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CONCEPTS OF GENERALIZED BOUNDED VARIATION AND THE THEORY OF FOURIER SERIES

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ABSTRACT. The aim of this article is to unify a large part of present knowledge about the behaviour of Fourier series of functions of generalized bounded variation. The connections between various concepts are discussed, particular attention being payed to those due to Waterman and Chanturiya. Exploiting the existing interactions and utilizing the power of one or another approach to some typical questions of the Fourier theory, a number of previously unnoticed results are obtained in the course of this exposition.

KEY WORDS AND PHRASES: Fourier series, generalized bounded variation, uniform and absolute convergences, Gibb's phenomenon.
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1. INTRODUCTION.

One of the features which distinguish Jordan's class BV among the other standard classes of functions in analysis is that the Fourier program can be carried out in it in the "most classical sense": the Fourier series of any function of this class is everywhere pointwise convergent; in case of a continuous function the convergence is uniform; continuity of a function may be characterized in terms of its Fourier coefficients, etc. (As it is well known, the Fourier program in its "original sense" failed dramatically already in 1876, when du Bois-Raymond constructed a continuous function whose Fourier series diverges at a point.) Another interesting occurrence is that already in order to characterize continuity, Wiener came out of the class BV and introduced the concept of variation of a higher order. Later investigations in Fourier analysis on the one hand, and a mathematician's always present desire for more elegance and/or more generality in treating a particular problem on the other hand, have lead to further interesting generalizations and new classes of functions. Domains of validity of many classical theorems were extended, sometimes to their "natural borders". After the section on notation, we survey some of these concepts and relationships between them in sections 3 - 5. In 10 sections which follow thereafter we try to picture the present state of the Fourier theory of functions of generalized bounded variation, treating topics as: pointwise, uniform, and absolute convergence, summability, order of magnitude of Fourier coefficients, continuity, determination of a jump, Gibb's phenomenon, conjugate series, Parseval's identities.

It is often indicated how to derive some recent results from each other and from the older ones, usually enriching their substance. However what is being said in this direction is, as a rule, a sample rather than a detailed account of such possibilities.

2. NOTATION AND CONVENTIONS.

2.1. We consider real-valued 2π-periodic functions. If I is an interval with the endpoints a, b $(a<b)$ we write $f(I) = f(b) - f(a)$. For bounded functions, $m(f) = \inf f$, $M(f) = \sup f$.

2.2. $a_n = a_n(f)$ $(b_n = b_n(f))$ denotes, as usual, the n-th cosine (sine) Fourier coefficient of a function f and $\rho_n = (a_n^2 + b_n^2)^{1/2}$. $S(f)$ is the Fourier series of f and $\tilde{S}(f)$ is its conjugate series. Partial sums of $S(f)$ are denoted by $S_n(f)$, $S_n(x,f)$ or simply $S_n(x)$ and those $\tilde{S}(f)$ analogously - $\tilde{S}_n(f)$, $\tilde{S}_n(x,f)$ or $\tilde{S}_n(x)$.

2.3. The modulus of continuity of f is

$$\omega_f(\delta) = \omega_\infty(\delta,f) = \sup_{0<h\le\delta} |f(x+h) - f(x)|.$$

Given an arbitrary nondecreasing continuous function ω defined on $[0,\pi]$, $\omega(0) = 0$ and ω subadditive, we set

$$H^\omega = \{f \in C: \omega_f(\delta) = O(\omega(\delta)) \text{ as } \delta \to 0\}.$$

If $\omega(\delta) = \delta^\alpha$, $0 < \alpha \le 1$, this class is denoted Lip α, as usual, and lip $\alpha = \{f \in C: \omega_f(\delta) = o(\delta^\alpha)\}$. The norm of $f \in C$ is

$$||f||_\infty = \sup_{0\le x\le 2\pi} |f(x)|.$$

2.4. Integral moduli of continuity of $f \in L^p$ are given by

$$\omega_p(\delta,f) = \sup_{0\le h\le\delta} \left(\int_0^{2\pi} |f(x+h) - f(x)|^p dx\right)^{1/p}, \quad 0 < p < \infty.$$

For $0 < \alpha \le 1$, Lip $(\alpha,p) = \{f \in L^p: \omega_p(\delta,f) = O(\delta^\alpha)\}$

lip $(\alpha,p) = \{f \in L^p: \omega_p(\delta,f) = o(\delta^\alpha)\}$.

2.5. The best approximation of order n to $f \in L^p$ in L^p is

$$E_p(n,f) = \inf \left(\int_0^{2\pi} |f(x) - T_n(x)|^p dx\right)^{1/p},$$

where the infimum is taken over all trigonometric polynomials T_n of degree not higher than n, $n = 1,2,\dots$.

2.6. (C,α), $\alpha \le -1$, denotes the Cesáro summability method of order α ([1], pp. 96-97; [2], p.76) and $\delta_n^\alpha(f)$ $(\tilde{S}_n^\alpha(f))$ the corresponding Cesáro means of $S(f)$ $(\tilde{S}(f))$.

2.7. A is the class of functions with absolutely convergent Fourier series. W is the class of regulated functions (Dieudonné [3], p. 139), i.e. functions possessing the one-sided limits at each point. For $f \in W$ we always suppose $f(x) = (f(x+0) - f(x-0))/2$. W is actually the universal class of this paper.

2.8. Symbols for the classes of generalized bounded variation and related notions are to be found in Sections 3 and 5.

3. GENERALIZATIONS DUE TO N. WIENER, L.C. YOUNG, D. WATERMAN AND Z.A. CHANTURIYA.

The classes of functions of bounded variation of higher orders were for the first time introduced by Wiener in his paper [4]. Various definitions of this notion appear in the literature, but they all yield the same classes of functions, [5]. We choose

DEFINITION 3.1. A function f is said to be of bounded p-variation on $[0,2\pi]$, $p\leq 1$ and to belong to the class V_p if

$$V_p(f) = \sup \{ \sum_i |f(I_i)|^p \}^{1/p} < \infty,$$

where the supremum is taken over all finite collections of nonoverlapping subintervals I_i of $[0,2\pi]$. The quantity $V_p(f)$ is called the p-variation of f on $[0,2\pi]$.

For a detailed study of p-variation see also [6-12].

Wiener's concept has been genralized by Young [13] in the following way.

DEFINITION 3.2. Let ϕ be a continuous function defined on $[0,\infty)$ and strictly increasing from 0 to ∞. The ϕ-variation of a function f on $[0,2\pi]$ is the supremum $V_\phi(f)$ of the sums

$$\sum_i \phi(|f(I_i)|)$$

over all systems $\{I_i\}$ of nonoverlapping subintervals of $[0,2\pi]$. $V_\phi = \{f: \ V_\phi(f)<\infty\}$.

Clearly, $\phi(u) = u$ gives the Jordan's class BV and $\phi(u) = u^p$ gives Wiener's V_p . It is customary to consider convex functions ϕ satisfying the conditions: $\phi(0) = 0$, $\phi(x)/x \to 0$ $(x\to 0+)$ and $\phi(x)/x\to \infty$ $(x\to\infty)$. Such a function ϕ is necessarily nonnegative, continuous and strictly increasing from the point $x_o = \inf \{x: \ \phi(x)> 0\} = \sup \{x: \ \phi(x) = 0\}$ on. We suppose that ϕ is nondegenerate, i.e. $x_o = 0$. The function ψ complementary to ϕ in the sense of Young is defined by $\psi(y) = \max_{x\geq 0}\{xy - \phi(x)\}$.

It is also convenient to suppose that ϕ satisfies the condition Δ_2: There exist positive constants x_o and d $(d \geq 2)$ such that

$$\phi(2x) \leq d\phi(x) \quad \text{for} \quad 0 < x \leq x_o$$

since this condition is necessary and sufficient for the space V_ϕ to be linear.

The classes V_ϕ have been thoroughly studied in Musielak and Orlicz [14] and Lesniewicz and Orlicz [15].

Another type of generalization, directly influenced by the convergence problems in the theory of Fourier series, appeared in Waterman's paper [16] in 1972.

DEFINITION 3.3. Let $\Lambda = \{\lambda_n\}$ be a nondecreasing sequence of positive numbers tending to infinity, such that $\Sigma\, 1/\lambda_n$ diverges. A function f is said to be of Λ-bounded variation on $[0,2\pi]$ (or to belong to the class ΛBV) if

$$\Sigma\ |f(I_n)|/\lambda_n < \infty$$

for every choice of nonoverlapping intervals $I_n \subset [0,2\pi]$. The supremum of these sums, is called the Λ-variation of f . In case $\Lambda = \{n\}$ one speaks of harmonic bounded variation and the class HBV.

Let $\Lambda^m = \{\lambda_{n+m}\}$, $m = 0,1,2,\ldots$. A function $f \in \Lambda BV$ is said to be <u>continuous in Λ-variation</u> (to belong to $\Lambda_c BV$) if $V_{\Lambda^m}(f) \to 0$, as $m \to \infty$ ([17]).

Properties of functions of the class ΛBV, properties of Λ-variation function, $\Lambda_c BV$, ΛBV as Banach space, etc., have been investigated in [18],[19], [20], [21]. Perlman, e.g., has shown in [18] that BV is precisely the intersection and W the union of all ΛBV spaces and that no one of these results can be improved by taking countable intersections or unions of ΛBVs.

If ϕ and ψ are Young's complementary functions, it follows directly from the inequality $xy \le \phi(x) + \psi(y)$ (Young's inequality) that $V_\phi \subset HBV$ if $\Sigma\,\psi(1/n) < \infty$. Actually, more can be said.

THEOREM 3.1. $\Sigma\,\psi(1/n) < \infty$ implies $V_\phi \subset H_c BV$.

PROOF. There exists a decreasing sequence of positive numbers ε_n tending to 0 so slowly that $\Sigma\,\psi(\frac{1}{n\varepsilon_n}) < \infty$ ([22]). Let $f \in V_\phi$ and $\{I_n\}$ be a sequence of nonoverlapping subintervals of $[0,2\pi]$. For every $m = 0,1,2,\ldots$ one has

$$\frac{1}{\varepsilon_{m+1}} \sum_n \frac{|f(I_n)|}{n+m} \le \sum_n \frac{|f(I_n)|}{(n+m)\varepsilon_{n+m}} \le$$

$$\le \sum_n \phi(|f(I_n)|) + \sum_n \psi(\frac{1}{(n+m)\varepsilon_{n+m}}) \le V_\phi(f) + C.$$

Hence

$$\sum_n \frac{|f(I_n)|}{n+m} \le \varepsilon_{m+1}(V_\phi(f)+C) \to 0 \quad (m \to \infty),$$

implying $f \in H_c BV$.

Recently Schramm and Waterman [23] have combined concepts of ϕ and Λ variation into

DEFINITION 3.4. A function f is said to be of $\phi\Lambda$-bounded variation ($f \in \phi\Lambda BV$) if for every system $\{I_n\}$ of nonoverlapping subintervals of $[0,2\pi]$

$$\Sigma\,\phi(|f(I_n)|)/\lambda_n < \infty \quad .$$

The supremum of these sums, $V_{\phi\Lambda}(f)$, is called the $\phi\Lambda$-variation of f on $[0,2\pi]$.

An intermediate step was Shiba's [24] introduction of the classes $\Lambda BV^{(p)}$ ($\phi(u) = u^p$, $p > 1$).

Finally, Chanturiya's generalization [25], whose connections with that of Waterman are going to play the central role in the course of our narrative, is described in

DEFINITION 3.5. The modulus of variation of a bounded function f is the function ν_f with domain the positive integers, given by

$$\nu_f(n) = \sup_{\Pi_n} \sum_{k=1}^{n} |f(I_k)|,$$

where $\Pi_n = \{I_k\colon\ k=1,\ldots,n\}$ is an arbitrary finite collection of n nonoverlapping subintervals of $[0,2\pi]$.

The modulus of variation of any function is nondecreasing and concave. Given a function ν of an integral argument with such properties, then by $V[\nu]$ one denotes the class of functions f for which $\nu_f(n) = O(\nu(n))$ $(n \to \infty)$. We note that $V_\phi \subset V[n\phi^{-1}(1/n)]$ and $W = \{f: \nu_f(n) = o(n)\}$ ([25]).

4. RELATIONSHIPS BETWEEN WATERMAN'S AND CHANTURIYA'S CONCEPTS.

Using a simple averaging argument the possibility of which lies in the heart of Definition 3.3 (Independent permutations of $\{I_n\}$ and $\{I_k\}$ are admissable!) we have pointed out in [26] the following inclusion relation between Waterman's and Chanturiya's classes.

THEOREM 4.1. $\Lambda BV \subset V\left[\frac{n}{\sum_{i=1}^{n} 1/\lambda_i}\right]$.

Analogously we obtain for the new classes given by Definition 3.4.

THEOREM 4.2. $\phi\Lambda BV \subset V\left[n\phi^{-1}\left(\frac{1}{\sum_{i=1}^{n} 1/\lambda_i}\right)\right]$.

PROOF. Let $\{I_i\}$ be an arbitrary collection of n overlapping intervals $I_i \subset [0,2\pi]$ and $f \in \phi\Lambda BV$. By Jensen's inequality we have

$$\phi\left(\sum_{i=1}^{n} \frac{|f(I_i)|/\lambda_i}{\sum_{k=1}^{n} 1/\lambda_k}\right) \le \sum_{i=1}^{n} \frac{\phi(|f(I_i)|)/\lambda_i}{\sum_{k=1}^{n} 1/\lambda_k} \le \frac{V_{\phi\Lambda}(f)}{\sum_{k=1}^{n} 1/\lambda_k} .$$

This yields

$$\sum_{i=1}^{n} \frac{|f(I_i)|}{\lambda_i} \le C\phi^{-1}\left(\frac{1}{\sum_{k=1}^{n} 1/\lambda_k}\right) \cdot \sum_{k=1}^{n} 1/\lambda_k$$

and in the next step, proceeding as in Theorem 4.1.,

$$\sum_{i=1}^{n} |f(I_i)| \le Cn\phi^{-1}\left(\frac{1}{\sum_{k=1}^{n} 1/\lambda_k}\right) .$$

The result in the opposite direction we shall need in a slightly more precise form than it was stated in [26].

THEOREM 4.3. If $\sum_{k=1}^{\infty} \Delta(1/\lambda_k)\nu_f(k) < \infty$, then f is continuous in Λ variation, i.e., $f \in \Lambda_c BV$.

PROOF. By Abel's partial summation and the facts that ν_f is nondecreasing and $\lambda_n \uparrow \infty$, one obtains

$$\sum_{k=1}^{n} \frac{|f(I_k)|}{\lambda_{k+m}} = \sum_{k=1}^{n-1} \left(\Delta\frac{1}{\lambda_{k+m}}\right) \sum_{i=1}^{k} |f(I_i)| + \frac{1}{\lambda_{n+m}} \sum_{k=1}^{n} |f(I_k)| \le$$

$$\le \sum_{k=1}^{n-1} \left(\Delta\frac{1}{\lambda_{k+m}}\right)\nu_f(k) + \nu_f(n) \sum_{k=n+m}^{\infty} \Delta\frac{1}{\lambda_k} \le$$

$$\le \sum_{k=1}^{n-1} \left(\Delta\frac{1}{\lambda_{k+m}}\right)\nu_f(k+m) + \sum_{k=n+m}^{\infty} \left(\Delta\frac{1}{\lambda_k}\right)\nu_f(k) = \sum_{i=m+1}^{\infty} \left(\Delta\frac{1}{\lambda_i}\right)\nu_f(i) = o(1)$$

as $m \to \infty$.

COROLLARY 1. i) $\sum_{k=1}^{\infty} \nu(k)/k^{1+\alpha} < \infty$, $0 < \alpha < 1$, implies $V[\nu] \subset \{n^\alpha\}_c BV$;

ii) $\sum_{k=1}^{\infty} \nu(k)/k^2 < \infty$ implies $V[\nu] \subset H_c BV$.

COROLLARY 2. i) If $\int_0^1 \frac{dx}{\phi^{1-\alpha}(x)} < \infty$, $0 < \alpha < 1$, then $V_\phi \subset \{n^\alpha\}_c BV$;

ii) If $\int_0^1 \log\frac{1}{\phi(x)}\, dx < \infty$, then $V_\phi \subset H_c BV$.

PROOF. i) $\int_0^1 \frac{dx}{\phi^{1-\alpha}(x)} < \infty$ yields $\sum_{n=1}^{\infty} \frac{1}{n^\alpha} \phi^{-1}(\frac{1}{n}) < \infty$. Now the assertion follows from Corollary 1. i), since $V_\phi \subset V[n\phi^{-1}(1/n)]$.

ii) Follows in a similar way from Corollary 1. ii).

COROLLARY 3. i) If $f \in C$ and $\sum_{k=1}^{\infty} \Delta(1/\lambda_k)k\omega_f(1/k) < \infty$, then $f \in \Lambda_c BV$;

ii) Lip $\alpha \subset \{n^\beta\}_c BV$ for $\beta > 1-\alpha$, $0 < \alpha < 1$.

PROOF. i) Follows from Theorem 4.3 and the fact that $\nu_f(k) = o(k\omega_f(1/k))$ ([25],Th. 4).

ii) Immediately from i).

An especially interesting case is given by

THEOREM 4.4. For $0 < \alpha < \beta < 1$ one has

$$\{n^\alpha\}BV \subset V_{1/(1-\alpha)} \subset V[n^\alpha] \subset \{n^\beta\}_c BV .$$

All inclusions are strict. In particular, Lip $(1-\alpha) \not\subset \{n^\alpha\}BV$.

PROOF. i) Let $f \in \{n^\alpha\}BV$ and let $\{I_k\}$ be an arbitrary finite collection of nonoverlapping subintervals of $[0,2\pi]$. We may suppose that they are denumerated so that the numbers $d_k = |f(I_k)|$ descend. From $\sum_{k=1}^{m} \frac{1}{k^\alpha} \sim \frac{m^{1-\alpha}}{1-\alpha}$ it follows $Cm^{1-\alpha}d_m \le d_m \sum_{k=1}^{m} \frac{1}{k^\alpha} \le \sum_{k=1}^{m} \frac{d_k}{k^\alpha} \le V_{\{n^\alpha\}}(f)$, for every m. Hence $m^{1-\alpha}d_m \le K$ for every m and the constant K is independent of the choice of intervals $\{I_k\}$. For $p = 1/(1-\alpha)$ one has $d_k^p = k^{(1-\alpha)(p-1)}d_k^{p-1}\frac{d_k}{k^\alpha}$. Therefore $\Sigma\, d_k^p \le K^{p-1} \Sigma \frac{d_k}{k^\alpha} \le K^{p-1}V_{\{n^\alpha\}}(f)$, what implies $f \in V_p$.

ii) The inclusion $\mathrm{Lip}(1-\alpha) \subset V_{1/(1-\alpha)}$ is obvious. Let us show that $\mathrm{Lip}(1-\alpha) \not\subset \{n^\alpha\}BV$. In the interval $[0,\pi]$ we shall take the points $x_0 = 0$, $x_n = \pi s_n/s$, where $s_n = \sum_{k=1}^{n} \frac{1}{k\log^{1/(1-\alpha)}(k+1)}$, $s = \lim_{n\to\infty} s_n$. The function f is defined by

$$f(0) = 0,$$

$$f(x_n) = (\pi/s)^{1-\alpha} \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k^{1-\alpha}\log(k+1)},$$

$$f(\pi) = (\pi/s)^{1-\alpha} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^{1-\alpha}\log(k+1)}$$

and to be linear on each $I_k = [x_{k-1}, x_k]$, $k = 1,2,\dots$. f is extended to $[0,2\pi]$ by $f(x+\pi) = f(\pi-x)$ for $0 < x \leq \pi$. Now

$$|f(I_k)| = (\pi/s)^{1-\alpha} \frac{1}{k^{1-\alpha}\log(k+1)} = |I_k|^{1-\alpha} .$$

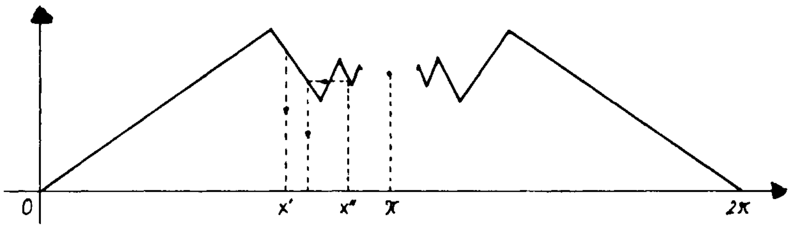

Figure 1

Hence $\sum_{k=1}^{n} \frac{|f(I_k)|}{k^{\alpha}} = (\pi/s)^{1-\alpha} \sum_{k=1}^{n} \frac{1}{k\log(k+1)} \to \infty \quad (n \to \infty)$ so that $f \notin \{n^{\alpha}\}BV$. However, $f \in Lip(1-\alpha)$. Really, for any two points x', $x'' \in [0,2\pi]$, not contained in the same interval I_k , one can always find I_{k_o} such that $|I_{k_o}| \leq |x' - x''|$ and $f(x')$, $f(x'') \in \{f(x) : x \in I_{k_o}\}$. (Geometrically it is obvious. See Figure 1.) Since $f \in Lip(1-\alpha)$ on I_k (with the Lipschitz constant independent of k), the assertion follows.

iii) $V_{1/(1-\alpha)} \subset V[n^{\alpha}]$ follows straightforwardly from the Hölder inequality. An example that the inclusion is strict is provided by the function f: $f(1/2^{i-1}) = 1/i^{1-\alpha}$, $f(3/2^{i+1}) = f(0) = f(2\pi) = 0$ and f linear on the intervals $[1/2^{i}, 3/2^{i+1}]$, $[3/2^{i+1}, 1/2^{i-1}]$ and $[1,2\pi]$, where $i=1,2,\dots$.

iv) $V[n^{\alpha}] \subset \{n^{\beta}\}_c BV$ follows from Theorem 4.3. This inclusion is obviously strict, since the values of β lie in the open interval.

5. BANACH INDICATRIX.

In 1925 Banach [27] introduced, for a continuous function f , the function N_f: $[m(f),M(f)] \to [0,\infty]$ setting $N_f(y)$ equal to the number of values of $x \in [0,2\pi]$ for which $f(x) = y$, if this number is finite, and $N_f(y) = \infty$ otherwise. He proved that $f \in BV$ if and only if $\int_{m(f)}^{M(f)} N_f(y)dy < \infty$ (see also Natanson [28], p. 253 ff., who called N_f the Banach indicatrix of f). If $f \in W$ there exist an increasing function ϕ on $[0,2\pi]$ and a continuous function F on $[\phi(0),\phi(2\pi)]$ such that $f = F\circ\phi$ (Sierpiński [29]; rediscovered in Tsereteli [30]). Defining $N_f = N_F$ one obtains that the Banach theorem is also valid for f , since that variation of a function does not vary for monotone transformations of the argument. This generalization of Banach's result is due to Lozinski [31]. The approach presented here is from [32].

Now let Ω be an increasing concave function on $[0,\infty)$, $\Omega(0)$, $\Omega(x)\to\infty$ $(x\to\infty)$, $\Omega(x)/x\to 0$ $(x\to\infty)$. Asatiani and Chanturiya [32] have recently proved the following theorem.

THEOREM 5.1. If $f\in W$ and

$$\int_{m(f)}^{M(f)} \Omega(N_f(y))dy < \infty , \tag{5.1}$$

then the modulus of continuity of f satisfies

$$\sum_{n=1}^{\infty}[2\Omega(n) - \Omega(n-1) - \Omega(n+1)]\ \nu_f(n) < \infty. \tag{5.2}$$

There exists a $f\in C$ for which (5.2) is valid but (5.1) is not.

Theorem 5.1 and Theorem 4.3 imply

THEOREM 5.2. $\{f\in W: \int_{m(f)}^{M(f)} \Omega(N_f(y))dy < \infty\} \subset \Lambda_c BV$, where $\lambda_n = \dfrac{1}{\Omega(n)-\Omega(n-1)}$, $n\in N$.

PROOF. It suffices to check that the sequence $\{\lambda_n\}$, so defined, possesses all the properties required by Definition 3.3. Now $\lambda_{n+1}\ge\lambda_n$ follows from the fact that Ω is concave. If there were a constant K such that $\lambda_n\le K$ for every n, then we would have $\Omega(n)-\Omega(n-1)\ge 1/K$ and $\Omega(n) = \sum_{k=1}^{n}[\Omega(k)-\Omega(k-1)]\ge n/K$ for every n, contrary to the assumption that $\Omega(n)/n\to 0$ $(n\to\infty)$. Hence $\lambda_n\uparrow\infty$. Finally $\sum_{k=1}^{n} 1/\lambda_k=\Omega(n)\to\infty$.

COROLLARY 1. If $p>1$, $\{f\in W: \int_{m(f)}^{M(f)} N_f^{1/p}(y)dy<\infty\}\subset\{n^{1-1/p}\}_c BV$.

COROLLARY 2. (Zerekidze [33]; Garsia and Sawyer [34], p.591)

$$\{f\in W: \int_{m(f)}^{M(f)} N_f^{1/p}(y)dy<\infty\}\subset V_p ,\ p>1.$$

COROLLARY 3. $\{f\in W: \int_{m(f)}^{M(f)} \log N_f(y)dy < \infty\}\subset H_c BV$.

6. POINTWISE CONVERGENCE OF FOURIER SERIES.

In the question of convergence of the Fourier series of functions belonging to the classes V_ϕ and $V[\nu]$, Salem and Chanturiya, respectively, have concentrated their attention on continuous functions of the class and uniform convergence of their series. The concept of Λ-variation again (and harmonic variation, in particular) has originated from Goffman's and Waterman's investigation [35] of the conditions under which a Fourier series converges everywhere for every change variable. (According to their own words, they were inspired by the possibilities hidden in Salem's method of [22].) The next result, obtained by Waterman ([16],[36]), is the farthest reaching result on the convergence of functions with bounded generalized variation.

THEOREM 6.1. If $f\in HBV$, then the partial sums of its Fourier series are uniformly bounded. The series converges everywhere and converges uniformly on closed intervals of points of continuity. If $\Lambda BV \supsetneq HBV$, then there is a continuous $f\in\Lambda BV$ whose Fourier series diverges at a point.

Using Theorem 4.2 and 4.3 we state three corollaries, the first two of them in partial form.

COROLLARY 1. (Tevzadze [37]), All funcitons of the class $V[\nu]$ with $\Sigma\ \dfrac{\nu(n)}{n^2}<\infty$ have everywhere convergent Fourier series.

COROLLARY 2. If $f \in V_\phi$ and $\Sigma \frac{1}{n}\phi^{-1}(\frac{1}{n}) < \infty$, then the Fourier series of f converges everywhere.

COROLLARY 3. If $\Sigma \frac{1}{n}\phi^{-1}\left(\frac{1}{\sum_{i=1}^{n} 1/\lambda_i}\right) < \infty$, the partial sums of the Fourier series of $f \in \phi\Lambda BV$ are uniformly bounded, the series converges everywhere and the convergence is uniform on the closed intervals of points of continuity.

In virtue of Chanturiya's result that $\Sigma \frac{1}{n}\phi^{-1}(\frac{1}{n}) < \infty$ is equivalent to Salem's condition $\Sigma\, \psi(\frac{1}{n}) < \infty$, Corollary 2 is the known result, obtained in an interesting manner (Koethe space setting) by Goffman [38]. However his method gives no information regarding uniform convergence (what is the initial Salem's result and the omitted part of Corollary 2.) By a result of Sevast'yanov (see Section 7 for more details) Corollaries 1 and 2 are in fact equivalent.

Corollary 3 is new. Unfortunately, the substance of Shiba's paper [39] on uniform convergence of funcitons of the class $\Lambda BV^{(p)}$ is unknown to us, but we expect that the corresponding result is in accordance with Corollary 3, i.e.

<u>If</u> $f \in \Lambda BV^{(p)} \cap C$ <u>and</u> $\Sigma \frac{1}{n(\sum_{i=1}^{n} 1/\lambda_i)^{1/p}} < \infty$, <u>then the Fourier series of f converges uniformly.</u>

In view of Theorem 4.4, everywhere convergence of the Fourier series of a function $f \in \{n^\alpha\}BV$, $0 < \alpha < 1$, follows already from Hardy-Littlewood's result concerning the classes $Lip(1/p,p)$ ([40]) and uniform convergence, if f is continuous, from Yano's observation [41]. (For other proofs in case of Wiener class $V_p \subset Lip(1/p,p)$ see Young [9] and Marcinkiewicz [8].) There is an interesting estimate, due to Bojanić and Waterman [42] of the rate of convergence in this case.

THEOREM 6.2. If $f \in \{n^\alpha\}BV$, $0 < \alpha < 1$, then

$$|S_n(x)-f(x)| \leq \frac{(2-\alpha)(1+2/\pi)}{(n+1)^{1-\alpha}} \sum_{k=1}^{n} \frac{1}{k^\alpha} V(\frac{\pi}{k}) ,$$

where $V(\frac{\pi}{k})$ denotes $\{n^\alpha\}$-variation of $g_x(t) = f(x+t) + f(x-t) - 2f(x)$.

For $\alpha=0$ (i.e. $f \in BV$) this is an earlier result of Bojanić [43]. For a more general result (classes ΛBV which can be closer to HBV) see Waterman [44].

7. UNIFORM CONVERGENCE.

As an application of his convergence criterion, Salem [22] has proved (as it was already noted in Section 6) that all continuous functions of the class V_ϕ have uniformly convergent Fourier series if the function ψ, complementary to ϕ, satisfies the condition

$$\Sigma\, \psi(1/n) < \infty . \tag{7.1}$$

(Later on, we shall refer to such classes V_ϕ as Salem's classes.) For a long time it was an open question whether the condition (7.1) is definitive. It was answered in positive independently by Baerstein [45] and Oskolkov [46] in 1972. Oskolkov has shown that (7.1) is equivalent to

$$\int_0^1 \log \frac{1}{\phi(\xi)} d\xi < \infty \tag{7.2}$$

and this is further, according to the result of Chanturiya [47], equivalent to

$$\Sigma \frac{1}{n}\phi^{-1}\left(\frac{1}{n}\right) < \infty . \tag{7.3}$$

For the deviation of $f \in V_\phi \cap C$ from the partial sums of its Fourier series Oskolkov gave the estimate

$$||S_n(f)-f||_\infty \leq C \int_0^{\omega(\pi/n)} \log\frac{V_\phi(f)}{\phi(\xi)}\, d\xi \quad .$$

Here he supposes merely that ϕ is an increasing function on $[0,\infty)$, with $\phi(0) = 0$, not necessarily convex. In case $\phi(\xi) = \xi$ one obtains the earlier result of Stechkin [48]:

If $f \in BV \cap C$ then $||f-S_n(f)||_\infty \leq C\omega(\pi/n) \log(V(f)\omega(\pi/n))$.

Using a nonincreasing rearrangement ([2], p.30) N_f^* of the Banach indicatrix of a function f, he constructed a convex function ϕ $(\phi(\xi) = \int_0^\xi \frac{du}{N_f^*(u)})$ such that $f \in V_\phi$ and showed that the criterion of Garsia and Sawyer $(\int_{m(f)}^{M(f)} \log N_f(y)\, dy < \infty$; [34], Theorem 1) is a consequence of Salem's result. (Till then, the cases of the Garsia-Sawyer class and Salem classes were treated separately. See, e.g., Goffman [38], Waterman [16].)

In view of the fact that $\nu_f(n) = O(n\phi^{-1}(1/n))$ for $f \in V_\phi$ and equivalence of (7.1) and (7.3), Salem's theorem follows from Chanturiya's result [47]:

Fourier series of all functions of the class $C \cap V[\nu]$ converge uniformly if (and only if)

$$\Sigma \frac{\nu(k)}{k^2} < \infty . \tag{7.4}$$

However, (7.4) is equivalent to (7.1). (If $\Sigma\, \nu(k)/k^2 < \infty$, one can construct ϕ such that $f \in V_\phi$ and (7.1) is satisfied. Sevast'yanov [49]).

Since all the conditions above on f yield $f \in HBV$, corresponding sufficient part statements are corollaries of Theorem 6.1. What is new here, are the estimates, the most general one being

THEOREM 7.1. (Chanturiya, [47]) If $f \in C[0,2\pi]$, then

$$||f-S_n(f)||_\infty \leq C \min_{1\leq m \leq [\frac{n-1}{2}]} \{\omega_f(1/n) \sum_{k=1}^{m} 1/k + \sum_{k=m+1}^{[(n-1)/2]} \nu_f(k)/k^2 \},\ n\geq 3.$$

It implies the estimates of Lebesque and Oskolkov.

8. SUMMABILITY OF FOURIER SERIES.

The usual task of summability in Fourier analysis is to recover a function from its Fourier series by some regular method, since, in general, the series fails to converge. However, in the class HBV this is not the case and our question here is quite different: Can one state some assertion, stronger than convergence, about the behaviour of the Fourier series of functions belonging to certain subclasses of HBV, and, if so, to what extent stronger? It is natural to expect that this depends on the "order of variation" of f. The methods which we consider are Cesáro methods (C,∞), $-1 < \infty < 0$. The central result in this direction is the following theorem due to Waterman [17].

THEOREM 8.1. The Fourier series $S(f)$ of a function $f \in \{n^\alpha\}BV$, $0 < \alpha < 1$, is everywhere $(C,\alpha-1)$ bounded and is uniformly $(C,\alpha-1)$ bounded on each closed interval of continuity of f. If $f \in \{n^\alpha\}_c BV$, then $S(f)$ is everywhere $(C,\alpha-1)$ summable to f and the summability is uniform on each closed interval of continuity.

From $(C,\alpha-1)$ boundedness and convergence, by a well known convexity theorem ([1], p. 127), (C,β) summability for $\beta > \alpha-1$ follows.

Analogous to Theorem 3.1 one realizes that $V_\phi \subset \{n^\alpha\}_c BV$ if $\Sigma\, \psi(1/k^\alpha) < \infty$, $0 < \alpha < 1$. Therefore, as the first corollary of Theorem 8.1 we can state a theorem of Akhobadze [50], proved independently and at the same time as Theorem 8.1.

COROLLARY 1. If $f \in V_\phi$ and $\Sigma\, \psi(1/k^{1-\alpha}) < \infty$, $0 < \alpha < 1$, then the Fourier series of f is $(C,-\alpha)$ summable everywhere and the summability is uniform if f is continuous.

Akhobadze establishes also that the condition $\Sigma\, \psi(1/k^{1-\alpha}) < \infty$ is necessary and sufficient in order that the assertion holds for every $f \in V_\phi$ and, on the other hand, Waterman proves that there exists $f \in C \cap \Lambda BV$ whose Fourier series is not $(C,\alpha-1)$ bounded at some point, if $\Lambda BV \supsetneq \{n^\alpha\}BV$.

COROLLARY 2. (Asatiani, [51] If $f \in C \cap V[\nu]$ and $\Sigma\, \nu(k)/k^{2-\alpha} < \infty$, $0 \le \alpha < 1$, then $S(f)$ is uniformly $(C,-\alpha)$ summable to f.

According to Theorem 8.1, if in Corollary 2 we let out the assumption of continuity of f, the we have $(C,-\alpha)$ everywhere summability in conclusion. Corollary 2, so enriched, and Theorem 4.2 imply

COROLLARY 3. If $f \in \phi\Lambda BV$ and

$$\sum_{n=1}^{\infty} \frac{1}{n^{1-\alpha}}\, \phi^{-1}\left(\frac{1}{\sum_{i=1}^{n} 1/\lambda_i}\right) < \infty ,$$

then $S(f)$ is everywhere $(C,-\alpha)$ summable to f and the summability is uniform on each closed interval of continuity.

In a similar way, using the results of of Section 5, one can state corresponding assertions for the classes defined by means of the Banach indicatrix. (C,β) summability for $f \in V_p$, $\beta > -1/p$, which is an immediate consequence of Theorems 8.1 and 4.4, was proved by Young [9]. It follows also from a more general result of Hardy and Littlewood [40] on the classes $\mathrm{Lip}(1/p,p)$ (see also [2], p. 66).

In the end we mention Asatiani's extension of Chanturiya's theorems on uniform convergence.

THEOREM 8.2. If $f \in C$ and

$$\lim_{n\to\infty} \min_{1\le m\le n-1} \{\omega_f(1/n) \sum_{k=1}^{m} 1/k^{1-\alpha} + \sum_{k=m+1}^{n-1} \nu_f(k)/k^{2-\alpha}\} = 0,\ 0 \le \alpha < 1,$$ then the Fourier series of f is uniformly $(C,-\alpha)$ summable to f.

The condition above (with ω and ν fixed) is also necessary for uniform $(C,-\alpha)$ summability of all Fourier series of functions of the class $H^\omega \cap V[\nu]$.

9. ABSOLUTE CONVERGENCE.

Bernstein has proved that the Fourier series of f is absolutely convergent if $\Sigma\, n^{-1/2}\, \omega_f(1/n) < \infty$ (see [2], p. 241). The condition is best possible in the sense that if $\Sigma\, n^{-1/2}\omega(1/n) = \infty$, then there exists $f \in H^{\omega}$ such that $f \notin A$ (Stechkin [52]; see [53], p. 14 f.). In particular $\mathrm{Lip}\,\alpha \subset A$ for $\alpha > 1/2$ and there exists $f \in \mathrm{Lip}\,1/2 \setminus A$ (This was already proved by Bernstein.) Now if $f \in BV$, the condition posed on its modulus of continuity may be significantly weakened. Namely, by a theorem of Zygmund ([2], p. 241) in this case

$$\Sigma\, 1/n\omega_f^{1/2}(1/n) < \infty$$

is sufficient for absolute convergence of $S(f)$. (Hence $BV \cap \mathrm{Lip}\,\alpha \subset A$ for all $\alpha > 0$.) That this condition is also definitive (and for every uniformly bounded complete orthonormal system!) was proved by Bochkarev [54] ([54a]) in 1973. (It is understandable that his method has become fundamental for necessity type results of this kind.)

The question arises: What is an appropriate condition if f is of generalized bounded variation? Trivially, $\mathrm{Lip}\,1/2 \subset V_2$ and by Hölder inequality $V_2 \subset \Lambda BV$ if $\Sigma\, 1/\lambda_n^2 < \infty$. Hence, the fact that $f \in \Lambda BV$ with $\Sigma\, 1/\lambda_n^2 < \infty$ does not contribute to absolute convergence of the Fourier series of $f \in H^{\omega}$. (Both in [20] and [16] theorems for $HBV \cap H^{\omega} \subset A$ have been stated!)

Śzasz has noted that what is actually needed in Bernstein's theorem is the convergence of the series

$$\Sigma\, n^{-1/2}\, \omega_2(1/n, f).$$

Stechkin [55] gave a criterion in terms of the best approximations to f in L^2-metric and deduced Śzasz's theorem from it by means of the Jackson type relation

$$E_2(k,f) = O[\omega_2(1/k,f)]. \tag{9.1}$$

In [56] McLaughlin has generalized Stechkin's result to:

THEOREM 9.1. Let $\{\psi_k\}$ be an orthonormal system. Assume that $\{m(k)\}$ is an increasing sequence of natural numbers and $0 < \beta \le 2$. If

$$\Sigma\, k^{\delta-\beta/2}[E_2(m(k),f)]^{\beta} < \infty\ , \text{ then } \Sigma\, k^{\delta}|a_{m(k)}|^{\beta} < \infty\ ,$$

where $\{a_k\}$ denotes the sequence of Fourier coefficients of f for $\{\psi_k\}$.

The theorem and the relation (9.1) stress the importance of the estimate for $\omega_2(1/k,f)$. The essential result for our purposes here is due to Chanturiya [57]:

$$\omega_2(1/k,f) \le Cn^{-1/2}\Big(\sum_{k=\phi(n)}^{\phi(n)+n} \nu_f^2(k)/k^2\Big)^{1/2}\ , \tag{9.2}$$

where $f \in W$ and $\phi(n) = \max\,\{m\colon \nu_f(m)/m \ge \omega_f(1/n)\}$.

Now from Theorem 9.1, (9.1) and (9.2) one obtains immediately Chanturiya's theorem announced in [58].

THEOREM 9.2. Let $f \in H^{\omega} \cap V[\nu]$, $\nu(n) = o(n)$. If $0 < \beta \le 2$ and

$$\sum_{n=1}^{\infty} n^{\delta-\beta}\Big(\sum_{k=\phi(n)}^{n+\phi(n)} \nu^2(k)/k^2\Big)^{\beta/2} < \infty\ , \text{ then } \sum_{n=1}^{\infty} n^{\delta}\rho_n^{\beta} < \infty.$$

For $\delta = 0$, $\beta = 1$ this yields the sufficiency part of

THEOREM 9.3. ([57]) For a Fourier series of the class $H^{\omega} \cap V[n^{\alpha}]$, $0 < \alpha < 1/2$, to be absolutely convergent it is necessary and sufficient that

$$\sum_{n=1}^{\infty} \frac{1}{n}\Big[\omega\Big(\frac{1}{n}\Big)\Big]^{\frac{1-2\alpha}{2(1-\alpha)}} < \infty\ .$$

Among a number of corollaries which one can state for particular classes, we point to

COROLLARY 1. ([59]) Cor. 1.) $\mathrm{Lip}\,\alpha \cap V[n^\beta] \subset A$ for $0 < \beta < 1/2$, $\alpha > 0$.

By Theorem 4.4 this is equivalent to $\mathrm{Lip}\,\alpha \cap \{n^\beta\}BV \subset A$ (α,β as above) and further to $\mathrm{Lip}\,\alpha \cap V_p \subset A$ $(p < 2)$ what was an earlier result due to Hirschman ([60], Lemma 2d). Similarly

COROLLARY 2. ([59], Theorem 3) If $f \in V[n^\delta]$, $1/2 \le \delta < 1$, then $\Sigma\, \rho_k^\beta < \infty$ for $\beta > 2/(3-2\delta)$, is equivalent to Golubov's result on V_p in [6] (a special case of Th. 6 there). (Corollary 2 may be obtained from Theorem 9.1 via (9.1) and 10(3).)

Using Theorem 4.2 and 4.4 it is also possible to deduce some informations (in various forms) about the classes $\phi\Lambda BV$.

10. ORDER OF MAGNITUDE OF FOURIER COEFFICIENTS.

A standard argument (see [61], p. 80 f.) leads to the relation $\omega_1(\delta,f) = O(\delta\nu_f([1/\delta]))$ for bounded f. We repeat it. Let $0 < h \le \delta$. Then

$$\int_0^{2\pi} |f(x+h)-f(x)|dx = \frac{1}{[1/h]}\sum_{k=1}^{[1/h]} \int_0^{2\pi} |f(x+kh)-f(x+(k-1)h)|dx \le \frac{2\pi}{[1/h]}\,\nu_f([1/h]+1). \tag{10.1}$$

Since $\nu_f(n)/n$ is nonincreasing, we have

$$\nu_f(n)/n \le \nu_f(n+1)/n \le (1+1/n)\nu_f(n+1)/(n+1) \le (1+1/n)\nu_f(n)/n.$$

Further $[1/h] \ge [1/\delta]$ implies $\nu_f([1/h])/[1/h] \le \nu_f([1/\delta])/[1/\delta]$.

Hence

$$\omega_1(\delta,f) = \sup_{0<h\le\delta} \int_0^{2\pi} |f(x+h)-f(x)|dx = O(\delta\nu_f([1/\delta])).$$

Since the absolute value ρ_n of the n-th Fourier coefficient of f is always dominated by $\omega_1(\pi/n,f)/(\pi/\sqrt{2})$, it follows that the Fourier coefficients of functions of the classes $V[\nu]$ are of the order $\nu(n)/n$ (see also Chanturiya [59]). From this and the results of Section 4, follow all known estimates for Fourier coefficients of functions of generalized bounded variation. In case of ΛBV, e.g., one obtains $\rho_n = O(1/\sum_{i=1}^{n} 1/\lambda_i)$. This was independently proved by Wang [20], Schramm and Waterman [23] and this author [62] in 1982. For $f \in \mathrm{Lip}(\alpha, p)$ $\rho_n(f) = O(1/n^\alpha)$ (Hardy and Littlewood [40]) and hence $\rho_n(f) = O(n^{-1/p})$ if $f \in V_p \subset \mathrm{Lip}(1/p,p)$ (Marcinkiewicz [8]).

Waterman's result on summability (Theorem 8.1) implies ([2], p.78) that $\rho_n(f) = o(n^{\alpha-1})$, if $f \in \{n^\alpha\}_c BV$, $0 < \alpha < 1$. In particular, $\rho_n(f) = o(n^{-1/p})$ $(p > 1)$ if $\int_{m(f)}^{M(f)} N_f^{1/p}(y)dy < \infty$, or if $\Sigma\, \nu_f(n)/n^{2-1/p} < \infty$ or if $f \in V_\phi$ with $\Sigma\, \psi(1/n^{1-1/p}) < \infty$.

Wang [20] has proved $\rho_n(f) = o(1/\sum_{i=1}^{n} 1/\lambda_i)$ for $f \in \Lambda_c BV$, in general. Hence functions of Salem classes and of the class of Garsia and Sawyer have Fourier coefficients of order $o(1/\log n)$. (See Theorem 3.1 and Corollary 3 in Section 5).

REMARK. In case of Fourier coefficients one can use the presence of the factor cos nx (resp. sin nx) in a relation analogous to (10.1) to eliminate π on the right hand side (a modification due to Izumi [63]). The group theoretic nature of the argument was noted already by Vilenkin ([64],3.22) in 1947. Vilenkin's observation escaped notice of Taibleson [65], Edwards ([66], p.35), Benedetto ([67], p. 120).

Another information on the magnitude of Fourier coefficients is contained in the estimate of the rate of approximation of a function by trigonometric polynomials in L^2-norm

$$\left(\sum_{k=n}^{\infty} \rho_k^2\right)^{1/2} = O(\omega_2(1/n,f)). \tag{10.2}$$

(See section 9, relation (9.1).)

Clearly, $n \sum_{k=n} \rho_k^2 = O(1)$ if $f \in \mathrm{Lip}(1/2,2)$. Hence one has the latter estimate for all V_p, $p \le 2$ and then, via Theorem 4.4, for all $\{n^\alpha\}BV$, $\alpha \le 1/2$ (and $V[n^\alpha]$, $\alpha < 1/2$). This is significantly better than Shiba's result in [68]. A simple application of the estimate for $\omega_1(1/n,f)$ and the fact that f is bounded (eventually continuous) does not lead to satisfactory results in (10.2). We have already seen the advantages of Chanturiya's approach in matters of absolute convergence. Sacrificing $\phi(n)$, we illustrate the principle on which his estimate of $\omega_2(1/n,f)$ (see relation (9.2) in Section 9) relies. Let us denote

$d_{k,n}(x) = |f(x+k\pi/r) - f(x+(k-1)\pi/n)|$, $k=1,2,\dots,n$. Then

$$\int_0^{2\pi} |f(x+\pi/n)-f(x)|^2 dx = 1/n \int_0^{2\pi} \left(\sum_{k=1}^{n} d_{k,n}^2(x)\right)dx = O\left(\frac{1}{n} \sum_{k=1}^{n} \nu_f^2(k)/k^2\right),$$

since $d_{k,n}(x) \le \nu_f(k)/k$, if we denumerate $\{d_{k,n}(x)\}_{k=1}^n$, for each particular x, so that these numbers decrease when k increases. Therefore

$$\omega_2\left(\frac{1}{n},f\right) = O\left[\left(\frac{1}{n} \sum_{k=1}^{n} \nu_f^2(k)/k^2\right)^{1/2}\right], \tag{10.3}$$

and, in general,

$$\omega_2(\delta,f) = O\left[\left(\delta \sum_{k=1}^{[1/\delta]} \nu_f^2(k)/k^2\right)^{1/2}\right].$$

((9.3) follows, of course, from (9.2), for $\nu_f(k)/k$ decreases.)

For $\{n^\alpha\}BV$, $\alpha > 1/2$, this gives us, for example, (via Theorem 4.4 again)

$$n^{2-2\alpha} \sum_{k=n}^{\infty} \rho_k^2 = O(1)$$

and this is also better than the corresponding result of Shiba.

11. CHARACTERIZATION OF CONTINUITY.

Of the following five conditions

$$n \sum_{k=1}^{\infty} \rho_k^2 \sin^2 \frac{k\pi}{2n} = o(1) \tag{11.1}$$

$$n \sum_{k=n}^{\infty} \rho_k^2 = o(1) \tag{11.2}$$

$$\sum_{k=1}^{\infty} k^2\rho_k^2 = o(n) \tag{11.3}$$

$$\sum_{k=1}^{n} k\rho_k = o(n) \tag{11.4}$$

$$\sum_{k=1}^{n} \rho_k = o(\log n) \tag{11.5}$$

the first three are equivalent (Tanović-Miller [69]). They imply (11.4) and this one implies further (11.5). By a well known theorem of Lukács ([2], p. 60) (11.5) is sufficient to insure continuity of f if f is supposed merely to be regulated. Wiener [4] has shown that (11.1) and (11.3) are also necessary for continuity of $f \in BV$. Sidon [70] has independently established the same for (11.4), which is, in case BV, trivially equivalent to (11.3), because of $k\rho_k = O(1)$. Hence (11.5) is also necessary in this case, what is known as Lozinski's theorem [71]. Actually, Wiener has proved the necessity of (11.1) if $f \in C \cap V_p$, $1 \leq p < 2$ (see [4], p.77 the bottom, and p.78, relation (21); see also Golubov [72]). This is contained in a more general result

$$\lim_{h\to 0^+} \left(\frac{1}{h}\int_0^{2\pi} |f(x+h)-f(x)|^q dx\right)^{1/q} = \left(\sum_k |f(x_k+0)-f(x_k-0)|^q\right)^{1/q},$$

which holds for $f \in V_p$, $1 \leq p < q < \infty$ (Golubov [7]).

In view of (10.2) condition (11.2) is obviously fulfilled if $f \in Lip(1/2,2)$. In particular, by (9.2), (11.1) - (11.5) are necessary and sufficient conditions for continuity of functions of the class $\{f : \sum_{k=1}^{\infty} \nu_f^2(k)/k^2 < \infty\}$ (Chanturiya [73]). Therefore also for the classes $\phi\Lambda BV$ with $\sum_{n=1}^{\infty} [\phi^{-1}(1/\sum_{i=1}^{n} 1/\lambda_i)]^2 < \infty$, e.g. (See Theorem 4.2)

On the other hand, there is no necessary condition for the continuity of $f \in V_2$ in terms of the absolute value ρ_k of its Fourier coefficients. (The function $f_1(x) = \sum_{k=1}^{\infty} \frac{\sin kx}{k} \in BV \subset V_2$, with a jump at 0, and Hardy-Littlewood function $f_2(x) = \sum_{k=1}^{\infty} \frac{\sin k(x+\log k)}{k} \in Lip\ 1/2 \subset V_2$ have $\rho_k(f_2) = 1/k$.)

Let us now prove

THEOREM 11.1. Each of the conditions (11.1) - (11.5) is necessary and sufficient for the continuity of $f \in \{n^{1/2}\}_c BV$.

PROOF. Sufficiency is clear. For the necessity part it is enough to prove $\sum_{k=1}^{n} |f(x+k\pi/n)-f(x+(k-1)\pi/n)|^2 = o(1)$ uniformly in x, as $n \to \infty$, since

$$4\pi n \sum_{k=1}^{n} \rho_k^2 \sin^2\frac{k\pi}{2n} = \int_0^{2\pi} \left(\sum_{k=1}^{n} |f(x+k\pi/n)-f(x+(k-1)\pi/n)|^2\right)dx$$

and then (11.1) will hold. For every x and n let $\{d_{k,n}(x)\}_{k=1}^{n}$ be as in Section 10. Then $k^{1/2}d_{k,n}(x) = O(1)$ uniformly in x. (See Theorem 4.4, part i) of the proof.) Given $\varepsilon > 0$, there exists n_o (independent of x) such that

$$\sum_{k=n_o+1}^{n} d_{k,n}(x)/k^{1/2} \leq V_{\{k^{1/2}\}}^{(n_o)}(f;[0,4\pi]) < \varepsilon .$$

(See Definition 3.3) Further there exists n_1 such that $\omega_f(1/n) < \varepsilon$ for $n \geq n_1$. Let $n_2 = \max\{n_o,n_1\}$. For $n > n_2$ we have then

$$\sum_{k=1}^{n} d_{k,n}^2(x) = \sum_{k=1}^{n_o} d_{k,n}^2(x) + \sum_{k=n_o+1}^{n} k^{1/2}d_{k,n}(x)\,\frac{d_{k,n}(x)}{\kappa^{1/2}} \leq$$

$$\leq n_o\,\omega_f^2(1/n) + C\sum_{k=n_o+1}^{n} d_{k,n}(x)/k^{1/2} < n_o\varepsilon^2 + C\varepsilon, \text{ uniformly in } x.$$

COROLLARY ([32], Theorem 6) In the class $\{f \in W: \int_{m(f)}^{M(f)} N_f^{1/2}(y)dy < \infty\}$ conditions (11.1) - (11.5) characterize the continuous functions.

Still easier than Theorem 11.1 one proves the theorem due to Cohen [74], saying that such characterization is possible in the class V_ϕ if ϕ satisfies the condition $u^2/\phi(u) = o(1)$ $(u\to 0+)$. In fact, given $\varepsilon > 0$, there exists u_o such that $u^2 < \varepsilon\phi(u)$, $0 < u \le u_o$. For $f \in C \cap V_\phi$ take n_o so that $\omega_f(1/n) \le u_o$ for $n \ge n_o$. Then $\sum_{k=1}^{n} d_{k,n}^2(x) < \varepsilon V_\phi(f)$ for every x and $n \ge n_o$ and the assertion follows immediately.

12. DETERMINATION OF A JUMP.

By the theorem of Lukács the jump of an integrable function at a point x_o of its discontinuity of the first kind may be determinated from its Fourier series using the first logarithmic mean ([2], p. 106, f.) of the sequence $\{nb_n \cos nx_o - na_n \sin nx_o\}$. In case $f \in BV$ the logarithmic mean may be replaced by (C,1) method (essentially Fejér [75]; see Czillag [76]) and indeed by any (C,α) $\alpha > 0$. In some sense we have completed this picture proving

THEOREM 12.1. ([77], Theorems 1 and 2) If $f \in HBV$, then the sequence $\{kb_k(f)\cos kx - ka_k(f)\sin kx\}$ is (C,α) summable to $(f(x+0)-f(x-0))/\pi$ for every x.

If $\Lambda BV \supsetneq HBV$, then there exists a continuous function $f \in \Lambda BV$ such that $\sum_{k=1}^{n} kb_k(f) \neq O(n)$.

THEOREM 12.2. ([78], Theorem 3) If $f \in V_p$, $p > 1$, ($f \in \{n^\beta\}BV$ or $f \in V[n^\beta]$, $0<\beta<1$) then the sequence $\{kb_k \cos kx_o - ka_k \sin kx_o\}$ is (C,α) summable to $(f(x+0)-f(x-0))/\pi$ for every $\alpha > 1-1/p$ $(\alpha > \beta)$ and every x.

(C,1)-limit of $\{kb_k \cos kx_o - ka_k \sin kx_o\}$ is, of course, equal to $\lim_{n\to\infty} \frac{S_n'(x_o)}{n+1}$.

Therefore, once the (C,1) result is established, it turns out that it is possible to take any of expressions:

$$\lim_{n\to\infty} \frac{S_n^{(2r-1)}(x_o)}{n^{2r-1}}, \quad \lim_{n\to\infty} \frac{\tilde{S}_n^{(2r)}(x_o)}{n^{2r}} \qquad r=1,2,\ldots$$

for the determination of the jump at x_o, since the summability methods here involved are implied by (C,1), [77]. ($S_n^{(k)}(x)$ is the k-th derivative of $S_n(x)$. A similar definition applies to other cases.)

An earlier result on the effectivness of (C,1) in the case of V_p classes is due to Golubov [79]. It is a corollary of our Theorem 12.2.

13. GIBB'S PHENOMENON.

A sequence of functions $\{f_n\}$ defined in the neighbourhood of a point x_o and converging at x_o (but not necessarily for $x \neq x_o$) is said to converge uniformly at x_o to limit s if to every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon)$ and a $n_o = n_o(\varepsilon)$ such that

$$|f_n(x)-s| < \varepsilon \quad \text{for} \quad |x-x_o| < \delta \quad \text{and} \quad n > n_o.$$

An equivalent definiton is that $f_n(x_n)\to s$ for each sequence $x_n\to x_o$ (Zygmund [2], p.58). If $f = \lim_{n\to\infty} f_n$ is defined in the neighbourhood of x_o and continuous at x_o, the absence of the Gibb's phenomenon at the point x_o is equivalent to the uniform convergence of $\{f_n\}$ at x_o (ibid., p. 61). We have

THEOREM 13.1. If f is of harmonic bounded variation, $S(f)$ shows Gibb's phenomenon at every point of discontinuity of f and only there.

PROOF. $f(x) = \lim_{n\to\infty} S_n(x,f)$ for every x, by Theorem 6.1. An inspection of Waterman's proof of this fact in [36] shows that if f is continuous at x_o, $S_n(x,f)$ is arbitrarily close to $f(x_o)$, provided n is large enough and x close enough to x_o. (Waterman's idea is repeated in the proof of Theorem 14.1.) Hence $S(f)$ converges uniformly at x_o and Gibb's phenomenon is absent.

Let us suppose now that f has a jump $f(x_o+0)-f(x_o-0)=d$ $(\neq 0)$ at x_o. The function $g(x) = f(x) - \frac{d}{\pi}\sum_{k=1}^{\infty} \frac{\sin k(x-x_o)}{k}$ is continuous at x_o and belongs to HBV. Hence $S(g)$ converges uniformly at x_o. The behaviour of $S_n(x,f)$ near x_o is then the same as the behaviour of the partial sums of the series $\frac{d}{\pi}\sum_{k=1}^{\infty} \frac{\sin k(x-x_o)}{k}$, which is known to show Gibb's phenomenon. Hence so does $S(f)$.

14. CONJUGATE SERIES.

Another among the classical theorems about functions of bounded variation which admit to the whole class HBV is Young's theorem on the convergence of the conjugate series.

THEOREM 14.1. If $f \in$ HBV, a necessary and sufficient condition for the convergence of $\tilde{S}(f)$ at x is the existence of the conjugate function $\tilde{f}$ at x, i.e. of the limit

$$\tilde{f}(x) = -\frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\pi} \frac{f(x+t)-f(x-t)}{2\tan(t/2)}\,dt,$$

which represents then the sum of $\tilde{S}(f)$ at x.

PROOF. Let us denote $\psi_x(t) = f(x+t)-f(x-t)$, $\tilde{f}(x,h) = \int_h^{\pi} \frac{\psi_x(t)}{2\tan(t/2)}\,dt$.

We may assume $\psi_x(+0)=0$, for otherwise both the series and the integral are known to diverge.

$$\tilde{S}_n(x) - \tilde{f}(x,\pi/n) = -\frac{1}{\pi}\int_0^{\pi} \psi_x(t)\,\frac{1-\cos nt}{2\tan(t/2)}\,dt + o(1) + \frac{1}{\pi}\int_{\pi/n}^{\pi} \frac{\psi_x(t)}{2\tan(t/2)}\,dt =$$

$$= \frac{1}{\pi}\int_{\pi/n}^{\pi} \psi_x(t)\,\frac{\cos nt}{2\tan(t/2)}\,dt - \frac{1}{\pi}\int_0^{\pi/n} \psi_x(t)\,\frac{1-\cos nt}{2\tan(t/2)}\,dt + o(1) = I_1 + I_2 + o(1).$$

Given $\varepsilon > 0$, we can chose n such that $|\psi_x(t)| < \varepsilon$ for $|t| \le \pi/n$. Then

$$|I_2| \le \frac{\varepsilon}{\pi}\int_0^{\pi/n} \left|\frac{1-\cos nt}{2\tan(t/2)}\right| dt \le \frac{2\varepsilon}{\pi}\int_0^{\pi/n} \frac{\sin^2 nt/2}{t}\,dt \le \frac{\varepsilon n^2}{2\pi}\int_0^{\pi/n} t\,dt < \varepsilon.$$

Further

$$I_1 = \frac{1}{\pi}\left(\int_{\pi/n}^{\delta} + \int_{\delta}^{\pi}\right) \psi_x(t)\,\frac{\cos nt}{2\tan(t/2)}\,dt \quad\text{and}\quad \int_{\delta}^{\pi}\ldots = o(1) \quad (n\to\infty) \text{ for every}$$

$0 < \delta < \pi$ by Riemann-Lebesgue.

The method of estimating $\int_{\pi/n}^{\delta} \dots$ is essentially that of Waterman's proof of Theorem 6.1 in [36].

$$\int_{\pi/n}^{\delta} \psi_x(t)\,\frac{\cos nt}{2\tan(t/2)}\,dt = \sum_{k=1}^{N} \int_{k\pi/n}^{(k+1)\pi/n} \psi_x(t)\,\frac{\cos nt}{2\tan(t/2)}\,dt + \int_{(N+1)\pi/n}^{\delta} \dots = I_1' + I_1'' ,$$

where $N+1 = [n\delta/\pi]$. Clearly $I_1'' = o(1)$ and removing the last term in I_1' in the same way, if necessary, we may assume N to be even. After an obvious change of variable one obtains

$$I_1' = \int_0^{\pi} \frac{1}{n} \sum_{k=1}^{N} \psi_x\left(\frac{t+k\pi}{n}\right) \frac{(-1)^k \cos nt}{\tan\left(\frac{t+k\pi}{2n}\right)}\,dt .$$

The absolute value of the integrand here is dominated by

$$\frac{1}{2n} \left|\sum_{k=1}^{N-1}{}' \left[\psi_x\left(\frac{t+k\pi}{n}\right)\cot\frac{t+k\pi}{2n} - \psi_x\left(\frac{t+(k+1)\pi}{n}\right)\cot\frac{t+(k+1)\pi}{2n}\right]\right| ,$$

where $\sum'$ indicates summation over odd indices. Let us write the general term of the sum in the form

$$\frac{1}{2n}\left[\psi_x\left(\frac{t+k\pi}{n}\right) - \psi_x\left(\frac{t+(k+1)\pi}{n}\right)\right] \cot\left(\frac{t+k\pi}{2n}\right) + \frac{1}{2n}\,\psi_x\left(\frac{t+(k+1)\pi}{n}\right)\left[\cot\frac{t+k\pi}{2n} - \cot\frac{t+(k+1)\pi}{2n}\right].$$

By the mean value theorem

$$\frac{1}{2n}\left[\cot\frac{t+k\pi}{2n} - \cot\frac{t+(k+1)\pi}{2n}\right] = \frac{1}{\sin^2\xi_k}\cdot\frac{\pi}{4n^2} \le \frac{1}{\frac{4}{\pi^2}\left(\frac{t+k\pi}{2n}\right)^2}\cdot\frac{\pi}{4n^2} \le \frac{\pi}{4k^2} < \frac{1}{k^2} .$$

Choosing N_o such that $\sum_{k=N_o+1}^{\infty} 1/k^2 < \varepsilon$, we have

$$\frac{1}{2n}\sum_{k=1}^{N-1}{}' \left|\psi_x\left(\frac{t+(k+1)\pi}{n}\right)\left[\cot\frac{t+k\pi}{2n} - \cot\frac{t+(k+1)\pi}{2n}\right]\right| \le \sup_{\substack{0\le t\le\pi \\ 1\le k\le N_o}} \left|\psi_x\left(\frac{t+(k+1)\pi}{n}\right)\right| \times \sum_{k=1}^{N_o}{}' 1/k^2 + \varepsilon.$$

The first summand after the inequality sign is clearly $o(1)$ as $n\to\infty$ since ψ is continuous at 0. Further

$$\frac{1}{2n}\left|\sum_{k=1}^{N-1}{}' \left[\psi_x\left(\frac{t+k\pi}{n}\right) - \psi_x\left(\frac{t+(k+1)\pi}{n}\right)\right]\cot\frac{t+k\pi}{2n}\right| \le \sum_{k=1}^{N-1}{}' \left|\psi_x\left(\frac{t+k\pi}{n}\right) - \psi_x\left(\frac{t+(k+1)\pi}{n}\right)\right| \frac{1}{k\pi} \le$$

$$\le \frac{1}{\pi} V_H(\psi_x;[0,\delta]) < \varepsilon,$$

if δ is sufficiently small (Waterman [21]).

Hence $\tilde{S}_n(x) - \tilde{f}(x,\pi/n) = o(1)$. Since $|\tilde{f}(x,h) - \tilde{f}(x,\pi/n)| = O(1/n)$ for $\pi/(n+1) < h < \pi/n$, the proof is complete.

The earlier extension of Young's theorem to the classes V_p, $p>1$, is due to Marcinkiewicz [8].

THEOREM 14.2. If $f \in HBV$ and $\tilde{f} \in W$, then both $S(f)$ and $\tilde{S}(f)$ are uniformly convergent.

PROOF. Already $f, \tilde{f} \in W$ imply that f and $\tilde{f}$ are continuous. Really, if, e.g., $f(x_o+0)-f(x_o-0) > 0$, then, by Lukác's theorem $\tilde{S}_n(x_o,f)\to -\infty$. Hence $\tilde{\sigma}_n(x_o,f) \to -\infty$ what contradicts the fact that $\tilde{\sigma}(x_o,f) = \sigma_n(x_o,\tilde{f}) \to (\tilde{f}(x_o+0)-\tilde{f}(x_o-0))/2$. The proof for $\tilde{f}$ is completely analogous. Now $f \in HBV \cap C$ implies uniform convergence of $S(f)$ by Theorem 6.1. But if $f, \tilde{f} \in C$ and $S(f)$ converges uniformly, then so does $\tilde{S}(f)$ ([61], p. 592).

15. PARSEVAL IDENTITIES.

Zygmund ([2], p. 157) calls two classes of functions, K and K_1, complementary if for any pair $f \in K$, $g \in K_1$ the Parseval formula

$$\frac{1}{\pi}\int_0^{2\pi} f(x)g(x)dx = \frac{1}{2}a_o(f)a_o(g) + \sum_{k=1}^{\infty}\{a_k(f)a_k(g) + b_k(f)b_k(g)\} \tag{15.1}$$

holds, in the sense that the series on the right is summable by some method of summation. If we require that this series converges, as we shall do, examples of complementary classes are (L,BV), (L^p,L^q) for $1/p+1/q=1$, $p > 1$, and $(L\log L,L^{\infty})$([2], pp.159 and 267). Waterman [36] has generalized the first result to

THEOREM 15.1. L and HBV are complementary classes. If $\Lambda BV \supsetneq HBV$, then there exist $f \in L$, $g \in \Lambda BV$ such that the series in (15.1) diverges.

We would like to point out another, less known, type of "Parseval identity" due to Young [9].

THEOREM 15.2. If $f \in V_p$, $g \in V_q$, $1/p+1/q \geq 1$, and f and g have no common points of discontinuity, then the Stieltjes integral $\int fdg$ exists in the Riemann sense and

$$\frac{1}{\pi}\int_0^{2\pi} fdg = \sum_{n=1}^{\infty} n[a_n(f)b_n(g)-a_n(g)b_n(f)]. \tag{15.2}$$

Taking the pair f, $\tilde{f}$ we obtain the interesting

COROLLARY. If $f \in V_p$, $\tilde{f} \in V_q$, $1/p+1/q > 1$, then

$$\sum_{n=1}^{\infty} n\rho_n^2 < \infty . \tag{15.3}$$

It is a classical theorem due to Hardy and Littlewood (see [2], p.286 f.) that the conditions f, $\tilde{f} \in BV$ imply $f \in A$. (Clearly, if $f \in A$ and $k\rho_k = O(1)$, (15.3) is satisfied.)

Generalizations of Theorem 15.2 to the classes V_ϕ have been investigated by Lesniewicz and Orlicz [15].

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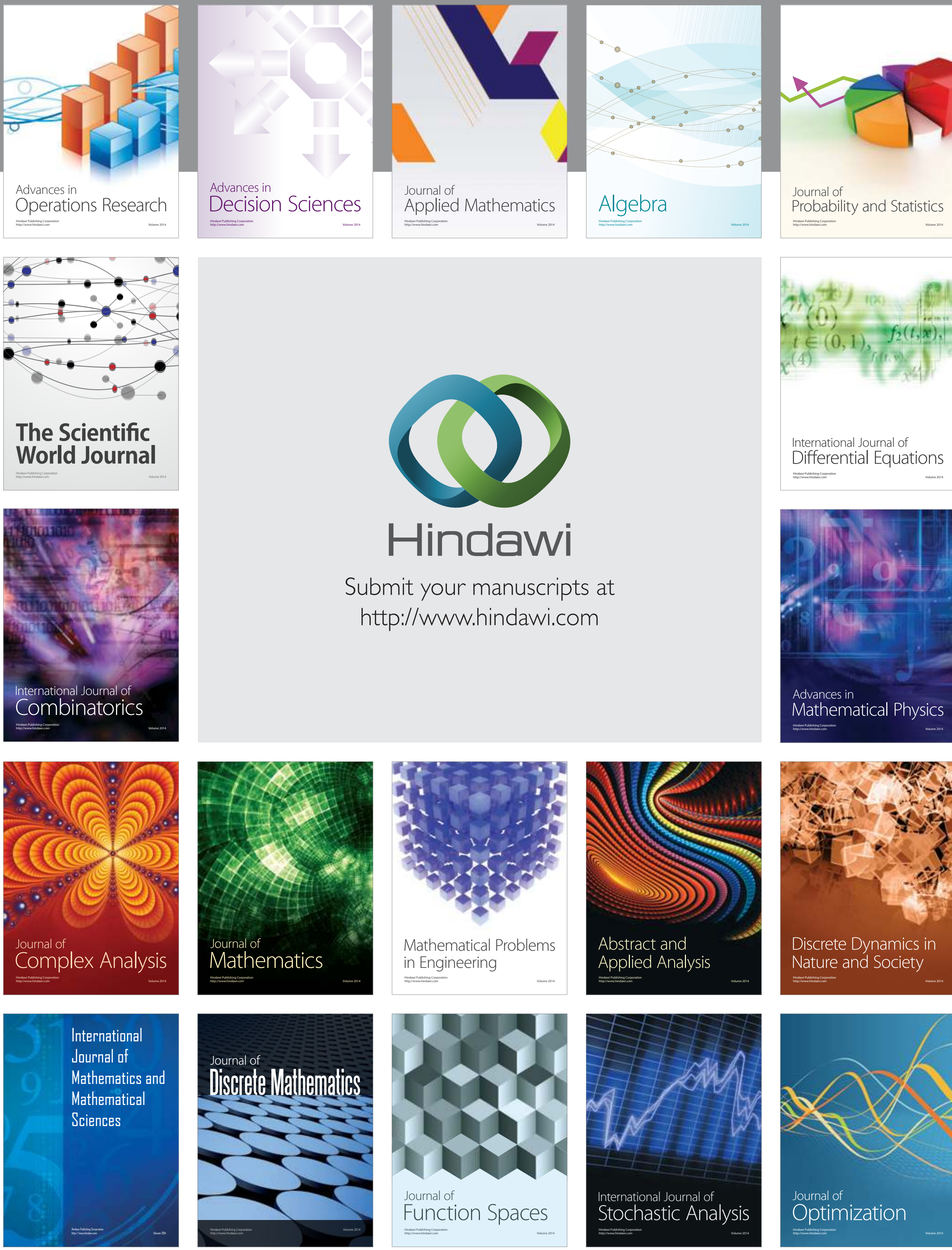
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