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CONCERNING CONGRUENCES ON SYMMETRIC INVERSE SEMIGROUPS

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The lattice of congruences on a symmetric inverse semigroup $\mathscr{I}_X$ has been determined by A. E. LIBER [2], using techniques very similar to those of A. I. MALCEV [3] for characterizing the congruences on a full transformation semigroup $\mathscr{T}_X$. The purpose of this note is to derive and extend these results using more recent theorems on any inverse semigroup. In the course of events, it will be shown that $\mathscr{I}_X$ is embedded in $\mathscr{T}_{X^0}$, the congruences on $\mathscr{I}_X$ are not just those induced by congruences on $\mathscr{T}_{X^0}$, but $\Lambda(\mathscr{I}_X) \cong \Lambda(\mathscr{T}_X)$.

I. INTRODUCTION, DEFINITIONS, AND PRELIMINARIES

As usual, the basic notation and terminology will be that of CLIFFORD and PRESTON [1]. Some familiarity with that notation is assumed. Specifically, if X is a set, then $|X|$ denotes the cardinal number of X, and $|X|'$ is the successor of $|X|$. X^0 will mean $X \cup \{0\}$ where $0 \notin X$. $\mathscr{I}_X$ will denote the symmetric inverse semigroup on X and $\mathscr{T}_X$ will denote the full transformation semigroup on X. Whenever a is a function, then $|a|$ means the cardinal number of a viewed as a set of ordered pairs, and rank (a) means $|\text{image }(a)|$. Finally, whenever S is a semigroup, $\Lambda(S)$ denotes the lattice of congruences on S.

The general approach of Clifford and Preston's treatment [1, Vol. 2, Chapter 10] of Malcev's results [3] will be followed. This involves using the usual Rees congruences and also using sequences of cardinal numbers when X is infinite. The situation here is somewhat more simple, however, for the following reason. If S is an inverse semigroup and E is its set of idempotents, then E is a commutative subsemigroup and there are theorems which guarantee which congruences on E can be extended to all of S.

First, using a device due to V. V. VAGNER [6] (and described in 1, vol. 2, p. 254), it may be seen that if $X^0 = X \cup \{0\}$, then $\mathscr{I}_X$ is embedded in $\mathscr{T}_{X^0}$ in the following way. For each $a \in \mathscr{I}_X$, let $U(a)$ be the domain of a, and let $a^0 \in T_{X^0}$ by

$$xa^0 = \begin{cases} xa & \text{if } x \in U(a) \\ 0 & \text{if } x \notin U(a) . \end{cases}$$

Let $K = \{\alpha \in \mathscr{T}_{X^0} : 0\alpha = 0, \text{ and } x\alpha = y\alpha \neq 0 \text{ implies } x = y\}$. Then $a \in \mathscr{I}_X$ implies $a^0 \in K$. Conversely if $\alpha \in K$, then $\alpha \mid X = \alpha \cap X \times X \in \mathscr{I}_X$. Furthermore, $a \to a^0$ and $\alpha \to \alpha \mid X$ are mutually inverse isomorphisms of $\mathscr{I}_X$ onto K and of K onto $\mathscr{I}_X$, respectively. Notice that if $a \in \mathscr{I}_X$, then $|a| + 1 = \operatorname{rank}(a^0)$.

Some definitions and theorems will now be stated. All are from [5], with the exception of Theorem 1.4, which first appeared in [4].

Definition 1.1. Let S be an inverse semigroup and let $P = \{E_\alpha : \alpha \in J\}$ be a partition of $E_S = E$. P is a *normal* partition of E if

(1) $\alpha, \beta \in J$ implies there exists $\gamma \in J$ such that $E_\alpha E_\beta \subseteq E_\gamma$;

(2) $\alpha \in J$ and $a \in S$ implies there exists $\beta \in J$ such that $aE_\alpha a^{-1} \subseteq E_\beta$.

Theorem 1.2. *Let $P = \{E_\alpha : \alpha \in J\}$ be a normal partition of the semilattice of idempotents of an inverse semigroup S. Let $\sigma = \{(a, b) \in S \times S$: there exists $\alpha \in J$ with $aa^{-1}, bb^{-1} \in E_\alpha$ and $ea = eb$ for some $e \in E_\alpha\}$ and let $\varrho = \{(a, b) \in S \times S : \alpha \in J$ implies there exists $\beta \in J$ such that $aE_\alpha a^{-1}, bE_\alpha b^{-1} \subseteq E_\beta\}$. Then σ and ϱ are respectively the smallest and largest congruences on S such that $\sigma \mid E = \varrho \mid E = \pi_p$ (the equivalence relation on E induced by P).*

Definition 1.3. Let S be an inverse semigroup. $\mathscr{N}$ is a kernel normal system of S if $\mathscr{N}$ is a collection of inverse subsemigroups of S, $\mathscr{N} = \{N_\alpha : \alpha \in J\}$ such that, if $E_\alpha = E_{N_\alpha}$, then

(1) $\{E_\alpha : \alpha \in J\}$ is a normal partition of E_S;

(2) $aa^{-1}, bb^{-1} \in E_\alpha$ and $a, ab^{-1} \in N_\alpha$ imply that $b \in N_\alpha$;

(3) $aa^{-1}, bb^{-1} \in E_\alpha$ and $ab^{-1} \in N_\alpha$ and $aE_\beta a^{-1} \subseteq E_\gamma$ implies that $aN_\beta b^{-1} \subseteq N_\gamma$.

Theorem 1.4. *Let S be an inverse semigroup and let $\mathscr{N} = \{N_\alpha : \alpha \in J\}$ be a kernel normal system of S. Let $\varrho_{\mathscr{N}} = \{(a, b) \in S \times S : aa^{-1}, bb^{-1} \in E_\alpha$ and $ab^{-1} \in N_\alpha$ for some $\alpha \in J\}$. Then $\varrho_{\mathscr{N}}$ is a congruence on S and $\{N_\alpha : \alpha \in J\}$ is the set of idempotents in $S/\varrho_{\mathscr{N}}$.*

Conversely, let ϱ be a congruence on S. Then $\mathscr{N} = \{e\varrho : e \in E\}$ is a kernel normal system of S and $\varrho = \varrho_{\mathscr{N}}$.

Theorem 1.5. *Let S be an inverse semigroup and $P = \{E_\alpha : \alpha \in J\}$ be a normal partition of E. For each $\alpha \in J$, let T_α be the largest inverse subsemigroup of S such* *that* $E_{T_\alpha} = E_\alpha$, *let* $M_\alpha = \{x \in T_\alpha : ex = e$ *for some* $e \in E_\alpha\}$, *and let* $N_\alpha = \{x \in T_\alpha : E_\alpha E_\beta \subseteq E_\gamma$ *implies* $xE_\beta x^{-1} \subseteq E_\gamma\}$. *Then* $\mathcal{M} = \{M_\alpha : \alpha \in J\}$ *and* $\mathcal{N} = \{N_\alpha : \alpha \in J\}$ *are kernel normal systems of* S, $\varrho_{\mathcal{M}} = \sigma$ *and* $\varrho_{\mathcal{N}} = \varrho$ *where* σ *and* ϱ *are defined as in Theorem* 1.2.

Theorem 1.6. *Let* S *be an inverse semigroup and let* $\theta = \{(\varrho, \sigma) \in \Lambda(S) \times \Lambda(S) : \varrho \mid E = \sigma \mid E\}$. *Then*

(1) θ *is a congruence on* $\Lambda(S)$;

(2) *each* θ*-class is a complete modular sublattice of* $\Lambda(S)$;

(3) *the natural homomorphism of* $\Lambda(S)$ *onto* $\Lambda(S)/\theta$ *is a complete lattice homomorphism.*

II. FINITE PRIMARY CARDINALS

Definition 2.1. Let S be an inverse semigroup with E as its semilattice of idempotents. A congruence ϱ on E is a *normal* congruence on E provided that $(e, f) \in \varrho$ and $a \in S$ imply $(a\,e\,a^{-1}, a\,f\,a^{-1}) \in \varrho$.

In the light of Definition 1.1 and Theorem 1.2, the normal congruences on E are just those congruences on E which may be extended to all of S.

Lemma 2.2. *Let* E *be the semilattice of idempotents of* $\mathcal{I}_X$, *and let* ϱ *be a normal congruence on* E. *Then there exists a cardinal number* $\eta(\varrho)$, $1 \leqq \eta(\varrho) \leqq |X|'$, *such that if* $e \in E$, *then* $(e, 0) \in \varrho$ *if and only if* $|e| < \eta(\varrho)$.

Proof. Let $A = \{\xi : \xi$ is a cardinal number and there exists $e \in E$ such that $|e| = \xi$ and $(e, 0) \in \varrho\}$. Choose $\eta(\varrho)$ minimal with respect to $\xi < \eta(\varrho)$ for each ξ in A. To show that $\eta(\varrho)$ has the property asserted, assume first that $e \in E$ and $(e, 0) \in \varrho$. Then $|e| \in A$ and so $|e| < \eta(\varrho)$.

Conversely, assume that $e \in E$ with $|e| < \eta(\varrho)$. Then there exists $f \in E$ such that $(f, 0) \in \varrho$ and $|e| \leqq |f|$. Let g be an extension of e such that $|g| = |f|$ and let a map $U(g)$ one-to-one onto $U(f)$. Then $(f, 0) \in \varrho$ implies $(a\,f\,a^{-1}, a\,0\,a^{-1}) = (g, 0) \in \varrho$, and hence $(eg, e0) = (e, 0) \in \varrho$, concluding the proof.

The cardinal number $\eta(\varrho)$ of Lemma 2.2 will be called the primary cardinal of ϱ.

If ξ is a cardinal number such that $1 \leqq \xi \leqq |X|'$, then $I_\xi = \{e \in E : |e| < \xi\}$ is an ideal of E. Let I_ξ^* denote the congruence on E such that E/I_ξ^* is the Rees quotient semigroup E/I_ξ. I_ξ^* is, in fact, a normal congruence on E since $a \in \mathcal{I}_X$ and $e \in E$ imply $|a\,e\,a^{-1}| \leqq |e|$. Similarly, $J_\varepsilon = \{a \in \mathcal{I}_X : |a| < \xi\}$ is an ideal of $\mathcal{I}_X$.

Let $D_\xi = \{a \in \mathcal{I}_X : |a| = \xi\}$ for $0 \leqq \xi \leqq |X|$. It is a simple matter to compute that $(a, b) \in \mathcal{L}$ if and only if $V(a)$ (the range of a) $= V(b)$, and $(a, b) \in \mathcal{R}$ if and only if $U(a) = U(b)$. Consequently, $(a, b) \in \mathcal{D}$ if and only if $|a| = |b|$ so that the $\mathcal{D}$ classes of $\mathcal{I}_X$ are just the sets D_ξ.

Lemma 2.3. *Let ϱ be a normal congruence on E such that $\eta(\varrho)$ is finite. Then* $\varrho = I^*_{\eta(\varrho)}$.

Proof. According to Lemma 2.2, if $|e|, |f| < \eta(\varrho)$, then $(e, 0), (f, 0) \in \varrho$ and so $(e, f) \in \varrho$ and hence $I^*_{\eta(\varrho)} \subseteq \varrho$.

In order to show that $\varrho \subseteq I^*_{\eta(\varrho)}$, let $(e, f) \in \varrho$. Lemma 2.2 guarantees that if either of $|e|$ and $|f|$ is less than $\eta(\varrho)$, then both of $|e|$ and $|f|$ are less than $\eta(\varrho)$ and so $(e, f) \in I^*_{(\eta)\varrho}$. Assume then that $\eta(\varrho) \leqq |e| \leqq |f|$. If $|ef| < \eta(\varrho)$, then $(ef, f) \in \varrho$ implies that $(0, f) \in \varrho$, a contradiction. Assume then that $\eta(\varrho) \leqq |ef|$ and $e \neq f$. Choose $G \subseteq U(ef)$ such that $|G| + 1 = \eta(\varrho)$ and let $x \in U(f) \setminus U(e)$. Let $H = G \cup \{x\}$ and let g, h be the identity mappings on G and H, respectively. Then $(he, hf) = (g, h) \in \varrho$ and so $(0, h) \in \varrho$, again a contradiction.

Lemma 2.4. *Let n be an integer such that* $1 \leqq n \leqq |X|$. *Then J_{n+1}/J_n is a completely 0-simple inverse semigroup. Consequently, the set of nontrivial congruences on J_{n+1}/J_n is isomorphic to $\Lambda(G_n)$* where *G_n is the symmetric group on n symbols.*

Proof. Since J_{n+1} is an ideal of $\mathscr{I}_X$, then J_{n+1} is itself an inverse semigroup and hence J_{n+1}/J_n is an inverse semigroup. But $J_{n+1} \setminus J_n$ is a $\mathscr{D}$ class of $\mathscr{I}_X$ from which it follows that J_{n+1}/J_n is 0-bisimple and hence 0-simple. Since n is finite, it follows that J_{n+1}/J_n is completely 0-simple. But any nontrivial congruence on a completely 0-simple inverse semigroup must separate idempotents. But when S is any completely 0-simple semigroup, the sublattice $\{\lambda \in \Lambda(S) : \lambda \subseteq \mathscr{H}_S\} \cong \Lambda(G)$ where G is any group $\mathscr{H}$ class of S.

Lemma 2.5. *Let n be an integer,* $1 \leqq n \leqq |X|$. *Let $\sigma \in \Lambda(J_{n+1}/J_n)$ with σ nontrivial, and let $\sigma^\dagger = i \cup [\sigma \mid D_n] \cup [J_n \times J_n]$. Then $\sigma^\dagger \in \Lambda(\mathscr{I}_X)$.*

Proof. Certainly $\sigma^\dagger$ is an equivalence relation on $\mathscr{I}_X$. Let $(a, b) \in \sigma^\dagger$ and $c \in \mathscr{I}_X$.

To see that $(ac, bc) \in \sigma^\dagger$, the only nontrivial cases are when $(a, b) \in \sigma \mid D_n$ and $c \in \mathscr{I}_X \setminus J_{n+1}$. Assume that this is the case.

Since $(a, b) \in \sigma \mid D_n$, then $(a, b) \in \mathscr{H}$ and so $V(a) = V(b)$. But $ac = a(c \mid V(a))$ and $bc = b(c \mid V(b))$. Consequently, $(ac, bc) \in J_n \times J_n$ if $V(a) = V(b) \nsubseteq U(c)$ and $(ac, bc) \in \sigma \mid D_n$ if $V(a) = V(b) \subseteq U(c)$. In any event, $(ac, bc) \in \sigma$.

Similarly, $(ca, cb) \in \sigma^\dagger$ and so $\sigma^\dagger \in \Lambda(\mathscr{I}_X)$.

Lemma 2.6. *Let $\varrho \in \Lambda(\mathscr{I}_X)$ such that $\eta(\varrho \mid E) = n$ is finite. Assume further that ϱ is nontrivial (i.e., that $n \leqq |X|$). Then $\varrho = \sigma^\dagger$ where $\sigma \in \Lambda(J_{n+1}/J_n)$, σ is nontrivial, and $\sigma^\dagger$ is defined as in Lemma* 2.5.

Proof. Since $\eta(\varrho \mid E) = n$, then $\varrho \mid E = I^*_n$ by Lemma 2.3. Since 0ϱ is an ideal of $\mathscr{I}_X$, it follows that $0\varrho = J_n$. Let τ denote the maximal extension of I^*_n to $\mathscr{I}_X$. Let $e \in E$ such that $n < |e|$. According to Theorem 1.5, the proof will be finished when it is shown that $e\tau = \{e\}$. T_e, the largest inverse subsemigoup of $\mathscr{I}_X$ such that $T_e \cap E = \{e\}$, is H_e. Let $a \in H_e$, $a \neq e$. Then there exists $x \in U(a)$ such that $(x, x) \notin a$. Let $f = \{(y, y) : y \in U(e) \text{ and } y \neq x\}$. Then $f \in E$, $ef = f$ and $n \leqq |f|$. But $a f a^{-1} \neq f$ since $x \in U(a f a^{-1})$ and $x \notin U(f)$. Hence $a \notin e\tau$ and so $e\tau = \{e\}$.

The preceeding lemma shows that if n is an integer, $1 \leqq n \leqq |X|$, then the set of extensions of I_n^* to $\mathscr{I}_X$ is $\Lambda(G_n)$, when G_n is the symmetric permutation group on n symbols, and that the set of all $\varrho \in \Lambda(\mathscr{I}_X)$ such that $\eta(\varrho \mid E)$ is finite forms a chain. Consider a congruence ϱ on $\mathscr{T}_X$ such that the primary cardinal of ϱ, $\eta(\varrho)$, is finite [1, Lemma 10.64]. The set of all such ϱ forms a chain and if n is a positive integer such that $1 \leqq n \leqq |X|$, then $\{\varrho \in \Lambda(\mathscr{T}_X) : \eta(\varrho) = n\} \cong \Lambda(G_n)$ [1, Theorem 10.68]. Hence, if X itself is finite, then $\Lambda(\mathscr{I}_X) \cong \Lambda(\mathscr{T}_X)$.

Recall that $\mathscr{I}_X$ is embedded in $\mathscr{T}_{X^0}$. Suppose ϱ is a congruence on $\mathscr{T}_{X^0}$ such that $\eta(\varrho) = n$ where $1 < n \leqq |X|$. If $a \in \mathscr{I}_X$, then $(0^0, a^0) \in \varrho$ if and only if rank $a^0 < n$ [1, vol 2, p. 231], that is, if and only if $|a| < n - 1$. This shows that $\varrho \mid E$ (where E is the semilattice of idempotents of $\mathscr{I}_X$) is I_{n-1}^*. Hence if $n = 5$, there are three such congruences ϱ on $\mathscr{T}_{X^0}$, but I_4^* has four extensions to $\mathscr{I}_X$. Thus the congruences on $\mathscr{I}_X$ are not precisely those induced by congruences on $\mathscr{T}_{X^0}$.

Since there is a one-to-one correspondence between the θ classes of $\Lambda(\mathscr{I}_X)$ and the normal congruences on E, when n is an integer such that $1 \leqq n \leqq |X|'$, let θ_n denote the θ class which corresponds to I_n^*. Thus $|\theta_n| = n$ if $1 \leqq n \leqq 4$; 3 if $5 \leqq n \leqq |X|$; and 1 if $n = |X|'$.

If S is any semigroup in which E is not empty, then θ may be defined on $\Lambda(S)$ as in Theorem 1.6 and θ is always an equivalence relation. S is called θ reduced if each θ class is a singleton. If S is an inverse semigroup and $\varrho, \sigma, \tau \in \Lambda(S)$ with $\varrho \subseteq \sigma, \tau$, then $(\sigma, \tau) \in \theta_S$ if and only if $(\sigma/\varrho, \tau/\varrho) \in \theta_{S/\varrho}$. Hence a congruence ϱ is θ reduced if and only if ϱ is the sup of the θ class to which it belongs, and $\varrho \subseteq \tau$ implies that τ is the sup of the θ class to which it belongs.

Returning to $\mathscr{I}_X$, suppose X is finite, say $|X| = k$. Let ϱ denote the maximal extension of I_k^*. Then ϱ is the minimum θ reduced congruence on S and S/ϱ is also a semilattice.

Some of the results of this section will be summarized in the following theorem.

Theorem 2.7. *Let X be a finite set. Then $\Lambda(\mathscr{I}_X) \cong \Lambda(\mathscr{T}_X)$, a finite chain. The minimum θ reduced congruence ϱ is the minimum semilattice congruence η, and $\mathscr{I}_X/\varrho$ contains just two elements.*

III. INFINITE PRIMARY CARDINALS

In this section several lemmas will be proved which will be of assistance in characterizing all congruences on a symmetric inverse semigroup when the underlying set X is infinite. Throughout, $S = \mathscr{I}_X$ where X is infinite, and ϱ is a normal congruence on E such that $\eta(\varrho)$ is infinite. When $e, f \in E$, then difference (e, f) (abbreviated to dif (e, f)) is defined to be max $\{|e \setminus f|, |f \setminus e|\}$. Considering S as embedded in $\mathscr{T}_{X^0}$, it is routine to see that dif $(e, f) =$ difference rank $(e^0 . f^0)$ if $e \leqq f$ or $f \leqq e$; and dif $(e, f) + 1 =$ dr (e^0, f^0) if $e \nleqq f$ and $f \nleqq e$ [1, vol. 2, page 228]. A consequence of this definition is that if $|f| < |e|$ and $|e|$ is infinite, then dif $(e, f) = |e|$. Whenever ξ is a cardinal number, $\Delta_\xi = \{(e, f) = E \times E : \text{dif}(e, f) < \xi\}$. Further, $(e, f) \in \Delta_{\aleph_0}$ will sometimes be shortened to $e \doteq f$. As before, $I_\xi = \{e \in E : |e| < \xi\}$.

Lemma 3.1. *Let $(e, f) \in \varrho$ with $|e \setminus f| = |e| = \eta$ where η is an infinite cardinal. Then $I_{\eta'} \times I_{\eta'} \subseteq \varrho$.*

Proof. Let a map $U(e)$ one-to-one onto $U(e \setminus f)$. Then $aea^{-1} = e$, $afa^{-1} = 0$ and so $(e, 0) \in \varrho$. The rest follows from Lemma 2.2.

Lemma 3.2. *Let $(e, f) \in \varrho$, $e < f$. Then $g \in E$ with $g < e$ and $g \doteq e$ implies $(g, e) \in \varrho$.*

Proof. Suppose first that g satisfies the conditions stated and dif $(g, e) = 1$. Select $k \in E$ such that $e < k \leqq f$ and dif $(e, k) = 1$. Note that $(e, k) \in \varrho$ and choose $(x, x) \in k \setminus e$ and $(y, y) \in e \setminus g$. Let $a = g \cup \{(y, x)\}$. Then $aka^{-1} = e$, $aea^{-1} = g$ and so $(e, g) \in \varrho$. An obvious induction argument completes the proof of the lemma.

Lemma 3.3. *Let $(e, f) \in \varrho$, $|e| = |f| = \infty$, $e \neq f$, and $e \doteq f$. Let $g \in E$ with $g \doteq e$. Then $(e, g) \in \varrho$.*

Proof. Since $(e, ef) \in \varrho$, $ef < e$, and $gef \leqq ef < e$, then $(gef, e) \in \varrho$ by Lemma 3.2. If $g = gef$, the lemma is proved. Otherwise, select $h, k \in E$ with $gef < h \leqq e$ and $gef < k \leqq g$ and dif (h, gef), dif $(k, gef) = 1$. Let $(x, x) \in h \setminus gef$, $(y, y) \in k \setminus gef$, and let $a = gef \cup \{(y, x)\}$. Then $aha^{-1} = k$, $agefa^{-1} = gef$, and so $(gef, k) \in \varrho$.

If $k = g$, there is nothing more to show. If $k < g$, let $r \in E$ such that $k < r \leqq g$ and dif $(k, r) = 1$. Let $(z, z) \in r \setminus k$. Let b map $U(r)$ one-to-one onto $U(k)$ in such a way that $(z, y) \in b$. Then $bkb^{-1} = r$, $bgefb^{-1} = k$, and so $(k, r) \in \varrho$. An obvious induction argument completes the proof.

Corollary 3.4. *Let $(e, f) \in \varrho$, $|e| = |f| = \infty$, $e \neq f$, and $e \doteq f$. Then $(g, h) \in \varrho$ for each $g, h \in E$ with $|g| = |h| = |e|$ and $g \doteq h$.*

Proof. Let a map $U(g)$ one-to-one onto $U(e)$. Since $(e, ef) \in \varrho$, then $(aea^{-1}, aefa^{-1}) = (g, aefa^{-1}) \in \varrho$. Now $aefa^{-1} \neq g$, $|g| = |aefa^{-1}|$ and $aefa^{-1} \doteq g$. Lemma 3.3 completes the proof.

Lemma 3.5. *Let $(e, f) \in \varrho$, $e < f$, and* dif $(e, f) = \xi$ *where ξ is an infinite cardinal. Then $g \in E$ with $g < e$ and* dif $(g, e) = \xi$ *implies $(g, e) \in \varrho$.*

Proof. Let t map $U(e \setminus g)$ one-to-one onto $U(f \setminus e)$ and let $a = g \cup t$. Then $afa^{-1} = e$, $aea^{-1} = g$, and so $(g, e) \in \varrho$.

Lemma 3.6. *Let* $(e, f) \in \varrho$ *and* $\operatorname{dif}(e, f) = \xi$ *where* ξ *is an infinite cardinal. The* $g \in E$ *with* $\operatorname{dif}(e, g) \leqq \xi$ *implies* $(e, g) \in \varrho$.

Proof. Without loss of generality, assume $|e \setminus f| = \xi$. Since $(e, ef) \in \varrho$ and $\operatorname{dif}(e, efg) = \xi$, then $(e, efg) \in \varrho$ by Lemma 3.5. Let g' be an extension of g such that $\operatorname{dif}(g', efg) = \xi$. Let t map $U(g' \setminus efg)$ one-to-one onto $U(e \setminus efg)$ and let $a = efg \cup t$. Then $aea^{-1} = g'$, $aefga^{-1} = efg$, and so $(g', efg) \in \varrho$. The lemma follows immediately from this.

Corollary 3.7. *Let* $(e, f) \in \varrho$, $|e| = |f|$, *and* $\operatorname{dif}(e, f) = \xi$ *where* ξ *is an infinite cardinal. Then* $g, h \in E$ *with* $|g| = |h| = |e|$ *and* $\operatorname{dif}(g, h) \leqq \xi$ *implies* $(g, h) \in \varrho$.

Proof. Again, without loss of generality, assume $|e \setminus f| = \xi$. Let a map $U(g)$ one-to-one onto $U(e)$. Then $(aea^{-1}, aefa^{-1}) = (g, aefa^{-1}) \in \varrho$ and $\operatorname{dif}(g, aefa^{-1}) = \xi$. The rest follows from Lemma 3.6.

Lemma 3.8. *Let* $(e, f) \in \varrho$, $|e| = |f| = \eta$ *and* $\operatorname{dif}(e, f) = \xi$, *where* η *is an infinite cardinal. Then*

(i) *if* ξ *is infinite, then* $(I_{\eta'} \times I_{\eta'}) \cap \Delta_{\xi'} \subseteq \varrho$;

(ii) *if* $0 < \xi < \aleph_0$, *then* $(I_{\eta'} \times I_{\eta'}) \cap \Delta_{\aleph_0} \subseteq \varrho$.

Proof. Assume, without loss of generality, that $|e \setminus f| = \xi$. Let $g, h \in E$ such that $|h| \leqq |g| \leqq \eta$ and $\operatorname{dif}(g, h) \leqq \xi$. Since $(e, ef) \in \varrho$, then $((e \setminus f)\, e, (e \setminus f)\, ef) = (e \setminus f, 0) \in \varrho$. Thus $\xi < \eta(\varrho)$ by Lemma 2.2. and so if $|g| \leqq \xi$, then $(g, h) \in \varrho$. Also, if g is finite, then $(g, h) \in \varrho$ since $|g|, |h| < \eta(\varrho)$. For the remainder of this proof assume $\xi < |g|$ and g is infinite.

(i) Assume ξ is infinite. Notice that this guarantees $|h| = |g|$, since otherwise, $|h| < |g| = \operatorname{dif}(h, g) \leqq \xi < |g|$, a contradiction. Let p be an extension of g such that $|p| = \eta$ and select $m < g$ such that $|m| = \xi$. Then $|\varrho \setminus m| = |p|$ and $\operatorname{dif}(p, p \setminus m) = \xi$. Hence $(p, p \setminus m) \in \varrho$ by Corollary 3.7. Thus $(pg, (p \setminus m)\, g) = (g, g \setminus m) \in \varrho$. But $\operatorname{dif}(g, g \setminus m) = \xi$ and so $(g, h) \in \varrho$ by Lemma 3.6.

(ii) Assume $0 < \xi < \aleph_0$. Again, $|h| = |g|$, because otherwise $|h| < |g| = \operatorname{dif}(h, g) < \aleph_0 \leqq |g|$, a contradiction. Let p be an extension of g such that $|p| = \eta$ and let $(x, x) \in g$. Then $(p, p \setminus \{(x, x)\}) \in \varrho$ by Corollary 3.4. Hence $(gp, g(p \setminus \{(x, x)\})) = (g, g \setminus \{x, x\}) \in \varrho$. Hence $(g, h) \in \varrho$ by Lemma 3.3.

For each cardinal number λ such that $\lambda \in [\eta(\varrho), |X|]$ let $A_\lambda = \{\xi :$ there exists $(e, f) \in \varrho$ such that $|e| = |f| = \lambda$ and $\operatorname{dif}(e, f) = \xi\}$. Define λ^* by λ^* is minimal with respect to $\xi < \lambda^*$ for each $\xi \in A_\lambda$.

Lemma 3.9. *Suppose that* $\lambda, \mu \in [\eta(\varrho), |X|]$ *with* $\lambda < \mu$. *Then*

(i) $\lambda^* \leqq \eta(\varrho)$

(ii) $\mu^* \leqq \lambda^*$.

Proof. (i) The first part of the proof for Lemma 3.8 guarantees this.

(ii) For a contradiction, suppose $\lambda^* < \mu^*$. Then there exists $(e, f) \in \varrho$ such that $|e| = |f| = \mu$ and $\lambda^* \leqq \operatorname{dif}(e, f) = \xi < \mu^*$. But (i) says that $\mu^* \leqq \eta(\varrho)$ and so $\lambda^* \leqq \xi < \mu^* \leqq \eta(\varrho) \leqq \lambda < \mu$. Assume $|e \setminus f| = \xi$ and let $g \in E$ such that $e \setminus f < g < e$ and $|g| = \lambda$. Then $(ge, gf) = (g, gf) \in \varrho$. But $g \setminus gf = e \setminus f$ and so $\operatorname{dif}(g, gf) = \xi$ and $|g| = |gf| = \lambda$. This contradicts the definition of λ^*.

Following Clifford and Preston [1, vol. 2, page 234], the map $\lambda \to \lambda^*$ is a map of $[\eta(\varrho), |X|]$ into $[1, \eta(\varrho)]$. The range of this map is finite, say $\{\xi_1, \ldots, \xi_k\}$, with $\xi_k < \ldots < \xi_1$. For each i, $1 \leqq i \leqq k$, let η_i be the least cardinal such that $\eta_i^* = \xi_i$. Then

$$\xi_k < \ldots < \xi_1 \leqq \eta(\varrho) = \eta_1 < \ldots < \eta_k \leqq \eta_{k+1} = |X|',$$

and $\{\xi_k, \ldots, \xi_1, \eta_1, \ldots, \eta_k\}$ is called the sequence of cardinals of ϱ.

All ξ_i are infinite, except possibly ξ_k, and if ξ_k is finite, then $\xi_k = 1$. For, if $1 < \xi_i = r < \aleph_0$, then there exists $(e, f) \in \varrho$ with $|e| = |f| = \eta_i$ and $\operatorname{dif}(e, f) = r - 1 > 0$. Lemma 3.8 (ii) guarantees that $\xi_i \geqq \aleph_0$, a contradiction.

Lemma 3.10. *Let* ξ_i, η_i $(i = 1, \ldots, k)$ *be* $2k$ *cardinal numbers such that:*

(i) $\xi_k < \ldots < \xi_1 \leqq \eta_1 < \ldots < \eta_k \leqq |X|$,

(ii) *All* ξ_i *and* η_i *are infinite except possibly* ξ_k, *and if* ξ_k *is finite, then* $\xi_k = 1$.

Define τ *on* E *by*

$$\tau = I_{\eta_1}^* \cup (\Delta_{\xi_1} \cap I_{\eta_2}^*) \cup \ldots (\Delta_{\xi_{k-1}} \cap I_{\eta_k}^*) \cup \Delta_{\xi_k}.$$

Then τ *is a normal congruence on* E *and* (i) *is the sequence of cardinals of* τ.

Conversely, if ϱ *is a normal congruence on* E *such that* ϱ *is not the universal congruence,* $\eta(\varrho)$ *is infinite, and* (i) *is the sequence of cardinals of* $\varrho(\eta(\varrho) = \eta_1)$, *then* $\varrho = \tau$.

Proof. That τ is a normal congruence on E follows from Malcev's theorem [1, Theorem 10.72]. For τ may be viewed as the restriction to E of a congruence defined on all of $\mathscr{T}_{X^0}$.

It must be shown that (i) is the sequence of cardinals of τ. First $I_{\eta_1}^* \subseteq \tau$ and so $\eta_1 \leqq \eta(\tau)$ by Lemma 2.2. Suppose $\eta_1 < \eta(\tau)$. Let $e \in E$ with $|e| = \eta_1$. Then $(e, 0) \in \tau$ again by Lemma 2.2. Then $(e, 0) \notin I_{\eta_1}^*$ and $(e, 0) \notin \Delta_{\xi_k}$ since $\xi_k \leqq \eta_1 = |e| = \operatorname{dif}(e, 0)$. Thus $(e, 0) \in \Delta_{\xi_i} \cap I_{\eta_{i+1}}^*$ for some i, $1 \leqq i < k$. Then $\eta_1 = |e| = \operatorname{dif}(e, 0) < \xi_i \leqq \eta_1$, a contradiction. Hence $\eta(\tau) = \eta_1$.

Suppose that $\eta_i \leqq \eta < \eta_{i+1}$ (where $\eta_{k+1} = |X|'$). To see that $\eta^* = \xi_i$, let $(e, f) \in \tau$ with $|e| = |f| = \eta$. Then $(e, f) \in \Delta_{\varepsilon_j} \cap I^*_{\eta_{j+1}}$ for some j. The monotone properties of $\{\xi_i\}$ and $\{\eta_i\}$ imply that $(e, f) \in \Delta_{\xi_i} \cap I^*_{\eta_{j+1}}$ and hence $\eta^* \leqq \xi_i$. But if $\xi < \xi_i$, then certainly there exists $g, h \in E$ such that $|g| = |h| = \eta$ and $\operatorname{dif}(g, h) = \xi$. Hence $(g, h) \in \Delta_{\xi_i} \cap I^*_{\eta_{i+1}} \subseteq \tau$ and so $\eta^* = \xi_i$. This also shows that η_i is the least of all cardinals η such that $\eta^* = \xi_i$.

Assume now that ϱ is a normal congruence on E, not the universal congruence. Let (i) be the sequence of cardinals of ϱ with $\eta(\varrho) = \eta_1$.

To see that $\varrho \subseteq \tau$, let $(e, f) \in \varrho$. According to Lemmas 2.1 and 3.1, either $|e|, |f| < \eta_1$, in which case $(e, f) \in I^*_{\eta_1} \subseteq \tau$, or $|e| = |f| \geqq \eta_1$. Assume $|e| = |f| = \eta$, $\eta_i \leqq \eta < \eta_{i+1}$ and $\operatorname{dif}(e, f) = \xi$. Then $\xi < \eta^* = \xi_i$ and so $(e, f) \in \Delta_{\xi_i} \cap I^*_{\eta_{i+1}} \subseteq \tau$.

Finally, suppose $(e, f) \in \tau$. If $(e, f) \in I^*_{\eta_1}$, then $|e|, |f| < \eta_1 = \eta(\varrho)$ and so $(e, f) \in \varrho$. Assume $(e, f) \notin I^*_{\eta_1}$. Then $(e, f) \in \Delta_{\xi_i} \cap I^*_{\eta_{i+1}}$ for some i, $1 \leqq i \leqq k$. If $|e| \neq |f|$, say $|e| > |f|$, then $\operatorname{dif}(e, f) = |e|$ so $|e| < \xi_i < \eta_1$ and hence $(e, f) \in I^*_{\eta_1}$, a contradiction. Hence $|e| = |f| = \eta$, say. Without loss of generality, $\eta_i \leqq \eta < \eta_{i+1}$. The $\operatorname{dif}(e, f) < \xi_i$ and so $(e, f) \in \varrho$ by Lemma 3.8 and an argument which parallels the proof for [1, Lemma 10.71].

IV. THE LATTICE $\Lambda(\mathscr{I}_X)$ FOR INFINITE X

Again, S will denote the symmetric inverse semigroup on X with X infinite and E is the semilattice of idempotents of S.

Lemma 4.1. *Assume that ϱ is a normal congruence on E such that $\eta(\varrho)$ is infinite. Then ϱ has a unique extension to all of S.*

Proof. First, ϱ has an extension to S by the remarks just after Definition 2.1. If ϱ is the universal congruence on E, and ϱ^* is an extension of ϱ to S, it is immediate that ϱ^* is the universal congruence on S.

Suppose ϱ is not the universal congruence on E. Let $\{\xi_k, \ldots, \xi_1, \eta_1 = \eta(\varrho), \ldots, \eta_k\}$ be the sequence of cardinals of ϱ (Lemma 3.10). Let $E_e = e\varrho$ for each $e \in E$ so that $\{E_e : e \in E\}$ is the normal partition of E induced by ϱ. Let $\mathscr{M}$ and $\mathscr{N}$ be the kernel normal systems of Theorem 1.5. To prove the lemma, it suffices to show that $M_e = N_e$ for each $e \in E$.

Assume first that $e \in E$ and $|e| < \eta_1$. Then $T_e = \{a \in S : |a| < \eta_1\}$. Since $0 \in E_e$ and $Oa = 0$ for each $a \in T_e$, then $M_e = T_e$ and so $M_e = N_e$.

Suppose then that $\eta_1 \leqq |e| = \eta$, say $\eta_i < \eta \leqq \eta_{i+1}$. Then $M_e = \{a \in T_e : fa = f$ for some $f \in E_e\} = \{a \in T_e : |\{x \in D(a) \mid xa \neq x\}| < \xi_i\}$. Assume that $a \in T_e$, $a \notin M_e$. It will be shown that $a \notin N_e$, which guarantees that $M_e = N_e$.

Let $A = \{x \in U(a) : xa = x\}$ and $B = \{x \in U(a) : xa \neq x\}$. Since $a \notin M_e$, then $|B| \geqq \xi_i$. Let $\mathscr{F} = \{V \subseteq B : x \in V$ implies that $xa \notin V\}$. $\mathscr{F} \neq \square$ since $\{x\} \in \mathscr{F}$ for each $x \in B$; and $\subseteq$ is a partial order for $\mathscr{F}$. A routine Zorn's Lemma argument shows that $\mathscr{F}$ contains a maximal element C. Suppose, for a contradiction, that $|C| < |B|$. If $xa \in C$ for each $x \in B \setminus C$, then $a \mid (B \setminus C)$ is an injection of $B \setminus C$ into C, a contradiction. Hence, there exists $x \in B \setminus C$ such that $xa \notin C$. But then $C \cup \{x\} \in \mathscr{F}$, contradicting the maximality of C. Thus $|B| = |C|$.

Let $F = A \cup C$, and let f be the identity map on F. Notice that $|f| = |e|$ and $x \in C$ implies that $xa \notin A$. Thus $x \in C$ implies $xa \notin A \cup C = F$ and so $(x, x) \notin afa^{-1}$. Hence $|f \setminus afa^{-1}| \geqq \xi_i$. Now $E_e E_f = E_{aa^{-1}} E_f = E_{aa^{-1}f} = E_f$, but $afa^{-1} \notin E_f$. Hence $a \notin N_e$.

The preceeding lemma shows that if ϱ is a normal congruence on E such that $\eta(\varrho)$ is infinite, then the θ class which corresponds to ϱ is a singleton. Let $\varrho = I^*_{\aleph_0} \cup \Delta_1$. The unique extension ϱ^* of ϱ to S is the minimum θ reduced congruence on S, but ϱ^* is not a semilattice congruence. The construction of $\Lambda(S)$ makes it clear that $\Lambda(S) \cong \Lambda(\mathscr{T}_{X^0}) \cong \Lambda(\mathscr{T}_X)$ and so Clifford and Preston's result [1, Theorem 10.77] that $\Lambda(\mathscr{T}_X)$ is distributive applies to $\Lambda(S)$ as well.

Theorem 4.2. *Let X be an infinite set. Then $\Lambda(\mathscr{I}_X) \cong \Lambda(\mathscr{T}_X)$. The minimum θ reduced congruence ϱ on $\mathscr{I}_X$ is $\{(a, b) \in \mathscr{I}_X \times \mathscr{I}_X : |a|, |b| < \aleph_0$ or $a = b\}$. The universal congruence is the minimum semilattice congruence.*

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