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#### Abstract

Considered is a conceptual analog of buggy algorithms and rule-based mathematical development. The investigations consider whether children's offorts to make conceptual sense of new mathematics instruction in terms of their available knowledge may sometimes lead them to make systematic errors. In particular, the possibility is explored that children overgeneralize concepts from a familiar domain of mathematics in order to interpret a new domain. A total of 113 children in the early phases of school instruction on decimal fractions participated: an American sample from grade 5 ( $n=17$ ); an Iaraeli sample from grade $6(n=21)$; and Franch samples from grades $4(n=37)$ and $5(n=38)$. Children were individually interviewed using duseribed tasks. The findings are discussed in some detail. There seem to be fundamental differences in the kinds of conceptual understanding that produce the Whole Number rule and Fraction rule. Different patterns of rule categorization among the three countries indirate thet difforent curriculum sequences produce different patterns of rule invention. Errorful rules appear intrinsic to learning and cannot be avoided in instruction: they can best be regarded as useful diagnostic tools for instructors, to detect the nature of children's understanding of a mathematics topic. (MNS)


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## Conceptual Bases of Arithmetıc Errors: The Case of Decimal Fractions

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# Conceptual Bases of Arithmetic Errors: The Case of Decimal Fractions 

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In the past few years, there have been several demonstrations of the phenomenon of systematic but incorrect rules accounting for errors in performance. In the domain of mathematics, the best developed work is on "buggy algorithms" in subtraction (Brown $\&$ Burton, 1978; Brown \& VanLehn, 1982). Brown and VanLehn's theory of the origin of these algorithms suggests that they come from attempts to repair a procedure by relaxing certain constraints. The repair process operates on a representation of subtraction in terms of symbol manipulation, without reference to quantities and their relationships (Resnick, 1982; Resnick \& Omanson, 1987). The theory thus suggests that children construct erroneous rules without reference to the conceptual content or the meaning of arithmetic. Similar accounts of systematic errors in terms of strictly symbolic interpretations have been given for decimal fractions (Hiebert \& Wearne, 1985) and for elementary algebra (Matz, 1982; Sleeman, 1982). The work of Siegler and Vago (1078) demonstrating that a changing rule structure provides the basis for chlldren's interpretation of railos and proportions is somewhat more conceptual in content than the work on buggy algorithms. However, Slegler's work has typically examined only the content of the rules themselves. It has not attempted to situate the rules in a more general knowledge system or to ask about the origin of the rules.

In the present work we look at what might be termed a conceptual analog of buggy algorithms and rule-based mathematical development. We investigate whether children's cfforts to make concepfual sense of new mathematics instruction in terms of their already available knowledge may sometimes lead them to make systematic errors. In particular, we explore the possibility that children over-generalize concepts from a familiar domain of mathematics in order to interpret a new domain.

Our question requires that we select a topic in mathematics learning in which
children commonly invent incorrect rules in the course of learning, and then seek to specify the conceptual sources of these incorrect rules The topic should be one in which there is some conflict between earlier learned concepts and the new concepts to be acquired, so that already familiar ideas may interfere with learning the new ones. The topic of decimal fractions meets these criteria and also offers an opportunity to examine the possibillty that different ordering of tepics in the school curriculum may differentially affect the kinds of errorful rules that chlidren construct. This is possible because in some countries decimal fractions appear in the curriculum well before ordinary fractions are introduced, while in others ordinary fraction instruction precedes decimal instruction by a year or two.

Using a task in which a child is asked to order decimal fraction numbers, SackurGrisvard and Lennard (1985) found that in fourth and firth grade French classes about half of the children tested used a systematic but incorrect rule to decide which number is greater. There were three different incorrect rules, each used when the numbers to be ordered had the same whole number digit (e.g., 3.214 and 3.8).

According to Sackur-Grisvard and Leonard's Rule 1, the number with more decimal places is the larger one; for example, 3.214 is greater than 3.8 because 3.214 has more digits in the decimal part and because 214 as a whole number is larger than 8. Sackur-Cirisvard and Leonard suggest that classroom instruction may support Rule 1 by glving students practice mainly in comparing decimals with the same number of digits, in which case treating decimals as whole numbers always works. Sackur-Grisvard and Leonard found that Rule 1 was common; it was used by $\mathbf{4 0 \%}$ of their fourth graders and by about $\mathbf{2 5} \mathrm{c}$ of their fifth graders.

Sackur-Grisvard and Leonards Rule 2 specifles that the number with fewer
decimal places is the larger. Thus, given the pair 1.05 and 1.2 , Rule 2 chooses 1.2 as greater. Rule 2 was the least common in their sample, occurring in less than $\mathbf{6 \%}$ of their cases. Grossman (1083), however, has suggested that a similar rule (choosing the longest decimal number as the smallest) was commoniy applied by large numbers of entering U.S. college students.

Sackur-Grisvard and Leonard's Ruie 3 gives a correct judgment when one or more zeroes are immediately to the right of the decimal point in one of the numl.ers, and otherwise chooses as larger the number with the most digits to the right of the decimal point. Thus, given three numbers to order (e.g., 3.214, 3.09, 3.8), Ruie 3 correctly chooses the number with the zero as the smallest, but then uses Rule 1 to order the remaining pair: i.e., 3.09, 3.8, 3.214. Ruie $\mathbf{3}$ was used by about $\mathbf{8 \%}$ of Sackur-Grisvard and Leonard's fourth graders and by $\mathbf{1 4 \%}$ of their $\mathbf{\text { Ifth }}$ graders.

We designed this study to examine the sources of these consistent categories of errors in more detail and more directly than has been done previousiy. Generally, we were interested in the ways in which childien use preexisting knowledge in constructing a mental representation of a new domain of knowledge. Our working hypothesis was that children's errors may derive from their efforts to apply previousiy learned concepts or notational conventions to a new domain. A child who has just been exposed to instruction on decimals must build a representation of decimal numbers and relate decimals to other well or partiaily acquired number systems-notably whole numbers (for all children) and ordinary fractions (for chlidren in countries where ordinary fractions are taught earler than decimal fractions)-as well as to certain conceptions of measurement (cf. Kieren, 1975). Prior knowiedge of whole numbers and fractions can both support and Interfere with construction of a correct concept of decimals. Table 1 lists elements of
knowledge needed for a full decimal fraction concept (column 1) and indicates corresponding elements of whole number knowledge (column 2). For each pair of knowledge elements, column 3 shows whether the elements support or contradict each other.

Table 1 consists of three sectlons. The firsu (A) refers to basic conceptual elements of knowledge about column values. The last iwo (B and C) refer to knowledge of notational and naming conventions that may interact with conceptual knowledge to produce incorrect rules. As can be seen in the table, there are some similarities and some dissimilarities between whole numbers and decimals. The fact that the new decimal concepts are embedded in a structure that shares key features of place value with whole numbers may suggest to children that the new system is identical to the old one and lead them to ignore the differences between the two. This might even be heightened by teachers* attempts to help children use their prior knowledge of whole numbers to facilitate learning the new decimal system.

Table 1 about here

Section A of the table shows that with respect to basic principles of column values the structures of decimals and whole numbers are in accord. Each column is ten times larger in value than the column to its right (A1 and A2). Decimals and whole numbers are also conceptually similar in the way zero functions as a place holder (A3). These conceptual similarities, which are likely to be stressed in instruction, lead to a prediction that shildren first learning about decimal fractions might atteinpt to use their knowledge of whole number values in comparing decimal fractions. This would produce SackurGrisvard and Leonard's Rule 1 (the longer decimal is larger), which we will henceforih
call the Whole Number rule.
Fully integrated knowledge of the place value structure would lead children to recognize that, while the decimal point marks $\boldsymbol{2}$ transition from units to fractions, the basic relationship between columns is not changed. Other features of the decimal system, however, might lead chlidren to interpret the decimal point as a fundamental "breaking point" in the digit string. For example, the structure of column names (Section B of the table) encourages the interpretation of decimal fractions as near mirror Images of whole numbers. Only a difficult-to-hear <th> distinguishes the names (B1), ${ }^{1}$ and if that distinction is ignored it appears that the same sequence of column names moves to the right after the decimal point, to the left before It (B3). Indeed, with respect to column values, the values do decrease as one moves right from the decimal point (A5), while they increase as one moves left from the point. Furthermore, a zero at the extreme right on the fraction side of the point leaves the number's value unchanged, while a zero at the extreme left on the whole number side leaves the value unchanged (A4). A further complication is that nothing corresponds to the unlts column on the fraction side (B2). Failure to learn and apply these differences between whole numbers and decimals could lead to errors such as calling the tenths column a oneths column or not recognizing that decimal numbers are actually symmetrical around the units column rather than around the decimal point. The symmetry around the decimal point and the difficulty of distingulshing the whole number and fractional names can be expected to heighten chlidren's tendency to apply a Whole Number rule. The highilghting of the decimal point

[^1]by symmetry would make them recognize that things change after the point, while the name similarity could lead them to belleve that, for example, tenths are smalier than hundredths, thus reinforcing the whole number Judgments of relative size.

A final difficuity arises in composing a string of digits into a single decimal quantity (Section $\mathbf{C}$ of Table 1). In the whole number domain, one always reads a number as a set of units or ones. For example, the number 2,674 is read as "two thousand six hundred seventy-four," with the understanding tiat the reference is to that many ones. In decimals, there is no constant unit corresponding to ones. This complicates the task of composing a string of digits into a single decimal number. One strategy for thinking about a decimal as a single quantity ts to use the smallest unit of division as the conmon unit (e.g., thousandths in .374) and to name the number in terms of that unit (e.g., 374 thousandths). One problem with this strategy is that the naming process can further confuse a student about Individual column value. For example, saying "three hundred seventy-four thousandths" might suggest to children that there are three hunareds in the number and lead them to think about it as a whole number. And, as we shall see later, naming decimals in this way often makes it harder to compare different length decimals. Decimals can be more powerfully thought about as compositions of tenths, hundredths, and so forth; (e.g., 3 tenths plus 7 hundredths plus 4 thousandths). However, with this conception there is no convenlent, single label for the number. An important element of knowledge needed for understanding decimals as compositions is the equivalences: 1 tenth equals 10 hundredths, which equals 100 thousandths; 1 hundredth equals 10 thousandths. Because of the conflicting naming sequence for whole numbers ( 1 ten does not equal 10 hundreds), chlldren are likely to be confused for some time about such equivalences in the decimal fraction domain. As a
result they will invent rules for comparing values that do not depend upon knowledge of these equivalences.

Table 2 specifies the potential connection points between decimal and ordinary fractions. As shown In Section A, the quantity values, to which both decimal and ordinary fractions refer, correspond completely. However, important differences in the notational systems could cause confusion (Section B). In ordinary fractions, digits show both the size of the parts and the number of parts; but in decimal fractions, digits show only the number of parts, while the size of the parts are implicit in the column values (B1 and B2). We can predict that some children who have learned ordinary fraction notation before decimal fractions might attempt to apply their conceptual knowledge about the relation between size of parts and number of parts (A2) to an incomplete understanding of the referents of decimal notation. For example, if they know that thousandths are snıaller parts than hundredths, and that three digit decimals are read as thousandths while two digit decimals are read as hundredths, they mighi well infer that longer decimals, because they refer to smaller parts, must have lower values. This kind of reasoning would produce Sackur-Grisvard and Leonard's Rule 2 for comparing decimals, which we henceforth call the Fraction rule. We expect use of the Fraction rule to be higher in countries such as Israel and the United States, where fractions are taught before decimals, than in France and other countries where decimals are taught before fractions. In the U.S. and Israel we would expect children who are generally more advanced and whose fraction knowledge is stronger to show more Fraction rule use.

Table 2 about here

Sackur-Grisvard and Leonard's Rule 3 is a special case of the Whole Number rule
for numbers with a zero immedlately following the decimal point. We believe that this special case rule might be generated by children who become aware of the placeholder function of zero but do not have a fully developed place value structure. As a result they apply their knowledge that zero is very small (i.e., "nothing") to conclude that the entire decimal number must be small. For mnemonic ease we will henceforth call this rule the Zero rule. We would expect this rule to appear mostly among chlldren who are advancing beyond the Whole Number rule but are not yet expert. It should be especially frequent among the French children in our study, who, because of the way their curriculum is organized, do not have knowledge about ordinary fractions to draw upon as they advance.

In this study, we began by diagnosing children's rules for comparing decimal fractions. We predicted substantial use of the Whole Number rule in all three countries, because all children are familiar with whole numbers when they begin to learn decimals. The Zero rule was expected to replace the Whole Number rule among some more advanced children, especially in France where little ordinary fraction knowledge was avallable for refining decimal concepts. In Israel and the U.S., we expected heavier use of the Fraction rule, especially among more advanced children and those whose ordinary fraction knowledge is strong. Once children were categorized with respect to rule use, we employed various test and Interview items to estadilsh possible sources of these rules.

## Method

## Subjects

A total of 113 children in the early phases of school instruction on decimals particlpated in the study. The children's grade levels ranged from fourth to stxth grade. Each sample included children of varying ability levels. The American sample was from
the fifth grade in a single schooi (a middle class, parochial school) with $\mathrm{N}=17$. All children in the class were interviewed by authors of this paper. The Israell sample was from grade 6 with $\mathrm{N}=21$. The participating children were nominated by their teachers as "average" math students, and all were interviewed by authors of this paper. The French samples were from grades four with $\mathrm{N}=37$ and five with $\mathrm{N}=38$. French children, nominated as participants by their teachers following a request for average" math students, were drawn from several different schools in the region around Nice. The interviewers were 15 psychology students who earued academic credit for this work but did not participate In the questionnalre design or data analysis. They followed the questionnaire,more strictly than the Isracll and American interviewers, therefore, and did little probing. As will be seen, thls had some effect on the nature of the data from the French samples.

In both Israel and the U.S., ordinary fractions are introduced at least one year earlier than decimals. Interviews were conducted within three months of completion of a decimal fraction instructional unit; thus, all Israell and American children had had prior exposure to both ordinary fraction notation and the meaning of decimal iractions. In France. decimal fractions are taught in CM-1 (equivalent to 4th grade), but ordinary fractions arr not introduced untll CM-2 (equivalent to 5th grade). All French chlldren were interviewed after decimal fraction instruction but before the introduction of ordinary fractions.

## Procedure and Materials

Each child was Interviewed individually in a semi-structured format. Israell and U.S. interviewers engaged in some probing for Justlfication or clarlfication of the child's answers. Each interview was audiotaped, and detalled protocols were obtained from the
tapes and from the ciains written work. Fach interviewer followed a written Interview form which varied slightly from country to country but included the same basic content. We describe the American version of the interview here and mention relevant variations In the Israell and French interviews in the course of reporting data.

Establishing presence of ihe rules. Our primary task for Identifylng the children's underlying rules was comparison of two decimal fractions. Let us consider the universe of such comparisons and their possible answers. Given two decimal fractions, $\mathbf{A}$ and B , of the form

$$
\begin{aligned}
& A=0 . a_{1} \cdot a_{2} \cdots a_{n} \\
& B=0 . b_{1} \cdot b_{2} \ldots b_{k}
\end{aligned}
$$

$A$ and $B$ can differ in two ways. They might have different lengths, i.e., $n>k, n=k$, or $n<k$. Within each column, they also might differ in the actual digit, e.g., $\mathbf{a}_{1}>\mathbf{b}_{\mathbf{1}}$, $a_{1}=b_{1}, a_{1}<b_{1}$. Table 3 summarizes the Inferences about rules that can be made based on children's judgments for prototype cases in which $n=1, k=2$, and there is no zero Immediately after the decimal point. As can be seen, if the child chooses $\mathbf{A}$ in the first case (i.e., when A is in fact larger but shorter), we cannot know with certainty whether he or she was correct or was applying the Fraction rule. Similarly, a choice of B in the second case may imply use of either the correct rule or the Whole Number rule. Thus, only if the child chooses $B$ in the first case or $A$ in the second can a firm attribution of rules be mad ihls provides the basic logic for Interpreting comparison item responses i: our interviews; however, the attribution cannot be mace on the basis of a single item. If a child is a candidate for holding a Fraction rule, she or he will reply A in both cases and will be right on the first item ( $0.8 ; 0.64$ ) but wrong on the second one ( $0.2 ; 0.64$ ). Thus. we attribute a rule to a child on the basis of a series of responses, some of which are correct and some incorrect.

Table 3 about here

When zeroes occupy the first column after the decimal point (i.e., if $\mathbf{a}_{\mathbf{1}}=\mathbf{0}$ or $\mathbf{b}_{\mathbf{1}}$ $=0$ ), the Zero rule may be employed. As Sackur-Grisvard and Leonard have shown, this is a specialized version of the Whole Number rule; it specifies that a number with a zero in the first column is smaller than a number with a non-zero digit in that column. The attribution of the Zero rule to a child is made on the basis of a pattern of answers in which the Whole Number rule fits for all cases except for those having zero in the teniths position.

Table 4 shows the comparison items ised to detect the rules by extracting the patterns of answers. For each pair of fractions, the child was to indicate the larger number. For ease of Interpretation, the table always shows $\mathbf{A}$ as the correct answer; although in the actual interview, the position of the correct answer was randomized. A systematic Whole Number rule child would choose the B fraction in the first two sets of items but answer the third sets correctily. A Zero rule child would answer the first set with a Whole Number pattern (B answers) but would answer the second and the third sets correctly (A answers). A Fraction rule child would give correct (A) answers for the first two sets but would give incorrect (B) answers in the third set.

## Table 4 about here

Two additional tasks were used to confirm our classification of children to rule-use groups. In the Hidden Numbers Comparison task, two numbers were presented with the actual digits covered, but the number of digits and their relation to the decimal point visible. For example, the child was asked, "Which is larger 0 . or $0 . \quad$ ?" In each
case, the correct answer is that one cannot tel! For these items, Whole Number rule children would select the number with more decimal places, and Fraction rule children would select the number with fewer decimal places. The Zero rule could only be detected if children said specifically that they could not tell which number was larger because they did not know whether or not there was a zero immediately after the decimal point.

A Zero Insertion task also served to confirm Whole Number and Fraction rule classifications. In this task, we asied subjects to tell us whether and how the value of 2.35 changed when a zero was inserted in four different places to produce $2.305,02.35$, 2.035, and 2.350 .

Once children's ruies were established, we used a series of other items-along with children's verbalizations on all of the tasks-to try to uncover childrên's con ual framen urks, with particular attention to whether they were thinking of decimals in whole number or ordinary fraction terms. These data were used to infer probable sources of the rules. In one set of items, chlldren were asked the values of digits in different positions in written numbers (e.g., the value of the 5 in 1.54 and 2.45). In a second set. the Interviewer read aloud numbers by column name (e.g., "six tenths and two hundredths") and asked the child to write the numbers in decimal notation. A third set of items had children compare two ordinary fractions. Finally, in a fourth set of items, children "rote numbers in decimal form that they were shown in ordinary fraction notation.

## Results

## Use of the Rules

Table 5 presents response patterns for the basic rule detection items for the U.S., Israell, and French samples. Individual chiidren are listed at the top of each section of the table. and Items (which vary slightly froin sample to sample) are listed vertically at the left. Each child's response to each Item is shown; a blank space indicates no response or an uncodable response. For purposes of discussion the tables show subjects grouped according to their dominant pattern of response. For the most part, children assigned to the three incorrect rule categories gave responses perfectly consistent with the expected rules. A few children who do not appear consistent on the table (e.g., Israell subjects 4, 16. 6, 17; French fourth grade subject 9) verbally explained their choices by directly describing the rule into which they are classified. Furthermore, inconsistent responses usually came on specific items that permitted a different judgment strategy than the standard one used by the child. For example, in comparing 0.5 with 0.36 , some chlldren recognized that 0.5 was a half and used il as a reference point, saying that .36 was less than half. In comparing 0.25 with 0.100 , oue child said the two zeroes at the right of 0.100 do not make a difference; therefore, 25 is greater than 1 . The other exceptions occurred mainly on items where one of the numbers was an ordinary fraction $\because$ : the other a decimal fraction. Errors in conversion between the notation systems probably accounted for these responses.

Table 5 about here

Table 6 shows the distribution of rules in the various sampies. The general pattern of this distribution ror French chlldren is similar to the earller Sackur-Grisvard and Leonard findings. As expected, there was a low incidence of the Fraction rule overall
and a shift from the Whole Number rule toward the Zero rule between 4th and 5th grades. Also as expected, the Israell and U.S. children show a higher incidence of the Fraction rule, with the Israeli children, who were sixth graders, showing an especially high use of the rule. The absence of Zero rule cases among U.S. children should be noted, although we are not able to offer an explanation.

Table 6 about here

Responses on the Hidden Number Comparison task can be used to confirm the classificati of children. The relevant data for the comblned Israell and U.S. samples appear in Table 7, which shows frequencies of children in each rule use category who gave each answer. ${ }^{2}$ Clearly, responses are largely consistent with the child's rule classification. As stated earlier, Whole Number rule children were expected to choose the longer string as the larger value, and Fraction rule children were expected to choose the shorter string. The "I don't know" answer is correct, but the same answer might be given by children who did not have any idea how to respond. We, therefore, do not consider the "I don't know" responses diagnostic of rule use.

Table 7 about here

An additional item that confirmed our classification of children according to the rules was the Zero Insertion task, which called for comparing the number 2.35 with 2.305, 2.035, 2.350, and 02.35. Virtually all children in all rule categories knew that inserting a zero before the 2 or after the 5 did not change the value. However, comparisons of 2.35 with 2.305 and 2.035 discriminated among the rule use categorles.

[^2]Table 8 shows data on these items for the U.S. and Israell samples combined. ${ }^{3}$ As can be seen, Whole Number rule children almost alwayr chose the longer numbers (i.e., 2.305, 2.035) as the larger. This is the expected Whole Number response. A few Whole Number children judged 2.035 to be equal to 2.35 . This is consistent with a Whole Number judgment based on the integers value rather than on the number of digits (i.e., $035=35$ ). All but one Fraction rule child chose 2.35 as having the larger value, as would be expected. Since these are also correct answers, however, this pattern can be considered only weak support for the Fraction rule classification.

Table 8 about here

## Conceptual Bases Underlying the Rules

We turn now to the cognitive sources of the errorful rules. We have two kinds of data to draw on in making our inferences: the children's verbalizations as they worked on the comparison tasks, and the patterns of answers for the items that more directiy examined place value and fractional knowledge. We will consider these for the Whole Number and the Fraction rules in succession.

Whole Number Rule. We had hypothesized that the Whole Number rule results from children's attempts to apply their knowledge about whole numbers to the new kind of numbers they are learning without integrating information aboitt the fractional values. This was confirmed by typical verbalizations of the Whole Number childrer as they responded to comparison items. For example, here are explanations by two Israell children:

S 4: $0.5<0.25$, "because 25 is bigger."
$4.7<4.08$, "because the zero does not matter and 8 is bigger than $7 . "$

[^3]S 16: $4.8<4.63$, "since 63 is bigger than 8."
When comparing 2.35 with 2.305 and 2.035 , Whole Number children often referred to a number's decimal portion as a whole number, saying that "three hundred and five" or "three hundred fifty" was bigger than "thirty five."

Whole Number children also showed confusion about zero's placeholder function. One child, for example, equated 2.35 and 2.035 , because the zero is "just a place marker and that would still be thirty five." This generally fragile understanding of the place value system is further confirmed by items assessing explicit place value knowledge. Most Whole Number and Zero rule children could not correctly give the value of the 5 s in 1.54 and 2.45 as tenths and hundredths respectively. Some examples from the protocols give a flavor of the difficulty Whole Number children had in labeling the columns in decimal numbers. Four of the U.S. children said that the 5 in 2.45 was 5 ones, suggesting that they were thinking of .45 as the whole number 45 . (Some subsequently changed their answers and were given credit for a correct response on the table.) Subject 13 sald that the 5 In 2.45 was 5 tens, but then changed her mind, saying the 5 in 1.54 stood for 5 tens. She did not identify the 5 in 2.45 again, but may have been thinking it was in the ones column, because in her explanation of why 5 tenths was greater than 5 hundredths she sald, "You add 10 more from the ones row to get to the second, to get to the tens row." Responses of this kind dominated among the Whole Number and Zero rule children. The Fraction rule children and experts were mostly able to answer these questlons correctly.

Other items examined children's ability to write decimals from an oral reading. Whole Number and Zero rule children had difficulty in writing ${ }^{6} 6$ tenths and 2 hundredths" or " 3 ones and 6 hundredths" correctly, while Fractions rule children, like
the experts, mostly gave correct responses. Whole Number children reversed the order of the digits 6 (tenths) and 2 (hundredths), writing 0.026 or 2.6 , which would be the correct ordering if the number were 6 tens and 2 hundreds. Other errors, too, suggest that Whole Number children were thinking of decimals as whole numbers. For example, for "3 hundredths," one subject wrote .300 . Another subject, when she heard the Intervlewer say " 6 tenths and 2 hundredths," asked, "You mean 60 and 2 hundredths?" After the interviewer repeated the number, she wrote 2.60 . Interpreting $\mathbf{6}$ tenths as $\mathbf{6 0}$ suggests that she was thinking of 6 teus rather than 6 tenths.

Fraction Rule. We hypothesized that the Fraction rule results from chlldren's efforts to integrate knowledge about fractlonal parts and ordinary fraction notation with their place value knowledge. In partlcular we expected them to know that if a number is dlvided Into more parts, the parts are smaller. We expected, however, that Fraction rule chlldren might have some difficulty figuring out whether the digits stated explicitly in the decimal form correspond to the numerator or the denominator of an ordinary fraction (1.e., to the number of parts or to the size of the parts).

Several aspects of the data confirm these expectations. First, a few Fraction rule children directly stated their bases for comparing decimals. Here is a typical explanation from an Israell subject:

S 20: $4.7>4.08$, "since this has 8 hundredths and this has 7 tenths."
$7.457>4.4502$, "because this is hundredths and this is tenthousandths."

Intervlewer: "So what?"
S 20: "Hundredths are bigger than ten thousandths."
To further understand how fractlon knowledge might affect decimal understanding, we searched the protocols of Fraction rule subjects for overt evidence that they were applying their fraction knowledge to decimals. We found two such Instances. Both were
cases where fraction reasoning produced correct judgments. U.S. Subject 5, comparing 4/100 and 0.038 , said that $4 / 100$ was bigger because, "One hundred is smaller than one thousand in the decimals. Smaller numbers make the larger pieces." U.S. subject 11 's explanation, although less clear, also seemed to use fraction knowledge to guide correct decimal comparisons. When ordering ten numbers, she sald that 1.4 is larger than 1.067 , "because 1 and 4 tenths is more of a part than 1 and 67 thousandths . . . . Four tenths is like more of a part . . . . It's like 5 tenths is a part, so you almost have a part. But on this [1.067] you don't even have a part."

In addition, Fraction rule children showed rather good knowledge of place value, as already discussed. Some of their explanations on the zero insertion task ( $\mathbf{2} \mathbf{. 3 5}$ compared with 2.035 and 2.305) confirmed this. Of the Fraction rule children, U.S. subject 5 was the most vocal. She sald that 2.35 is greater than 2.305 , "because the 0 would take hundredths place, and 5 would get pushed down to thousandths." This child gave a similar justification for choosing 2.35 as greater than 2.035 , saying, "Zero takes the tens place, and 3 and 5 get pushed down. More generally she explained that zero makes a difference in the middle of a number but not at the beginning or at the end, where it is "Just a placeholder." Subject 5 thus limited the placeholder language to the beginning and end of the numbers, but applied the placeholder concept to all positions. U.S. subject 7 gave no justifications for his individual comparisons but did give a chlldilke version of the full placeholder theory. He sald that zero matters in between other numbers because it shows that, "Nothing is home in the middle space." (His teacher had used a similar formulation.)

Table 9 shows the frequency for U.S. and Israell children of responses on items comparing $1 / 333$ with $1 / 334$ and $1 / 3$ with $1 / 4$. These items test children's knowledge of
a feature of ordinary fraction notation that is frequently confusing to beginners: as the digit in the denominator grows, the size of the unit fraction shrinks. As can be seen, Whole Number rule children had great difficulty while Fraction rule children succeeded to a larger extent (although not perfectly). In particular, Whole Number chlldren judged as greater fractions in which the number in the denominator is larger. These items were not scored for the French sample because the children had not yet encountered ordinary fractions in their school instruction. Many French children simply refused to respond; others treated the slash in the fraction as equivalent to a decimal point (e.g., 3/10 $=$ 3.10). The French children did know, however, certaln common benchmark fractions and their decimal equivalents (e.g., $5=1 / 2$ ).

Table 9 about here

Table 10 shows results on the items requiring children to translate common fractions with denominators other than 10 or 100 (i.e., 3/4, 1/5, 3/2, 2/3) into decimals. ${ }^{4}$ None of the children could do this reliably, but their attempts to do so revealed their Ideas about how numerators and denominators of regular fractions correspond to decimal notation. There were three maln categories of incorrect translation:

1. Encode only the numerator in the decimal, and Ignore the denominator. For example, $3 / 4$ becomes .3 or $.003,3 / 100$ becomes .3 . This translation reveals knowledge of the correspondence between the written digits in ordinary and decimal fractlons (clement B1 in Table 2). It falls, however, to find a correspondence between the notation of the denominator in ordinary fractions and the place value system of decimal fractlons (element B2 In Table 2).

[^4]2. Encode only the denominator in the decimal, and ignore the numerato:. For example, $1 / 3$ becomes $.3,3 / 4$ becomes .04 or .4. This translation recognizes that oniy one part of the fraction actually is shown in the decimal notation, but it mistakes the part. It is a less advanced response than the zumeratoroniy encoding.
3. Keep both the numerator and the denominator of the fraction, and put a decimai in someplace. For example, $3 / 4$ becomes 3.4 or .34. These are "syntactic" translations. They produce a number that has the surface structure of a decimal but has no sensible relation to the quantity expressed in the fraction.

As shown in the tabie, Fraction rule subjects made entirely numerator-only errorsones in which they encoded the fraction's numerator in the decimal notation. Only one Whoie Number rule subject, arer made this type of error; instead, Whole Number rule subjects made mostly syntactic errors or encoded only the denominator.

## Table 10 about here

## Discussion

The study clearly replicates the earlier findings of Sackur-Grisvard and Leonard concerning the rules for comparing decimal fractions but goes beyond their study in two respects. First, it trics to understand the rationaie behind the rules in terms of the children's entire conceptual framework. Second, it tries to find the relative place of these rules in decimal knowledge development. The Whole Number rule occurs frequen!ly and carly in learning. The Zero rule appears as a variant mostly among the older French children and the sixth grade Israelis. The Fraction rule occurs mostiy in the Israell and the U.S. samples. It is strongest among the Israelis, who were oider than the U.S. children. This trend toward an increase in the Fraction rule as children progress in
learning has been replicated subsequent to cur study by Bllha Zuker from Israel, who in her M.A. thesis looked for developmental trends for these rule; (Zuker, 1985). Zuker tested 74 seventh graders, 100 eighth graders. and 60 ninth graders. She found that the Whole Number rule decilned from $\mathbf{1 8 \%}$ in the $\mathbf{7 t h}$ grade to $\mathbf{5 \%}$ in the $\boldsymbol{9 t h}$ grade, while the Fraction rule was more persistent and appeared in $\mathbf{2 3 \%}$ of the sample in $\mathbf{7 t h}$ and 9 th grades.

The present data alsc demonstrate substantial internal consistency in rule application. Overall, $88 \%$ of chlldren could be classified as consistently using one of the three rules under study. In this respect, each individual's "buggy ${ }^{\text {" }}$ performances in this task are much more systematle than they are in domains such as algebra, where an Indlvidual typlcally alternates between two or more incorrect transformation rules, and perhaps a correct one as well, within a single problem solving session (Carry, Lewis, \& Bernard, 1980; Greeno et al., 1985). There may be even more consistency in the decimal domaln than In buggy subtraction, where recent work has demonstrated considerable "mlgration* of bugs, leadlng to a revision of the view that buggy performances are normally very stable (VanLehn, 1982).

There seem to be fundamental differences in the kinds of conceptual understanding that produce the Whole Number rule and the Fraction rule. Whole Number rule chil:Iren appear to have a very Impoverished representation of decimal numbers. Apparently their representation of the place value system does not contaln the integration of the crucial Information of column values, column names, and the role of zero as a placeholder. In terms of the knowledge elements shown In Table 1, these children do not seem to think about the column values of the number's decimal portion at all; rather. they simply "Import" from whole numbers a comparison rule that may not

Itself even require that all of the elements shown in section A for whole numbers be represented fully. Their responses to column value comparisons and decimal writing tasks clearly reveal their weak or nonexistent coding of column values and their borrowing of whole number column names to apply to decimal fractions (Section B of Table 1). They also quite clearly do not have a complete representation of how zero works as a placeholder (element A3). Even more important, Whole Number rule chlldren show no signs of recognizing that the number's decimal portion represents a fractional part of a whole, and so none of the knowledge elements in Table 2 is invoked. Their fallure to represent the decimal portion of the number as a fraction means that nothing in these children's representations constrains their use of the Whole Number rule to compare decimal fractions.

Fraction rule children are most sharply distingulshed from Whole Number rule children by their use-albelt incompletely coordinated-of elements of fractional knowledge. In particular, Fraction rule children appear to know and apply the principle (congruent for decimal and ordinary fractions-scr: element A2 of Table 2) that the more parts 2 . whole is divided into, the smaller each part is. They also know that the number of places in a decimal fraction tells the size of the parts (element B1 of Table 2)specifically, that if there is one place after the decimal, the parts are tenths; if there are two places, the parts are hundredths; if there are three places, the parts are thousandths. Ifowever. these children could not coordinate information about the size of parts (element B1) with information about the number of parts (element B2). Thus, when attending in size of parts (specified by the number of columns), they ignored the number of parts (specified by the digits).

To achieve the necessary coordination requires a rather sophisticated
representation of place value notation. The glowal understanding of place value that Fraction rule children seem to apply specifies only that one place signals tenths, two places hundredths, and so on. However, with this representation of place value alone, it Is not possible to compare decimals that have different numbers of places. To deal witt this problem, the representation of place value would have to change to a form that reflects the additive structure of decimal numbers. Tnis representation specifies that if there are two places after the decimal point, the parts are tenths plus hundredths; if there are three places, the parts are tenths plus hundredths plus thousandths. Furthermore it specifles that the digit in each place tells how many parts of a given size are to be included in the addition. Such a representation would allow children to make column by column comparisons of values, thus producing correct judgments even for pairs with different numbers of digits.

We see from the different patterns of rule categorization among the three countries that different curriculum sequences produce different patterns of rule invention. This confirms our working hypothesis that errors derive from students' attempts to integrate new material that they are taught with already established knowledge. The fact that the French children by and large avoided the errors associated with the Fraction rule and instead seemed to pass directly to use of the correct decimal comparison rule might seem to suggest a superiority of the French curriculum sequence (which is shared by several other countries) in which decimal iraction instruction precedes ordinary fraction Instruction by a substantial period of time. Howc:er, it is important to note that we have no evidence that any of the children classified as experts in fact understood the conceptual basis for decimal comparisons. French and other chlldren who gave consistently correct answers may very well have arrived at their correct rules on the
basis of purely surface and syntactic considerations.
A commonly taught procedure for comparing decimals illustrates how this might happen. Children are often taught an essentially syntactic rule for comparing decimals in which they first add zeroes to the shorter number in order to give both numbers the same number of digits; they can then compare the decimals as if they were whole numbers. This is a relatively simple rule to learn and gives reliably correct answers. Children applying it would have been classified as experts in. our study. Yet they would have been applying the Whole Number rule, and their concepth.l understanding could have remained at the level we have attributed to Whole Number rule children. Such syntactic teaching would serve to suppress errors in performance without improving chlldren's conceptual understanding.

This brings us to a more general cunsideration of the status of systematic errors in mathematics learning and teaching. From a cognitive point of view, almost all instruction is incomplete in the sense that it is not possible in any single demonstration or explanation to cover all special cases or all possible implications of principles or rules that may be presented. Instruction, like all normal human communication, proceeds on the assumption that learners will use the presented material to make inferences and interretations that complete and make sense of what the teacher or text has said. In making these inferences and interpretations, children are very likely to make at least temporary errors. Errcri 1 rules, on this view, are intrinsic to all learning-at least as a temporary phe.sinenou-because they are a natural resuit of children's efforts to interpret what they are told and to go beyond the cases actually presented. Several analyses (e g., Resnick, 1987; VanLehn, 1986) have shown that these errorful rules are intelligent constructions based on what is more often incomplete than incorrect
knowledge. Errorful rules, then, cannot be avoided in instruction. In fact, they are best regarded as useful diagnostic tools for insiructors, who can often use children's systematic errors to detect tle nature of chlldren's understanding of a mathematics topic. Mathematics education researchers can support this instructional function by discovering and documenting common errors and the conceptual understaucilig that underlies them.

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Table 1
Comparison of Decimal Fraction and amole Number Knoyledge

| Elements of Deciaal | Corresponding Elements of | Siallar $(+)$ or |
| :--- | :--- | :--- |
| Fraction Knouledge | Whole Number Knowledge | Different $(-)$ |

A. Column Values:

1. Values decrease as move left to right
2. Each colunn is 10 tines greater than column to right
3. Zero serves as a place holder
4. Zero added to rightmost coluan does not change total value
5. Values decrease as nove ayay from decimal point
B. Column Names:
6. End in <ths>
7. Start yith tenths
8. Naming sequence (tenths, hundredths...) moves left to right
9. Reading sequence is tenths, huniredths, thousandths
C. Reading Rules:
10. The units must be explicitly specified and they vary
A. Column Values:
11. Values decrease as move left to right
12. Each colunn is 10 tines greater than coluan iw right +
13. Zero serves as a place holder +
14. Zero added to leftmost colunn does not change total value
15. Values increase as move ayay fron decinal point
B. Column Names:
16. End in <s>
17. Start vith units
18. Naning sequence (tens, hundreds...) moves right to left
19. Reading sequence is thousands, hundreds, tens, ones
C. Reading Rules:
20. The ones implicitly serve as the units in all cases

Table 2
Comparison of Decimal Fraction and Ordinary Fraction Knowledge

| Elements of Decimal | Corresponding Elements of | Similar $(+)$ or |
| :--- | :--- | :--- |
| Fraction Knovledge | Ordinary Fraction Knowledge | Different $(-)$ |

A. Fraction Values:

1 Expresses a value betveen 0 and 1
2. The more parts a whole is divided into, the smaller is each part
3. Therc are infinite decimal fractions between 0 and 1
B. Fraction Notation:

1. The number of parts a unit is divided into is given implicitly by the column position
2. The number of parts included in the fractional quantity are the only visible numerals
3. The whole is divided only Into powers of 10 parts
4. The ending <th> ("tenth") is typical for a fractional part
A. Fraction Values:
5. Expresses a value betveen 0 and 1
6. The more parts a whole is divided into, the smaller is each part
7. There are infinite ordinary fractions between 0 and 1
B. Fraction Notation:
8. The number of parts a unit is divided into is given explicitly by the denominator
9. The number of parts included in the fractional quantity is the numerator of the fraction
10. The whole is divided into any number of parts
11. The ending <th> ('fourth") is typical for a fractional part

Table 3
Relations Among Decimal Paırs and the Underlying Rules for Comparing Then


Table 4
Comparison Items That Detect Rules:

Question: For each pair, circle the numer that is bigger.

| Iten on Questionnaire | Number Pair* |  | Answer by Subject Grnup |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 1 | B Wh | Whole Number Rule | Zero Rule | Fraction Rule | Experts |
| Whole Number Rule: | 4.8 | 4.63 | $B$ | B | 1 | 1 |
|  | 0.5 | 0.36 | B | B | 1 | 1 |
|  | 0.25 | 0.100 | B | B | $A$ | 1 |
| Zero Rule: | 4.7 | 4.08 | B | A | A | $A$ |
|  | 2.621 | 2.0687986 | 6 B | 1 | $A$ | 1 |
|  | 4/100 | 0.038 | B | 1 | 1 | 1 |
| Fraction Rule: | 4.4502 | 4.45 | 1 | B | 1 | 1 |
|  | 0.457 | 4/10 | 1 | B | 1 | 1 |

[^5]Table 5
Items Detecting Underiying Rules.
A. U.S. Sample

| Student Responses (Grouped According to Rule Used) |  |  |  |  |  |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Item |  | Whole Number |  |  |  |  |  | Fraction |  |  | Experts |  |  |
| 1 | B | 12 | 13 | 14 | 15 | 16 | 17 | 5 | 7 | 11 | 1 | 2 | 3 |
| 4.8 | 4.63 | B | B | B | B | B | B | $A$ | 1 | 1 | 1 | $A$ | A |
| 0.5 | 0.36 | B | B | $B$ | B | B | B | $A$ | 1 | A | 1 | A | 1 |
| 0.25 | 0.100 | B | B | B | B | B | B | A | 1 | A | 1 | 1 | A |
| 13/100 | 0.125 | B | $A$ | B | B | B | B | 1 |  | B | 1 | A | B |
| 4.7 | 4.08 | B | B | B | B | B | 6 | $A$ | 1 | 1 | 1 | $A$ | 1 |
| 2.621 | 2.068796 | B | B | B | B | B | 8 | 1 | $A$ | 1 | $A$ | A | A |
| 4/100 0 | 0.038 | 1 | 1 | B | B | B | B |  | 1 | B | 1 | B | B |
| 4.4502 | 4.45 | 1 | $A$ | 1 | 1 | 1 | 1 | B | B | B | $A$ | $A$ | 1 |
| 0.457 | 4/10 | $n$ | 1 | $A$ | $A$ | 1 | 1 | B | B | B | $A$ | $A$ | 1 |

5 children could not be classified

Table 5, continued
Items Detecting Underiying Rules
B. Israeli Sample

| Student Responses (Grouped According to Rute Used) |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Iten |  | Whole Number |  |  |  | Zers |  |  | Fraction |  |  |  |  |  |  | Experts |
| $A$ | B |  | 10 | 12 |  |  | 172 |  | 3 | 5 | 8 | 11 | 13 | 18 |  | 121519 |
| 4.8 | 4.63 | B | B | B | B | B | B | B | 1 | $A$ | B | 1 | 1 | $A$ | 1 | A A A |
| 0.5 | 0.36 | $B$ | B | B | 1 | 1 | 1 | B | 1 | $A$ | B | 1 | 1 | 1 | 1 | A A A |
| 0.25 | 0.100 | $B$ | B | B | B | B | 1 | B | 1 | A | A | 1 | B | $A$ | 1 | A A A |
| 13/100 | 0.125 | $B$ | B | B | B | B | A | B | 1 | B | 1 | 1 | 1 | A | $B$ | A A A |
| 7/10 | 0.678 | 1 | B | B | 1 | 1 | 1 | B | 1 | 1 | A | 1 | 1 | 1 | 1 | A A A |
| 4.7 | 4.08 | 8 | B | B | B | 1 | 1 | 1 | 1 | 1 | B | 1 | 1 | ${ }^{\wedge}$ | 1 | A A A |
| 2.621 | 2.0687986 | 6 B | B | B | B | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | $\wedge$ | 1 | A A A |
| 4/100 0 | 0.038 | $A$ | B | 1 | $A$ | B | $A$ | B | 1 | B | $A$ | 1 | $\wedge$ | $A$ | B | A A A |
| 4.4502 | 4.45 | $A$ | 1 | 1 | 1 | 1 |  | $A$ | B | B | B | 1 | $B$ | B | B | A A A |
| 0.457 | 4/10 |  | $A$ | 1 |  | 1 |  | $A$ | $B$ | B | B | 8 | 8 | B | B | A A A |

3 children could not be classified

Table 5. continued
Iteas Detecting Halrules

C French Fourth Grade Sample

> Student Responses (Grouped According to Rule Used';

| Iten | Student Responses (Grouped According lo Rule Used) |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Whole Number |  |  |  |  |  |  |  |  |  |  |  |  |  | Zero |  |  |  | Fraction |  |  | Experts |  |  |  |  |  |  |  |  |  |  |
| $1 \quad 8$ | 1 | 7 | 8101 | 111 | 13 |  |  |  |  |  |  | 35 |  |  |  | 15 | 26 |  |  | 3 |  | 3 | 41 | 121 |  | 19 |  |  | 24 |  |  |  |
| $48 \quad 463$ | $\theta$ | 8 | 8 B | B | $\theta$ |  | 8 | B | 8 | $B$ | B | B | B | $B$ | $B$ | 8 | 8 | B |  | 1 | 1 | A | 1 | 1 | 1 | 1 | 1 | 1 | A | 1 | 1 | A |
| 05036 | $B$ | B | B B | B | B | B | 8 | 8 | 8 | 8 | 8 | B | 8 | 8 | 8 | B | 8 | 8 | 1 | 8 | 1 | 1 | 1 | 1 | 1 | A | A | 1 | 1 | 1 | 1 | $A$ |
| 0250100 | 8 | 8 | D B | B | B | 0 | 8 | 8 | 8 | 8 | 8 | B | B | B | B | B | 8 |  |  | B |  | 1 | A |  | $A$ | A | 1 | 1 | 1 | 1 | 1 |  |
| 47108 | B | 8 | 8 B | $\theta$ | 8 | 1 | 8 | B | B | B | 8 | 8 | 8 | B | A | 1 | 1 | 1 | 1 | 1 | 1 | 1 | B | 1 | A | A | A | 1 | A | A | 1 | 1 |
| 262120687986 | B | B | B B | B | 8 | 8 | $\theta$ | 8 | B | B | 8 | B | B | B | 1 |  |  |  |  | 1 | A | 1 | $A$ |  | 1 | 1 | 1 | A | 1 | $A$ | 1 |  |
| 44502445 | 1 | 1 | 1 A | 1 | 1 | 1 | 1 | A | 1 | 1 | 1 | 1 | 1 | A | A | $A$ | 1 | 1 | B | 8 | 8 | 1 | $A \cdot$ | A | 1 | 1 |  | A | 1 | 1 | 1 | A |

[^6]Table 5. continued
Items Detecting Underlyifg Rules
D French Fifth Grade Sample

| Student Responses (Grouped According to Rule Used) |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Iten | Whole Number |  |  |  |  |  |  | Zero |  |  |  |  |  |  |  |  | Fraction | Experts |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| $1 \quad$ B | 2 | 6 |  | 12 | 131 | 143 | 36 | 1 | 4 | 81 | 101 | 182 |  | 263 | 313 |  | 32 | 3 | 5 | 91 | 1115 |  |  | 1920 |  | 2324 |  |  | 2829 |  |  |  |  |
| 48163 | B | $B$ | B | B | B | B | B | B | B | $B$ | 8 | 8 | 8 | B | B | A | 1 | 1 | 1 | 1 | 11 | 1 | 1 | 11 | 1 | 11 | 1 | 1 | 1 A | 1 | 1 |  | $A$ |
| 05036 | B | B | $B$ | B | B | B | B | B | B | 8 | $B$ | 8 | 8 | B | B | B | $A$ | 1 | A | 1 | $A 1$ | A | A | $A 1$ | A | 1 A | 1 | 1 | $A$ A | 1 | A |  | 1 |
| 025.0 .100 | B | B | B | 1 | B | B | 1 | 1 | B | B | B | 1 | B | B | B | B | 1 | A | A |  | 11 | 1 |  | $A$ A | A | 11 | 1 | 1 | 1 A | 1 | 1 | 1 | 1 |
| 17108 | B | 8 | 8 | 8 | B | B | B | 1 | 1 | 1 | $\lambda$ | $A$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 11 | 1 | $A$ | $A 1$ | 8 | 1 A | 1 | 1 | 1 A | 1 | 1 | A | 1 |
| 262120687986 | B | B | B | B | B | B | B | B | 1 | 8 | $A$ | $B$ | 1 | 8 | 1 | 1 | 1 | 1 | 1 |  | 11 | 1 |  | $A 1$ | 1 | 1 A | 1 | 1 | $A 1$ | 1 | 8 |  |  |
| 44502445 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | $A$ | 1 | 1 | 1 | 1 | 1 | A |  | 1 | 1 | 1 | 11 |  | 1 | 1 A | 1 | 11 | 1 | 1 | 1 A | 1 | 1 | $\lambda$ | 1 |

One child could not be classified.

## Table 6

Proportion of Underiying Rules Used in the 3 Countries

| Sample | Number of Studerits | Rule Use Category |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | Whole <br> Number | Zero | Fraction | Experts | $\begin{gathered} \text { Not } \\ \text { Classified } \end{gathered}$ |
| United States | 17 | . 35 | . 00 | . 18 | . 18 | 29 |
| Israel | 21 | . 19 | . 14 | . 33 | . 19 | . 14 |
| France. |  |  |  |  |  |  |
| 4th graders | 37 | . 41 | . 11 | . 08 | 30 | . 11 |
| 5th graders | 38 | 18 | 24 | . 03 | 53 | . 03 |

Table 7
Hidden Numbers Comparison Task Number of Children in Each Rule Use Category Giving Each Type of Ansver (Combined U.S. annd Israeli Samples)

|  | Rule Use Category |  |  |
| :--- | :---: | :---: | :---: |
| Answer |  | Fraction | Experts |
| Choosing the long string | 10 | 0 | 0 |
| Choosing the short string | 1 | 18 | 0 |
| I don't knov | 13 | 2 | 14 |

Table 8
Zero Insertion Task Number of Children in Each Rule Use Category Giving Each Responsé (Combined U S and Israeli Samples)

## Rule Use Category

|  |  | Whole Number | Zero | Fraction | Experts |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | $2.35<2.305$ | 8 | 3 | 0 | 0 |
| 2.35 compared | $2.35>2.305$ | 1 | 0 | 9 | 6 |
| vith 2.305 | $235=2.305$ | 1 | 0 | 1 | 0 |
|  | unclear | 0 | 0 | 0 | 1 |
|  | $2.35<2.035$ | 6 | 1 | 0 | 0 |
| 2.35 compared | $2.35>2.035$ | 1 | 2 | 9 | 6 |
| vith 2.305 | $2.35=2.035$ | 3 | 0 | 0 | 0 |
|  | unclear | 0 | 0 | 1 | 1 |

This sample only of U.S and Israel.
The French formulated a different question on this item and, therefore, did not yield !nformation for this table.

Table 9
Ordinary Fraction Notation Frequency of Correct and Incorrect Responses Comparing $1 / 333$ and $1 / 324 ; 1 / 3$ and $1 / 4$ (Combined U S. and Israeli Samples)

## Rule Use Category

Whole Number Zero Fraction Experts

| Correct | 5 | 3 | 11 | 9 |
| :--- | :---: | :---: | :---: | :---: |
| Incorrect | 14 | 3 | 7 | 5 |

Table 10
Fraction to Decimal Translation for $3 / 4,1 / 5,3 / 2,2 / 3$ (U.S. Sample)

|  | Rule Use Category |  |
| :--- | :---: | :---: |
|  | Whole Nuaber | Fraction |
| Numerator encoded | 1 | 6 |
| Denominator encoded | 3 | 0 |
| Syntactic translation | 10 | 0 |


[^0]:    

    * Reproductions supplied by EDRS are the best that can be made $x$

    3 from the original document. $x$
    

[^1]:    ${ }^{1}$ Chlidren who spoke two languages other than English were Included In the present study. In French, the difference between <dizalnes> and <dixlemes>, <centalnes> and <centlemes> is easter to discriminate in writing, but many children confound the terms orally. In Hebrew, the terms <asaroc> and <asiriot>. <maot> and <malot> are easler tc discriminate than the corresponding English terms, yet slmilar enough to ceuse dimeulty for chlldren.

[^2]:    ${ }^{2}$ These ltems were administered to the French sample In a way that did not permit discrimination of the malrule, and those results are, therefore, omitted.

[^3]:    ${ }^{3}$ The French formulation of this question did not yield comparable data.

[^4]:    ${ }^{4}$ These Items were not included in the Israell Interview and were not Interpretable in the French sample because of the French children's Ilmited prior exposure to ordinary fraction notation.

[^5]:    * The numbers in Column $A$ are the larger numbers of the corresponding pair: therefore, an ' $A$ ' answer is a correct ansver

[^6]:    4 children could not be classified.

