \leq C\|h\|_{L^{2}}, \quad\left|c_{m, i}\right| \leq C\|h\|_{L^{2}}
$$

Before giving the proof, it is necessary to get an apriori estimate.
Lemma 3.2. If $\left(\phi, c_{m, i}\right)$ is a solution of the problem (14), then

$$
\|\phi\|_{H^{2}} \leq C\|h\|_{L^{2}}, \quad\left|c_{m, i}\right| \leq C\|h\|_{L^{2}}
$$

Proof. The proof is very standard and we here prove it briefly for the completion. First, we test the equation (14) by $\bar{Z}_{\ell, j}, 1 \leq \ell \leq K, 0 \leq j \leq N$ and get that

$$
\begin{equation*}
\left\langle L \phi, \bar{Z}_{\ell, j}\right\rangle=\operatorname{Re} \int_{\mathbb{R}^{N}} h \bar{Z}_{\ell, j}+c_{\ell, j} \int_{\mathbb{R}^{N}}\left|Z_{\ell, j}\right|^{2}+O\left(\mathrm{e}^{-\delta / \varepsilon} \sum_{i=0}^{N} \sum_{m=1}^{K}\left|c_{m, i}\right|\right) . \tag{15}
\end{equation*}
$$

Note that

$$
\begin{aligned}
\operatorname{Re} \int_{\mathbb{R}^{N}}(\mathrm{i} \nabla+A(\varepsilon y)) \phi \overline{(\mathrm{i} \nabla+A(\varepsilon y)) Z_{\ell, j}} & =\operatorname{Re} \int_{\mathbb{R}^{N}}(\mathrm{i} \nabla+A(\varepsilon y)) Z_{\ell, j} \overline{(\mathrm{i} \nabla+A(\varepsilon y)) \phi} \\
& =\operatorname{Re} \int_{\mathbb{R}^{N}}\left(\mathrm{i} \nabla+A\left(\zeta_{\ell}\right)\right) Z_{\ell, j} \overline{\left(\mathrm{i} \nabla+A\left(\zeta_{\ell}\right)\right) \phi}+O(\varepsilon)\|\phi\|_{H^{1}},
\end{aligned}
$$

and

$$
\left|\int_{\mathbb{R}^{N}} h Z_{\ell, j}\right| \leq C\|h\|_{L^{2}}
$$

Thus, it holds from (15) and the equation of $\bar{Z}_{\ell, j}$ that

$$
\begin{equation*}
c_{\ell, j}=O\left(\varepsilon\|\phi\|_{H^{1}}+\|h\|_{L^{2}}\right) \tag{16}
\end{equation*}
$$

Next we will prove $\|\phi\|_{H^{1}} \leq C\|h\|_{L^{2}}$ by contradiction. Suppose that for some sequence $\left\{\varepsilon_{n}\right\}$, there always exist $\phi_{n}$ and $h_{n}$ such that

$$
\left\|\phi_{n}\right\|_{H^{1}}=1 \quad \text { and } \quad\left\|h_{n}\right\|_{L^{2}}=o(1) \text { as } \varepsilon_{n} \rightarrow 0
$$

Testing (14) against $\eta_{m} \varphi\left(\varphi \in C_{c}^{\infty}\left(\mathbb{R}^{N}, \mathbb{C}\right)\right)$ where $\eta_{m}(y) \equiv 1$ in $\left|y-\zeta_{m}^{\prime}\right|<\frac{\delta}{\sqrt{\varepsilon}}$ and $\eta_{m}(y) \equiv 0$ in $\left|y-\zeta_{m}^{\prime}\right|>\frac{2 \delta}{\sqrt{\varepsilon}}$, one can obtain that

$$
\begin{aligned}
& \operatorname{Re} \int_{\mathbb{R}^{N}}\left(\mathrm{i} \nabla+A\left(\varepsilon_{n} y\right)\right) \phi_{n} \overline{\left(\mathrm{i} \nabla+A\left(\varepsilon_{n} y\right)\right) \eta_{m} \varphi}+\operatorname{Re} \int_{\mathbb{R}^{N}} \phi_{n} \eta_{m} \bar{\varphi} \\
& -(p-1) \operatorname{Re} \int_{\mathbb{R}^{N}}|\mathcal{W}|^{p-3}\left(\operatorname{Re}\left(\overline{\mathcal{W}} \phi_{n}\right)\right) \mathcal{W} \eta_{m} \bar{\varphi}-\operatorname{Re} \int_{\mathbb{R}^{N}}|\mathcal{W}|^{p-1} \phi_{n} \eta_{m} \bar{\varphi} \\
=\quad & \operatorname{Re} \int_{\mathbb{R}^{N}} h_{n} \eta_{m} \bar{\varphi}+\sum_{i=0}^{N} c_{m, i} \operatorname{Re} \int_{\mathbb{R}^{N}} \chi_{m} Z_{m, i} \bar{\varphi} .
\end{aligned}
$$

Note that $\phi_{n} \rightharpoonup \phi$ in $H_{\text {loc }}^{1}\left(\mathbb{R}^{N}, \mathbb{C}\right)$ up to a subsequence. Thus dominated convergence theorem tells us that

$$
\begin{aligned}
& \operatorname{Re} \int_{\mathbb{R}^{N}}\left(\mathrm{i} \nabla+A\left(\zeta_{m}\right)\right) \phi \overline{\left(\mathrm{i} \nabla+A\left(\zeta_{m}\right)\right) \varphi}+\operatorname{Re} \int_{\mathbb{R}^{N}} \phi \bar{\varphi} \\
& -(p-1) \operatorname{Re} \int_{\mathbb{R}^{N}}\left|U_{m}\right|^{p-3}\left(\operatorname{Re}\left(\bar{U}_{m} \phi\right)\right) U_{m} \bar{\varphi}-\operatorname{Re} \int_{\mathbb{R}^{N}}\left|U_{m}\right|^{p-1} \phi \bar{\varphi}=0 .
\end{aligned}
$$

This means that $\phi$ is a solution of

$$
\left(\mathrm{i} \nabla+\boldsymbol{A}\left(\zeta_{m}\right)\right)^{2} \phi+\phi-(p-1)\left|U_{m}\right|^{p-3} \operatorname{Re}\left(\bar{U}_{m} \phi\right) U_{m}-\left|U_{m}\right|^{p-1} \phi=0 \quad \text { in } \mathbb{R}^{N}
$$

Then one gets that $\phi=0$ from the orthogonal conditions, which further implies that

$$
\begin{equation*}
\phi_{n} \rightarrow 0 \quad \text { a.e. in } B_{R}\left(\zeta_{m}^{\prime}\right), \quad \forall R>0, m=1,2, \ldots, K \tag{17}
\end{equation*}
$$

On the other hand, note that

$$
\begin{align*}
& \int_{\mathbb{R}^{N}}\left|(\mathrm{i} \nabla+\boldsymbol{A}(\varepsilon y)) \phi_{n}\right|^{2}+\int_{\mathbb{R}^{N}}\left|\phi_{n}\right|^{2}-(p-1) \int_{\mathbb{R}^{N}}|\mathcal{W}|^{p-3}\left(\operatorname{Re}\left(\overline{\mathcal{W}} \phi_{n}\right)\right)^{2}-\int_{\mathbb{R}^{N}}|\mathcal{W}|^{p-1}\left|\phi_{n}\right|^{2} \\
= & \operatorname{Re} \int_{\mathbb{R}^{N}} \bar{h}_{n} \phi_{n}=o(1) . \tag{18}
\end{align*}
$$

From (17) and the exponential decay of $\left|U_{m}\right|$, we obviously have

$$
\int_{\mathbb{R}^{N}}|\mathcal{W}|^{p-1}\left|\phi_{n}\right|^{2}=\sum_{m=1}^{N} \int_{B_{R}\left(\zeta_{m}^{\prime}\right)}\left|U_{m}\right|^{p-1}\left|\phi_{n}\right|^{2}+\int_{\mathbb{R}^{N} \backslash \cup_{m=1}^{K} B_{R}\left(\zeta_{m}^{\prime}\right)}|\mathcal{W}|^{p-1}\left|\phi_{n}\right|^{2}+O(\varepsilon)=O\left(\mathrm{e}^{-R}\right)+o(1)
$$

So is $\int_{\mathbb{R}^{N}}|\mathcal{W}|^{p-3}\left(\operatorname{Re}\left(\overline{\mathcal{W}} \phi_{n}\right)\right)^{2}$, which together with (18) shows that

$$
\int_{\mathbb{R}^{N}}\left|\phi_{n}\right|^{2}=O\left(e^{-R}\right)+o(1) \quad \text { and } \quad \int_{\mathbb{R}^{N}}\left|(\mathrm{i} \nabla+\boldsymbol{A}(\varepsilon y)) \phi_{n}\right|^{2}=O\left(\mathrm{e}^{-R}\right)+o(1)
$$

Finally, it is derived that

$$
\begin{aligned}
O\left(\mathrm{e}^{-R}\right)+o(1) & =\int_{\mathbb{R}^{N}}\left|(\mathrm{i} \nabla+\boldsymbol{A}(\varepsilon y)) \phi_{n}\right|^{2} \\
& =\int_{\mathbb{R}^{N}}\left|\nabla \phi_{n}\right|^{2}+\int_{\mathbb{R}^{N}}\left|\boldsymbol{A}\left(\varepsilon_{n} y\right)\right|^{2}\left|\phi_{n}\right|^{2}+2 \operatorname{Re} \int_{\mathbb{R}^{2}} \mathrm{i} \boldsymbol{A}\left(\varepsilon_{n} y\right) \cdot \nabla \phi_{n} \phi_{n} \\
& \geq \frac{1}{2} \int_{\mathbb{R}^{N}}\left|\nabla \phi_{n}\right|^{2}-\int_{\mathbb{R}^{N}}\left|\boldsymbol{A}\left(\varepsilon_{n} y\right)\right|^{2}\left|\phi_{n}\right|^{2}=\frac{1}{2}\left\|\phi_{n}\right\|_{H^{1}}+O\left(\mathrm{e}^{-R}\right)+o(1)
\end{aligned}
$$

since $\boldsymbol{A}(x)$ is bounded. This leads to a contradiction to $\left\|\phi_{n}\right\|_{H^{1}}=1$. Hence

$$
\|\phi\|_{H^{1}} \leq C\|h\|_{L^{2}}
$$

At last, the regularity theory gives $\|\phi\|_{H^{2}} \leq C\|h\|_{L^{2}}$.
Proof of Proposition 3.1. Denote the Hilbert space

$$
\mathcal{H}=\left\{\phi \in H\left(\mathbb{R}^{N}\right) \mid \operatorname{Re} \int_{\mathbb{R}^{N}} \chi_{m} \phi \bar{Z}_{m, i}=0\right\}
$$

Then, from Riesz representation theorem the equation (14) is equivalent to

$$
\phi-T(\phi)=\tilde{h}, \quad \text { in } \mathcal{H}
$$

where $T$ is a compact operator on $\mathcal{H}$. Based on Proposition 3.1, Fredholm alternative tells us the unique existence of $\phi$. And $c_{m, i}$ can be given by $\phi$ using integration. Their estimates were given in the above proposition.

Also for $\phi=\widetilde{T}(h)$, it is important for later purposes to understand the differentiability of the operator $\widetilde{T}$ with respect to $\zeta_{j}^{\prime}$ and $\sigma_{j}, j=1, \ldots, K$. Recall that $\phi$ satisfies the equation

$$
L \phi=(\mathrm{i} \nabla+A(\varepsilon y))^{2} \phi+\phi-(p-1)|\mathcal{W}|^{p-3} \operatorname{Re}(\overline{\mathcal{W}} \phi) \mathcal{W}-|\mathcal{W}|^{p-1} \phi=h+\sum_{m, i} c_{m, i} \chi_{m} Z_{m, i}
$$

Thus

$$
\begin{aligned}
L\left(\partial_{\zeta_{j, k}^{\prime}} \phi\right) & =(\mathrm{i} \nabla+A(\varepsilon y))^{2}\left(\partial_{\zeta_{j, k}^{\prime}} \phi\right)+\partial_{\zeta_{j, k}^{\prime}} \phi-(p-1)|\mathcal{W}|^{p-3} \operatorname{Re}\left(\overline{\mathcal{W}} \partial_{\zeta_{j, k}^{\prime}} \phi\right) \mathcal{W}-|\mathcal{W}|^{p-1} \partial_{\zeta_{j, k}^{\prime}} \phi \\
& =O\left(|\mathcal{W}|^{p-2}|\phi|\left|\partial_{\zeta_{j, k}^{\prime}} \mathcal{W}\right|\right)+\sum_{i} c_{j, i} \partial_{\zeta_{j, k}^{\prime}}\left(\chi_{j} Z_{j, i}\right)+\sum_{m, i}\left(\partial_{\zeta_{j, k}^{\prime}} c_{m, i}\right) \chi_{m} Z_{m, i} .
\end{aligned}
$$

Moreover, the derivative of the orthogonal condition is

$$
\begin{gathered}
\operatorname{Re} \int_{\mathbb{R}^{N}} \chi_{m} \bar{Z}_{m, i}\left(\partial_{\zeta_{j, k}^{\prime}} \phi\right)=0, \quad \text { for } m \neq j, \\
\operatorname{Re} \int_{\mathbb{R}^{N}} \chi_{j} \bar{Z}_{j, i}\left(\partial_{\zeta_{j, k}^{\prime}} \phi\right)=-\operatorname{Re} \int_{\mathbb{R}^{N}} \partial_{\zeta_{j, k}^{\prime}}\left(\chi_{j} \bar{Z}_{j, i}\right) \phi
\end{gathered}
$$

Set $\varphi=\partial_{\zeta_{j, k}^{\prime}} \phi+\sum_{i=1}^{N} b_{j k, i} \chi_{j} Z_{j, i}$ and

$$
b_{j k, i}=\operatorname{Re} \int_{\mathbb{R}^{N}} \partial_{\zeta_{j, k}^{\prime}}\left(\chi_{j} \bar{Z}_{j, i}\right) \phi / \int_{\mathbb{R}^{N}} \chi_{j}\left|Z_{j, i}\right|^{2}
$$

Note that $\left|b_{j k, i}\right| \leq C\|\phi\|_{L^{2}} \leq C\|h\|_{L^{2}}$. Then $\varphi$ satisfies all the orthogonal conditions and

$$
L \varphi=b_{j i} L\left(\chi_{j} Z_{j, i}\right)+O\left(|\mathcal{W}|^{p-2}|\phi|\left|\partial_{\zeta_{j, k}^{\prime}} \mathcal{W}\right|\right)+\sum_{i} c_{j, i} \partial_{\zeta_{j, k}^{\prime}}\left(\chi_{j} Z_{j, i}\right)+\sum_{m, i}\left(\partial_{\zeta_{j, k}^{\prime}} c_{m, i}\right) \chi_{m} Z_{m, i}
$$

With Lemma 3.2 in hand,

$$
\|\varphi\|_{H^{2}} \leq C\left\|b_{j i} L\left(\chi_{j} Z_{j, i}\right)\right\|_{L^{2}}+C\left\||\mathcal{W}|^{p-2} \phi \partial_{\zeta_{j, k}^{\prime}} \mathcal{W}\right\|_{L^{2}}+C \sum_{i}\left\|c_{j, i} \partial_{\zeta_{j, k}^{\prime}}\left(\chi_{j} Z_{j, i}\right)\right\|_{L^{2}} \leq C\|h\|_{L^{2}}
$$

Therefore, we conclude that

$$
\left\|\partial_{\zeta_{j, k}^{\prime}} \widetilde{T}(h)\right\|_{H^{2}}=\left\|\partial_{\zeta_{j, k}^{\prime}} \phi\right\|_{H^{2}} \leq\|\varphi\|_{H^{2}}+\left\|b_{j i} \chi_{j} Z_{j, i}\right\|_{H^{2}} \leq C\|h\|_{L^{2}}
$$

The same process may be carried out for $\partial_{\sigma_{j}} \phi$. Based on the above discussion, the following proposition holds obviously.

Proposition 3.3. For the unique solution $\phi=\widetilde{T}(h)$ in Proposition 3.1, it holds that

$$
\left\|\partial_{\zeta_{j, k}^{\prime}} \widetilde{T}(h)\right\|_{H^{2}} \leq C\|h\|_{L^{2}}, \quad\left\|\partial_{\sigma_{j}} \widetilde{T}(h)\right\|_{H^{2}} \leq C\|h\|_{L^{2}}, \quad \forall j=1, \ldots, K, k=1, \ldots, N
$$

Now we can deal with the following nonlinear problem

$$
\left\{\begin{array}{l}
L \phi=-R(y)+N(\phi)+\sum_{i=0}^{N} \sum_{m=1}^{K} c_{m, i} \chi_{m} Z_{m, i}  \tag{19}\\
\operatorname{Re} \int_{\mathbb{R}^{N}} \chi_{m} \bar{Z}_{m, i} \phi=0, \quad i=0, \cdots, N, m=1, \ldots, K
\end{array}\right.
$$

Proposition 3.4. The nonlinear problem (19) admits a unique solution $\phi$ satisfying

$$
\|\phi\|_{H^{2}}=O\left(\varepsilon^{2}\right)
$$

Moreover, $\left(\boldsymbol{\sigma}, \boldsymbol{\zeta}^{\prime}\right) \rightarrow \phi$ is of class $C^{1}$ for $\boldsymbol{\sigma}=\left(\sigma_{1}, \ldots, \sigma_{K}\right), \boldsymbol{\zeta}^{\prime}=\left(\zeta_{1}^{\prime}, \ldots, \zeta_{K}^{\prime}\right)$, and

$$
\left\|\partial_{\zeta_{j, k}^{\prime}} \phi\right\|_{H^{2}}=O\left(\varepsilon^{2 \beta}\right) \quad \text { and } \quad\left\|\partial_{\sigma_{j}} \phi\right\|_{H^{2}}=O\left(\varepsilon^{2 \beta}\right), \quad \forall j, k .
$$

Proof. Recall that $\beta=\min \{p-1,1\}$. The proof is based on the contraction mapping theorem. First, for a large enough number $\gamma_{0}>0$, we set

$$
\mathcal{S}=\left\{\phi \in \mathcal{H} \mid\|\phi\|_{H^{2}} \leq \gamma_{0} \varepsilon^{2}\right\}
$$

The nonlinear problem (19) is transferred to solving

$$
\phi=\widetilde{T}(-R(y)+N(\phi)):=\mathcal{A}(\phi),
$$

which means to find a fixed point of the operator $\mathcal{A}$.
First, the operator $\mathcal{A}$ is from $\mathcal{S}$ to itself. In fact,

$$
\|\mathcal{A}(\phi)\|_{H^{2}}=\|\widetilde{T}(-R(y)+N(\phi))\|_{H^{2}} \leq C\|R(y)\|_{L^{2}}+C\|N(\phi)\|_{L^{2}} \leq C \varepsilon^{2}+C\|\phi\|_{H^{2}}^{1+\beta} \leq \gamma_{0} \varepsilon^{2}
$$

Next the operator $\mathcal{A}$ is a contraction mapping, since

$$
\begin{aligned}
\left\|\mathcal{A}\left(\phi_{1}\right)-\mathcal{A}\left(\phi_{2}\right)\right\|_{H^{2}} & =\left\|\widetilde{T}\left(N\left(\phi_{1}\right)-N\left(\phi_{2}\right)\right)\right\|_{H^{2}} \leq C\left\|N\left(\phi_{1}\right)-N\left(\phi_{2}\right)\right\|_{L^{2}} \\
& \leq C\left(\left\|\phi_{1}\right\|_{H^{2}}^{\beta}+\left\|\phi_{2}\right\|_{H^{2}}^{\beta}\right)\left\|\phi_{1}-\phi_{2}\right\|_{H^{2}} .
\end{aligned}
$$

Thus $\mathcal{A}$ has a unique fixed point in $\mathcal{S}$, which is the unique solution of problem (19).
Next we come to $\partial_{\zeta_{j, k}^{\prime}} \phi$ and $\partial_{\sigma_{j}} \phi$. The $C^{1}$-regularity in $\zeta_{j}^{\prime}$ and $\sigma_{j}$ is guaranteed by the implicit function theorem. One may refer to the proof of Lemma 4.1 in [10. It is easy to see

$$
\begin{equation*}
\partial_{\zeta_{j, k}^{\prime}} \phi=\partial_{\zeta_{j, k}^{\prime}} \widetilde{T}(-R(y)+N(\phi))+\widetilde{T}\left(-\partial_{\zeta_{j, k}^{\prime}} R(y)+\partial_{\zeta_{j, k}^{\prime}} N(\phi)\right) \tag{20}
\end{equation*}
$$

Notice that $\left|\partial_{\zeta_{j, k}^{\prime}} N(\phi)\right|=O\left(|\phi|^{\beta}\left|\partial_{j k}^{\prime} w\right|+|\phi|^{\beta}\left|\partial_{\zeta_{j, k}^{\prime}} \phi\right|\right)$. So we get

$$
\left\|\partial_{\zeta_{j, k}^{\prime}} N(\phi)\right\|_{L^{2}} \leq C\|\phi\|_{H^{2}}^{\beta}+C\|\phi\|_{H^{2}}^{\beta}\left\|\partial_{\zeta_{j, k}^{\prime}} \phi\right\|_{H^{2}} .
$$

Thus (20) and Proposition 3.3 lead to

$$
\left\|\partial_{\zeta_{j, k}^{\prime}} \phi\right\|_{H^{2}} \leq C\left\|\partial_{\zeta_{j, k}^{\prime}} R(y)\right\|_{L^{2}}+C\|R(y)\|_{L^{2}}+C\|N(\phi)\|_{L^{2}}+C\left\|\partial_{\zeta_{j, k}^{\prime}} N(\phi)\right\|_{L^{2}} \leq C \varepsilon^{2 \beta} .
$$

The estimate for $\left\|\partial_{\sigma_{j}} \phi\right\|_{H^{2}}$ may be gotten by the same process.

## 4. Variational Reduction

According to the above discussion, the remaining thing is to let $c_{m, i}=0$ in the nonlinear problem (19) in order to make $\mathcal{W}+\phi$ be a solution of the original problem. It can be done by the variational reduction process.

Note the energy functional of problem (5) is

$$
E(u)=\frac{1}{2} \int_{\mathbb{R}^{N}}|\mathrm{i} \nabla u+\boldsymbol{A}(\varepsilon y) u|^{2} \mathrm{~d} y+\frac{1}{2} \int_{\mathbb{R}^{N}}|u|^{2}-\frac{1}{p+1} \int_{\mathbb{R}^{N}}|u|^{p+1} .
$$

Define

$$
F\left(\boldsymbol{\sigma}, \boldsymbol{\zeta}^{\prime}\right)=E(\mathcal{W}+\phi)\left(\boldsymbol{\sigma}, \boldsymbol{\zeta}^{\prime}\right)
$$

then the existence of critical points to $E(u)$ may be reduced to find critical points of the finite dimensional function $F\left(\boldsymbol{\sigma}, \boldsymbol{\zeta}^{\prime}\right)$.

Proposition 4.1. If $\left(\boldsymbol{\sigma}, \boldsymbol{\zeta}^{\prime}\right)$ is a critical point of $F\left(\boldsymbol{\sigma}, \boldsymbol{\zeta}^{\prime}\right)$, then $c_{m, i}=0$ for all $m, i$.
Proof. It is easy to see that

$$
\begin{aligned}
\partial_{\zeta_{m, j}^{\prime}} F\left(\boldsymbol{\sigma}, \boldsymbol{\zeta}^{\prime}\right) & =\partial_{\zeta_{m, j}^{\prime}} E(\mathcal{W}+\phi)=E^{\prime}(\mathcal{W}+\phi)\left[\frac{\partial \mathcal{W}}{\partial \zeta_{m, j}^{\prime}}+\frac{\partial \phi}{\partial \zeta_{m, j}^{\prime}}\right] \\
& =\sum_{i, \ell} \operatorname{Re} \int_{\mathbb{R}^{N}} c_{\ell, i} \chi_{\ell} \bar{Z}_{\ell, i}\left[\frac{\partial U_{m}}{\partial \zeta_{m, j}^{\prime}}+\varepsilon \frac{\partial \Psi_{m}}{\partial \zeta_{m, j}^{\prime}}+\frac{\partial \phi}{\partial \zeta_{m, j}^{\prime}}\right]=c_{m, j} \int_{\mathbb{R}^{N}} \chi_{m}\left|Z_{m, j}\right|^{2}+o(1)
\end{aligned}
$$

Similarly, it is also true that

$$
\partial_{\sigma_{m}} F\left(\boldsymbol{\sigma}, \boldsymbol{\zeta}^{\prime}\right)=c_{m, 0} \int_{\mathbb{R}^{N}} \chi_{m}\left|Z_{m, 0}\right|^{2}+o(1)
$$

Thus $c_{m, i}=0$ if $\left(\boldsymbol{\sigma}, \boldsymbol{\zeta}^{\prime}\right)$ is a critical point of $F$ since the coefficient matrix of $c_{m, i}$ is diagonal dominant.
Next we should calculate $F\left(\boldsymbol{\sigma}, \boldsymbol{\zeta}^{\prime}\right)$ in view of Proposition 4.1.
Proposition 4.2. It holds that

$$
F\left(\boldsymbol{\sigma}, \boldsymbol{\zeta}^{\prime}\right)=E(\mathcal{W})+O\left(\varepsilon^{4}\right)
$$

and for any $j, k$,

$$
\partial_{\zeta_{j, k}^{\prime}} F\left(\boldsymbol{\sigma}, \boldsymbol{\zeta}^{\prime}\right)=\partial_{\zeta_{j, k}^{\prime}} E(\mathcal{W})+O\left(\varepsilon^{2+2 \beta}\right), \quad \partial_{\sigma_{j}} F\left(\boldsymbol{\sigma}, \boldsymbol{\zeta}^{\prime}\right)=\partial_{\sigma_{j}} E(\mathcal{W})+O\left(\varepsilon^{2+2 \beta}\right)
$$

Proof. Direct computation leads to

$$
\begin{aligned}
F\left(\boldsymbol{\sigma}, \boldsymbol{\zeta}^{\prime}\right)= & \frac{1}{2} \int_{\mathbb{R}^{N}}|\mathrm{i} \nabla(\mathcal{W}+\phi)+\boldsymbol{A}(\varepsilon y)(\mathcal{W}+\phi)|^{2}+\frac{1}{2} \int_{\mathbb{R}^{N}}|\mathcal{W}+\phi|^{2}-\frac{1}{p+1} \int_{\mathbb{R}^{N}}|\mathcal{W}+\phi|^{p+1} \\
= & E(\mathcal{W})+\frac{1}{2} \int_{\mathbb{R}^{N}} \operatorname{Re}((R(y)-N(\phi)) \bar{\phi})-\int_{\mathbb{R}^{N}}\left[\frac{1}{p+1}|\mathcal{W}+\phi|^{p+1}-\frac{1}{p+1}|\mathcal{W}|^{p+1}\right. \\
& \left.-|\mathcal{W}|^{p-1} \operatorname{Re}(\mathcal{W} \bar{\phi})-\frac{p-1}{2}|\mathcal{W}|^{p-3}(\operatorname{Re}(\mathcal{W} \bar{\phi}))^{2}-\frac{1}{2}|\mathcal{W}|^{p-1}|\phi|^{2}\right] .
\end{aligned}
$$

Then the proposition follows from Proposition 2.1 and Proposition 3.4 easily. By the computation in Proposition 3.4, it is easy to check that

$$
\partial_{\zeta_{j, k}^{\prime}} F\left(\boldsymbol{\sigma}, \boldsymbol{\zeta}^{\prime}\right)=\partial_{\zeta_{j, k}^{\prime}} E(\mathcal{W})+O\left(\|\phi\|^{1+\beta}\right)
$$

So is $\partial_{\sigma_{j}} F\left(\boldsymbol{\sigma}, \boldsymbol{\zeta}^{\prime}\right)$.
Since $E(\mathcal{W})$ is the main part of $F\left(\boldsymbol{\sigma}, \boldsymbol{\zeta}^{\prime}\right)$, it is important to get the expression of $E(\mathcal{W})$. Elegant computation shows the following proposition.

Proposition 4.3. It holds that for $\varepsilon$ small enough,

$$
E(\mathcal{W})=A_{0} K+B_{0} \varepsilon^{2} \sum_{m=1}^{K} \sum_{i, j=1}^{N}\left(\partial_{i} A_{j}\left(\zeta_{m}\right)-\partial_{j} A_{i}\left(\zeta_{m}\right)\right)^{2}+O\left(\varepsilon^{4}\right)
$$

Furthermore, the remainder term $O\left(\varepsilon^{4}\right)$ also holds for the derivatives in $\boldsymbol{\zeta}^{\prime}$, $\boldsymbol{\sigma}$. Here $A_{0}=\frac{p-1}{2(p+1)} \int_{\mathbb{R}^{N}} w^{p+1}(|y|) \mathrm{d} y$ and $B_{0}=\frac{1}{4} \int_{\mathbb{R}^{N}} y_{1}^{2} w^{2}(|y|) \mathrm{d} y$ are both universal positive constants.
Proof. It is easy to see that

$$
\begin{align*}
E(\mathcal{W})= & \frac{1}{2} \int_{\mathbb{R}^{N}}|\mathrm{i} \nabla \mathcal{W}+\boldsymbol{A}(\varepsilon y) \mathcal{W}|^{2}+\frac{1}{2} \int_{\mathbb{R}^{N}}|\mathcal{W}|^{2}-\frac{1}{p+1} \int_{\mathbb{R}^{N}}|\mathcal{W}|^{p+1} \\
= & \frac{1}{2} \operatorname{Re} \int_{\mathbb{R}^{N}} R(y) \overline{\mathcal{W}}+\frac{p-1}{2(p+1)} \int_{\mathbb{R}^{N}}|\mathcal{W}|^{p+1} \\
= & \sum_{m=1}^{K} \operatorname{Re} \int_{B\left(\zeta_{m}^{\prime} ; \frac{\delta}{\sqrt{\varepsilon}}\right)}\left[\frac{1}{2} R(y) \overline{\mathcal{W}}+\frac{p-1}{2(p+1)}|\mathcal{W}|^{p+1}\right]+\operatorname{Re} \int_{\mathbb{R}^{N} \backslash \cup_{m=1}^{K} B\left(\zeta_{m}^{\prime} ; \frac{\delta}{\sqrt{\varepsilon}}\right)}\left[\frac{1}{2} R(y) \overline{\mathcal{W}}+\frac{p-1}{2(p+1)}|\mathcal{W}|^{p+1}\right] \\
= & \sum_{m=1}^{K} \int_{B\left(\zeta_{m}^{\prime} ; \frac{\delta}{\sqrt{\varepsilon}}\right)}\left[\frac{R_{m, 1}}{2} w\left(\left|y-\zeta_{m}^{\prime}\right|\right)+\varepsilon \frac{R_{m, 2}}{2} \psi_{m}(y)+\frac{p-1}{2(p+1)}\left|U_{m}\right|^{p+1}\right. \\
& \left.+\frac{(p-1) \varepsilon^{2}}{4}\left|U_{m}\right|^{p-1}\left|\Psi_{m}\right|^{2}\right]+O\left(\varepsilon^{4}\right) \tag{21}
\end{align*}
$$

where $\psi_{m}, \Psi_{m}$ are given in (9), and $R_{m, 1}, R_{m, 2}$ are defined in Proposition 2.1. First, by the oddness of the terms in order $\varepsilon^{3}$ in (13), it can be obtained that

$$
\begin{aligned}
& \int_{B\left(\zeta_{m}^{\prime} ; \frac{\delta}{\sqrt{\varepsilon}}\right)} R_{m, 1} w\left(\left|y-\zeta_{m}^{\prime}\right|\right) \\
&= \varepsilon^{2} \sum_{i, j, k=1}^{N} \partial_{j} A_{i}\left(\zeta_{m}\right) \partial_{k} A_{i}\left(\zeta_{m}\right) \int_{B\left(\zeta_{m}^{\prime} ; \frac{\delta}{\sqrt{\varepsilon}}\right)}\left(y_{j}-\zeta_{m, j}^{\prime}\right)\left(y_{k}-\zeta_{m, k}^{\prime}\right) w^{2}\left(\left|y-\zeta_{m}^{\prime}\right|\right) \\
&-\varepsilon^{2} \sum_{j, k, \ell=1}^{N} \partial_{\ell} A_{k}\left(\zeta_{m}\right)\left[\partial_{j} A_{k}\left(\zeta_{m}\right)+\partial_{k} A_{j}\left(\zeta_{m}\right)\right] \int_{B\left(\zeta_{m}^{\prime} ; \frac{\delta}{\sqrt{\varepsilon}}\right)}\left(y_{j}-\zeta_{m, j}^{\prime}\right)\left(y_{\ell}-\zeta_{m, \ell}^{\prime}\right) w^{2}\left(\left|y-\zeta_{m}^{\prime}\right|\right) \\
&-\varepsilon^{2} \sum_{i, j, k, \ell}^{N} \partial_{\ell} A_{k}\left(\zeta_{m}\right) \partial_{j} A_{i}\left(\zeta_{m}\right) \int_{B\left(\zeta_{m}^{\prime} ; \frac{\delta}{\sqrt{\varepsilon}}\right)}\left(y_{i}-\zeta_{m, i}^{\prime}\right)\left(y_{j}-\zeta_{m, j}^{\prime}\right)\left(y_{k}-\zeta_{m, k}^{\prime}\right)\left(y_{\ell}-\zeta_{m, \ell}^{\prime}\right) \\
&-\frac{\varepsilon^{2}}{2}\left(\nabla_{x} \cdot \boldsymbol{A}\left(\zeta_{m}\right)\right) \sum_{i, j=1}^{N} \partial_{j} A_{i}\left(\zeta_{m}\right) \int_{B\left(\zeta_{m}^{\prime} ; \frac{\delta}{\sqrt{\varepsilon}}\right)}\left(y-\zeta_{m}^{\prime} \mid\right) \\
&\left|y-\zeta_{m}^{\prime}\right| \\
&\left(y_{i}-\zeta_{m, i}^{\prime}\right)\left(y_{j}-\zeta_{m, j}^{\prime}\right) w^{2}\left(\left|y-\zeta_{m}^{\prime}\right|\right) \\
&-\frac{(p-1) \varepsilon^{2}}{8} \sum_{i, j, k, \ell}^{N} \partial_{j} A_{i}\left(\zeta_{m}\right) \partial_{\ell} A_{k}\left(\zeta_{m}\right) \int_{B\left(\zeta_{m}^{\prime} ; \frac{\delta}{\sqrt{\varepsilon}}\right)}\left(y_{i}-\zeta_{m, i}^{\prime}\right)\left(y_{j}-\zeta_{m, j}^{\prime}\right)\left(y_{k}-\zeta_{m, k}^{\prime}\right)\left(y_{\ell}-\zeta_{m, \ell}^{\prime}\right) w^{p+1} \\
&+O\left(\varepsilon^{4}\right) .
\end{aligned}
$$

Since integration by parts gives

$$
\begin{aligned}
\int_{\mathbb{R}^{N}} y_{i} y_{j} y_{k} y_{\ell} w(|y|) \frac{w^{\prime}(|y|)}{|y|} \mathrm{d} y & =\frac{1}{2} \int_{\mathbb{R}^{N}} y_{i} y_{j} y_{k}\left(\partial_{\ell} w^{2}\right) \\
& =-\frac{1}{2} \int_{\mathbb{R}^{N}} \delta_{i \ell} \delta_{j k} y_{j} y_{k} w^{2}-\frac{1}{2} \int_{\mathbb{R}^{N}} \delta_{j \ell} \delta_{i k} y_{i} y_{k} w^{2}-\frac{1}{2} \int_{\mathbb{R}^{N}} \delta_{k \ell} \delta_{i j} y_{i} y_{j} w^{2}
\end{aligned}
$$

one gets

$$
\begin{aligned}
& \sum_{i, j, k, \ell=1}^{N} \partial_{\ell} A_{k}\left(\zeta_{m}\right) \partial_{j} A_{i}\left(\zeta_{m}\right) \int_{\mathbb{R}^{N}}\left(y_{i}-\zeta_{m, i}^{\prime}\right)\left(y_{j}-\zeta_{m, j}^{\prime}\right)\left(y_{k}-\zeta_{m, k}^{\prime}\right)\left(y_{\ell}-\zeta_{m, \ell}^{\prime}\right) \frac{w^{\prime}\left(\left|y-\zeta_{m}^{\prime}\right|\right)}{\left|y-\zeta_{m}^{\prime}\right|} w\left(\left|y-\zeta_{m}^{\prime}\right|\right) \\
=\quad & -2 B_{0} \sum_{i, j}\left[\partial_{i} A_{i}\left(\zeta_{m}\right) \partial_{j} A_{j}\left(\zeta_{m}\right)+\left(\partial_{j} A_{i}\left(\zeta_{m}\right)\right)^{2}+\partial_{i} A_{j}\left(\zeta_{m}\right) \partial_{j} A_{i}\left(\zeta_{m}\right)\right]
\end{aligned}
$$

It may be checked that

$$
\begin{align*}
& \int_{B\left(\zeta_{m}^{\prime} ; \frac{\delta}{\sqrt{\varepsilon}}\right)} \frac{R_{m, 1}}{2} w\left(\left|y-\zeta_{m}^{\prime}\right|\right) \\
= & B_{0} \varepsilon^{2}\left\{2 \sum_{i, j}\left(\partial_{j} A_{i}\left(\zeta_{m}\right)\right)^{2}-2 \sum_{i, j} \partial_{j} A_{i}\left(\zeta_{m}\right)\left(\partial_{j} A_{i}\left(\zeta_{m}\right)+\partial_{i} A_{j}\left(\zeta_{m}\right)\right)\right. \\
& \left.\quad+\sum_{i, j}\left[\partial_{i} A_{i}\left(\zeta_{m}\right) \partial_{j} A_{j}\left(\zeta_{m}\right)+\left(\partial_{j} A_{i}\left(\zeta_{m}\right)\right)^{2}+\partial_{i} A_{j}\left(\zeta_{m}\right) \partial_{j} A_{i}\left(\zeta_{m}\right)\right]-\left(\sum_{i} \partial_{i} A_{i}\left(\zeta_{m}\right)\right)^{2}\right\} \\
& -\frac{(p-1) \varepsilon^{2}}{16} \sum_{i, j, k, \ell}^{N} \partial_{j} A_{i}\left(\zeta_{m}\right) \partial_{\ell} A_{k}\left(\zeta_{m}\right) \int_{B\left(\zeta_{m}^{\prime} ; \frac{\delta}{\sqrt{\varepsilon}}\right)}\left(y_{i}-\zeta_{m, i}^{\prime}\right)\left(y_{j}-\zeta_{m, j}^{\prime}\right)\left(y_{k}-\zeta_{m, k}^{\prime}\right)\left(y_{\ell}-\zeta_{m, \ell}^{\prime}\right) w^{p+1} \\
& +O\left(\varepsilon^{4}\right) . \tag{22}
\end{align*}
$$

Also

$$
\begin{aligned}
& \int_{B\left(\zeta_{m}^{\prime} ; \frac{\delta}{\sqrt{\varepsilon}}\right)}\left|U_{m}\right|^{p-1}\left|\Psi_{m}\right|^{2} \\
= & \frac{1}{4} \sum_{i, j, k, \ell=1}^{N} \partial_{j} A_{i}\left(\zeta_{m}\right) \partial_{\ell} A_{k}\left(\zeta_{m}\right) \int_{B\left(\zeta_{m}^{\prime} ; \delta\right)}\left(y_{i}-\zeta_{m, i}^{\prime}\right)\left(y_{j}-\zeta_{m, j}^{\prime}\right)\left(y_{k}-\zeta_{m, k}^{\prime}\right)\left(y_{\ell}-\zeta_{m, \ell}^{\prime}\right) w^{p+1}\left(\left|y-\zeta_{m}^{\prime}\right|\right)+O\left(\mathrm{e}^{-\frac{\delta}{\varepsilon}}\right),
\end{aligned}
$$

and

$$
\int_{B\left(\zeta_{m}^{\prime} ; \frac{\delta}{\sqrt{\varepsilon}}\right)} R_{m, 2} \psi_{m}=O\left(\varepsilon^{3}\right)
$$

by the oddness. Obviously one has

$$
\int_{B\left(\zeta_{m}^{\prime} ; \frac{\delta}{\sqrt{\varepsilon}}\right)}\left|U_{m}\right|^{p+1}=\int_{\mathbb{R}^{N}} w^{p+1}(y) \mathrm{d} y+O\left(\mathrm{e}^{-\frac{\delta}{\varepsilon}}\right)
$$

Note that in (22) the term containing $w^{p+1}$ is canceled with $\frac{p-1}{4} \varepsilon^{2} \int_{\mathbb{R}^{N}}\left|U_{m}\right|^{p-1}\left|\Psi_{m}\right|^{2}$. So we conclude, from (21), that

$$
\begin{aligned}
E(\mathcal{W}) & =A_{0} K+\varepsilon^{2} B_{0} \sum_{m=1}^{K} \sum_{i, j=1}^{N}\left[\left(\partial_{j} A_{i}\left(\zeta_{m}\right)\right)^{2}-\partial_{j} A_{i}\left(\zeta_{m}\right) \partial_{i} A_{j}\left(\zeta_{m}\right)\right]+O\left(\varepsilon^{4}\right) \\
& =A_{0} K+\varepsilon^{2} \frac{B_{0}}{2} \sum_{m=1}^{K} \sum_{i, j=1}^{N}\left(\left(\partial_{j} A_{i}\left(\zeta_{m}\right)\right)-\partial_{i} A_{j}\left(\zeta_{m}\right)\right)^{2}+O\left(\varepsilon^{4}\right)
\end{aligned}
$$

The last equality is due to the symmetry of the indexes $i$ and $j$. The remainder $O\left(\varepsilon^{4}\right)$ also holds for the derivatives of $E(\mathcal{W})$ in ( $\left.\boldsymbol{\sigma}, \boldsymbol{\zeta}^{\prime}\right)$ from directly checking the expressions of $R_{m, 1}$ and $R_{m, 2}$ in the proof of Proposition 2.1.

## 5. Proof of the main theorems

This section devotes to the proof of main theorems.
Proof of Theorem 1.1. Proposition 4.2 and Proposition 4.3 mean

$$
F\left(\boldsymbol{\sigma}, \boldsymbol{\zeta}^{\prime}\right)=A_{0} K+B_{0} \varepsilon^{2} \sum_{m=1}^{K}\left\|\boldsymbol{B}\left(\zeta_{m}\right)\right\|_{F}^{2}+O\left(\varepsilon^{4}\right)
$$

We shall show that $F$ has a critical point under the assumption. Note that for any fixed $\boldsymbol{\zeta}^{\prime}, F\left(\boldsymbol{\sigma}, \boldsymbol{\zeta}^{\prime}\right)$ is periodic in $\boldsymbol{\sigma} \in([0,2 \pi])^{K}$. So there always exists a $\boldsymbol{\sigma}\left(\boldsymbol{\zeta}^{\prime}\right)$ such that $\partial_{\boldsymbol{\sigma}} F\left(\boldsymbol{\sigma}\left(\boldsymbol{\zeta}^{\prime}\right), \boldsymbol{\zeta}^{\prime}\right)=0$. Next consider the configuration set $\boldsymbol{\Omega}^{\prime}=\Omega_{1}^{\prime} \times \Omega_{2}^{\prime} \times \cdots \times \Omega_{m}^{\prime}$ of $\zeta^{\prime}=\left(\zeta_{1}^{\prime}, \zeta_{2}^{\prime}, \cdots, \zeta_{m}^{\prime}\right)$, where $\Omega_{m}^{\prime}=\varepsilon^{-1} \Omega_{m}$. Obviously

$$
\max _{\Omega^{\prime}} F\left(\boldsymbol{\sigma}\left(\boldsymbol{\zeta}^{\prime}\right), \boldsymbol{\zeta}^{\prime}\right) \geq A_{0} K+B_{0} \varepsilon^{2} \sum_{m=1}^{K}\left\|\boldsymbol{B}\left(P_{m}\right)\right\|_{F}^{2}+O\left(\varepsilon^{4}\right)
$$

On the other hand, for any $\zeta^{\prime}$ on the boundary $\partial \boldsymbol{\Omega}^{\prime}$, i.e. at least $\zeta_{1}^{\prime} \in \partial \Omega_{1}^{\prime}$ without loss of generality, then $\left\|B\left(\zeta_{1}\right)\right\|_{F}^{2} \leq\left\|B\left(P_{1}\right)\right\|_{F}^{2}-\delta_{0}$ for some fixed small $\delta_{0}>0$. Thus one finds that

$$
\left.F\left(\boldsymbol{\sigma}\left(\boldsymbol{\zeta}^{\prime}\right), \boldsymbol{\zeta}^{\prime}\right)\right|_{\boldsymbol{\zeta}^{\prime} \in \partial \boldsymbol{\Omega}^{\prime}} \leq A_{0} K+B_{0} \varepsilon^{2} \sum_{m=2}^{K}\left\|\boldsymbol{B}\left(P_{m}\right)\right\|_{F}^{2}+B_{0} \varepsilon^{2}\left(\left\|\boldsymbol{B}\left(P_{1}\right)\right\|_{F}^{2}-\delta_{0}\right)+O\left(\varepsilon^{4}\right)
$$

Therefore $\max _{\boldsymbol{\Omega}^{\prime}} F\left(\boldsymbol{\sigma}\left(\boldsymbol{\zeta}^{\prime}\right), \boldsymbol{\zeta}^{\prime}\right)>\max _{\partial \boldsymbol{\Omega}^{\prime}} F\left(\boldsymbol{\sigma}\left(\boldsymbol{\zeta}^{\prime}\right), \boldsymbol{\zeta}^{\prime}\right)$. It implies that $F\left(\boldsymbol{\sigma}, \boldsymbol{\zeta}^{\prime}\right)$ admits a critical point.
The same procedure can be carried out for the case of $K$ local minimum points. Theorem 1.1 concludes from Proposition 4.1.

Proof of Theorem 1.2. From Proposition 4.2 and Proposition 4.3, we see that

$$
\nabla_{\zeta_{m}^{\prime}} F\left(\boldsymbol{\sigma}, \boldsymbol{\zeta}^{\prime}\right)=B_{0} \varepsilon^{2} \nabla_{\zeta_{m}^{\prime}}\left(\left\|\boldsymbol{B}\left(\zeta_{m}\right)\right\|_{F}^{2}\right)+O\left(\varepsilon^{2+2 \beta}\right)=B_{0} \varepsilon^{3} \nabla_{\zeta_{m}}\left(\left\|\boldsymbol{B}\left(\zeta_{m}\right)\right\|_{F}^{2}\right)+O\left(\varepsilon^{2+2 \beta}\right)
$$

Assume $m=1$ for simplicity. Choose $\zeta_{1}=P_{1}+\varepsilon^{\alpha} \xi_{1}$ where $0<\alpha<2 \beta-1$. Here the assumption $p>\frac{3}{2}$ is used to let $\beta=\min \{p-1,1\}>1 / 2$. Then it is equivalent to find a $\left|\xi_{1}\right| \leq 1$ such that

$$
\begin{aligned}
0 & =\nabla_{\zeta_{1}}\left(\left\|\boldsymbol{B}\left(\zeta_{1}\right)\right\|_{F}^{2}\right)+O\left(\varepsilon^{2 \beta-1}\right) \\
& =\nabla_{\zeta_{1}}\left(\left\|\boldsymbol{B}\left(P_{1}\right)\right\|_{F}^{2}\right)+\varepsilon^{\alpha} \nabla_{\zeta_{1} \zeta_{1}}^{2}\left(\left\|\boldsymbol{B}\left(P_{1}\right)\right\|_{F}^{2}\right) \cdot \xi_{1}+O\left(\varepsilon^{2 \alpha}\left|\xi_{1}\right|^{2}\right)+O\left(\varepsilon^{2 \beta-1}\right)
\end{aligned}
$$

Thus, the nondegeneracy of the critical point $P_{1}$ leads to the existence of $\xi_{1},\left|\xi_{1}\right|=o(1)$ from the Brouwer fixed point theorem. Lastly, the existence of critical $\boldsymbol{\sigma}$ is guaranteed by the periodicity just like in the proof of Theorem 1.1. The proof is complete.

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