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CONCERNING SPACES OF CONTINUOUS FUNCTIONS

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If T is a completely regular topological space, we denote by $C(T)$ the linear space of all continuous functions defined on T. The space $C(T)$ is topologized by means of the family of pseudonorms $|x|_K = \max_{t \in K} |x(t)|$, where K runs over all compact subsets of the space T. In this way, $C(T)$ becomes a convex topological linear space. Let us denote by $M(T)$ the space of all linear functionals defined on $C(T)$. Now every completely regular topological space may be imbedded in a space hT (the Hewitt closure of T) so that T is dense in hT and every $x \in C(T)$ admits a continuous extension over hT. If follows that, as far as the algebraic structure is concerned, the spaces $C(T)$ and $C(hT)$ are identical. If the topological structure is taken into account as well, these spaces are easily distinguished, since, clearly, the space $M(T)$ may be considered as a subspace of $M(hT)$. Suppose now a space $C(T)$ is given. Since $C(T)$ and $C(hT)$ are algebraically isomorphic, every member of $M(hT)$ represents a linear form on $C(T)$. This form will not be continuous on $C(T)$ in the general case. It is natural to ask, however, whether it does not perhaps retain some weakened form of continuity which would enable us to characterize the members of $M(hT)$ in terms of $C(T)$ only. On the other hand, let $M(hT)$ be equipped with an arbitrary topology such that $C(hT)$ and $M(hT)$ are dual to each other. Then $M(T)$ is dense in $M(hT)$. The question presents itself whether $M(hT)$ cannot be considered in some sense as a completion of $M(T)$. Both these questions are closely connected together and are treated in the present paper, which is a continuation of the author's discussion [5] of pseudocompact subsets of convex topological linear spaces. Most of the results of the present paper are generalizations of corresponding theorems of [5] and are intended as a basis for further applications.

The present paper forms a continuation of the author's discussion of pseudocompact subsets of convex topological linear spaces. It turns out that the methods used in [5] and [6] for the case of a pseudocompact space may be adapted to the proof of similar results concerning the Hewitt closure of an arbitrary completely regular topological space T. The main question treated in the connection between the space $M(T)$ of all linear functionals on $C(T)$ and the analogous space corresponding to the Hewitt closure hT.

The first paragraph contains the essential points of the theory of the Hewitt closure. The theorems of this section are not essentially new. Most of them have been (explicitly or implicitly) proved already by HEWITT. In the present paper we intend to give a unified theory of the Hewitt closure based on a lemma that we have established in [5]. It turns out that the proofs can be considerably simplified if this lemma is consistently applied. It is only because of the shortness of the proofs that this paragraph has been included, as it seems to the author that these ideas are not without interest even if they are used to prove results which are more or less known. The second and third paragraphs are devoted to the proof of the main result; the proof is divided into several lemmas, the most important being a generalization of theorem (1,2) of [5]. In the fourth paragraph we summarize the results obtained in theorems (4,1) and (4,2). These results have interesting applications which will be published later.

1.

In this paragraph we prove some simple propositions concerning the Hewitt closure of a given completely regular topological space.

Let T be a completely regular topological space. We shall denote by $C(T)$ the linear space of all continuous real-valued functions defined on T. Now let $x \in C(T)$ be bounded on T. Then x admits a continuous extension over βT. Throughout this paper, the following convention will be adopted. The same symbol will be used for a bounded continuous function defined on T and its extension on βT. In this notation, the space of all bounded continuous functions on T coincides with $C(\beta T)$.

Let T be a completely regular topological space. A point $s \in \beta T$ will be called a Hewitt point of the space T if every continuous function defined on T can be continuously extended over the point s.

This, of course, is meant in the following sense. For every continuous function x defined on T there exists a function x^* defined and continuous on $T \cup (s)$ such that $x(t) = x^*(t)$ for every $t \in T$. Clearly every point $t \in T$ is a Hewitt point of the space T. The set of all Hewitt points of the space T will be denoted by hT. We shall call it the Hewitt closure of T.

For our purpose it will be convenient to state the definition of a Hewitt point in a slightly different form. The following lemma is a trivial modification of a result which has been used by the author in [5].

(1,1) *Let T be a completely regular topological space. Let $s \in \beta T$. Then the following three properties of s are mutually equivalent.*

(1) *s is a Hewitt point of the space T*

(2) *for every $x \in C(\beta T)$ there exists a point $t \in T$ such that*

$$x(t) = x(s)$$

(3) *for every countable subset $S \subset C(\beta T)$ there exists a point $t \in T$ such that*

$$x(t) = x(s)$$

for every $x \in S$.

Proof: Let $s \in hT$ and let x be an arbitrary bounded continuous function on T. Suppose that the relation $x(t) = x(s)$ is not fulfilled for any point $t \in T$. It follows that the function v defined on βT by the relation $v(t) = |x(t) - x(s)|$ is positive for every $t \in T$. The function w defined on T by the relation $w(t)\, v(t) = 1$ is clearly continuous on T. Now w can be continuously extended over the point s, since $s \in hT$. For this point we obtain $w(s)\, v(s) = 1$ which is impossible since $v(s) = 0$. The existence of a point $t \in T$ with the required property is thus proved.

Now let x_n be an arbitrary sequence of bounded continuous functions on T. There are positive numbers β_n such that $|x_n(t)| \leq \beta_n$ for every $t \in T$. For $t \in \beta T$ let us define

$$v(t) = \sum_n \frac{1}{2^n \beta_n} |x_n(t) - x_n(s)| \, .$$

We have $v(s) = 0$. Now if condition (2) is fulfilled, there exists a point $t \in T$ for which $v(t) = 0$. Clearly we have $x_n(t) = x_n(s)$ for every n.

Suppose now that the point s fulfills condition (3) and let x be an arbitrary continuous function defined on T. For every natural m let us define

$$x_m(t) = \tfrac{1}{2}|x(t) + m| - \tfrac{1}{2}|x(t) - m| \, .$$

According to our assumption concerning the point s, there exists a point $w \in T$ such that $x_m(w) = x_m(s)$ for every m. For any $t \in T$, let $x^*(t) = x(t)$ and let $x^*(s) = x(w)$. Take a natural number m and a positive ε so that $m > |x(w)| + \varepsilon$. Let

$$G = \underset{t}{E}[t \in T \cup (s),\, |x_m(t) - x_m(s)| < \varepsilon]$$

so that G is an open subset of the space $T \cup (s)$. It is easy to see that, for $t \in G$, we have $x^*(t) = x_m(t)$ so that x^* is continuous on the whole of $T \cup (s)$.

Let T be a completely regular topological space. A set $W \subset T$ will be called relatively pseudocompact in T if every $x \in C(T)$ is bounded on W.

(1,2) *Let $W \subset T$ be relatively pseudocompact. Let us denote by $\overline{W}$ the closure of W in βT. Then $\overline{W} \subset hT$.*

Proof: Let x be an arbitrary bounded continuous function on T, let s be an arbitrary point of $\overline{W}$. We are going to show that there is a point $t \in T$ such that $x(t) = x(s)$. According to (1,1) this is sufficient to prove our theorem. For every $t \in T$ let us put $v(t) = |x(t) - x(s)|$, so that v is a bounded continuous function on T. We have $v(t) \geqq 0$ for every $t \in T$ and $\inf_{w \in W} v(w) = 0$. Suppose $v(t) > 0$ for every $t \in T$. It follows that the function h defined on T by the relation $h(t)\, v(t) = 1$ is a continuous function on T which is not bounded on W. It follows that $v(t) = 0$ for some $t \in T$; for such a point t we have $x(t) = x(s)$.

For the sake of brevity, we shall use the following notations. If T is a completely regular topological space, we shall denote by $\mathbf{F}(T)$ or simply by $\mathbf{F}$ the family of all closed subsets of the space T. Similarly, $\mathbf{F}^*(T)$ is taken to mean the family of all sets of the form $\underset{t}{E}\,[t \in T,\, x(t) = 0]$ where x is a member of $C(T)$.

(1,3) *Let T be a completely regular topological space. Then the following properties of T are equivalent.*

(1) *the space T coincides with hT*

(2) *if $\mathbf{A} \subset \mathbf{F}(T)$ has the finite intersection property and for every $x \in C(T)$ a set $A \in \mathbf{A}$ can be found such that x is bounded on A, then the intersection of $\mathbf{A}$ is nonempty and compact.*

(3) *if $\mathbf{A} \subset \mathbf{F}^*(T)$ has the finite intersection property and for every $x \in C(T)$ a set $A \in \mathbf{A}$ can be found such that x is bounded on A, then the intersection of $\mathbf{A}$ is nonempty and compact,*

(4) *if $\mathbf{A} \subset \mathbf{F}(T)$ has the finite intersection property and if, for every $x \in C(T)$ and every positive ε, a set $A \in \mathbf{A}$ can be found such that the diameter of $x(A)$ is less than ε, then the intersection of $\mathbf{A}$ is nonempty,*

(5) *if $\mathbf{A} \subset \mathbf{F}^*(T)$ has the finite intersection property and if, for every $x \in C(T)$ and every positive ε, a set $A \in \mathbf{A}$ can be found such that the diameter of $x(A)$ is less than ε, then the intersection of $\mathbf{A}$ is nonempty.*

Proof: Let $T = hT$. Let $\mathbf{A}$ be a system of closed subsets of T which has the finite intersection property. Suppose that, for every $x \in C(T)$, an $A \in \mathbf{A}$ can be found such that x is bounded on A. We are to show that the intersection of $\mathbf{A}$ is nonvoid. Take an $s \in \beta T$ which lies in the closure of every $A \in \mathbf{A}$. Suppose there is a function $v \in C(\beta T)$ such that $v(s) = 0$ and $v(t) > 0$ for every $t \in T$. The function w defined on T by the relation $w(t)\, v(t) = 1$ is clearly continuous on T. According to our assumption concerning $\mathbf{A}$ there exists a set $A \in \mathbf{A}$ such that w is bounded on A. Consequently there exists a positive α such that $w(t) \leqq \alpha$ for $t \in A$. It follows that, for $t \in A$, we have $v(t) \geqq \frac{1}{\alpha} > 0$, which is impossible since $s \in \bar{A}$ and $v(s) = 0$. It follows that $s \in hT$. We have thus shown that the intersection of all sets $\bar{A}$, where A runs over $\mathbf{A}$, is contained in hT. Since $T = hT$ it follows that the intersection of $\mathbf{A}$ is compact.

It is easy to see that the proof of our theorem will be concluded if we show that the inclusion $T = hT$ is a consequence of property (5). To see that, let us take a space T which fulfills condition (5). Let $s \in hT$. For every $x \in C(T)$ let us put

$$A(x) = \underset{t}{E}[t \in T, x(t) = x(s)]$$

so that $A(x) \in \mathbf{F}^*(T)$. The system $\mathbf{A}$ consisting of all $A(x)$ has the finite intersection property. This in an immediate consequence of the fact that, for any finite system $x_1, \ldots, x_n$ of elements of $C(T)$ the function $\Sigma |x_i(t) - x_i(s)|$ must attain the value 0 in some point $t \in T$. If $x \in C(T)$, the diameter of $x(A(x))$ is zero, so that the system $\mathbf{A}$ fulfills the condition mentioned in (5). It follows that the intersection of $\mathbf{A}$ is nonempty. Let $t \in T$ be a point which lies in every $A(x)$. We have then $x(t) = x(s)$ for every $x \in C(T)$ so that $s = t \in T$ which concludes the proof.

We use this opportunity to say a few words concerning another question closely connected with the theory of the Hewitt closure.

It is a wellknown fact that spaces which coincide with their Hewitt closure (such spaces are termed Q-spaces by Hewitt) possess a lot of simple and important properties and that most of the spaces occuring in applications belong to this category of spaces. Anyhow, for the study of the Hewitt closure it is important to know examples of completely regular spaces which are not Q-spaces. An obvious way of obtaining such spaces is to take a pseudocompact*) space which is not compact. Clearly, if T is pseudocompact, we have $hT = \beta T$ so that T cannot be a Q-space unless $T = hT = \beta T$, in other words, unless T is compact. Hewitt has shown that there exist pseudocompact spaces which are not even countably compact. Other examples of such spaces have been given recently by J. Novák [4] and S. Mrówka [3]. We intend to conclude this section with another simple example of such a space.

(1,4) Let us denote by T the space consisting of all functions t defined on the interval $0 \leqq p \leqq 1$ and subject to the following two conditions:

1) $0 \leqq t(p) \leqq 1$
2) for every s the set $\underset{p}{E}[t(p) \neq s]$ is infinite.

Let P be the set of all real numbers p contained in the interval $0 \leqq p \leqq 1$. Let $t_0 \in T$ be given. If $p_1, \ldots, p_n$ is a finite set of points of the interval P, ε a positive number, let

$$U(t_0; p_1, \ldots, p_n; \varepsilon) = \underset{t}{E}[t \in T, |t(p_i) - t_0(p_i)| < \varepsilon, i = 1, 2, \ldots, n] .$$

Clearly the sets U define a topology on T so that T becomes a completely regular space. To see that, it is sufficient to note that T is a subspace of a cartesian product of line segments.

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*) A completely regular space T is said to be pseudocompact if every continuous function defined on T is bounded on T.

Let us denote by S the set consisting of all points t_n of the form

$$t_n(p) = p^n$$

where n runs over all natural numbers.

We are going to show that the set S is closed in T. Let us take for that purpose an arbitrary point $t_0 \in T$ which does not lie in S. There exists a point $p_0 < 1$ so that $t_0(p_0) > 0$. Now there are two cases possible.

(1) for every natural n, we have $t_0(p_0) \neq p_0^n$. Then there exists a positive ε so that the inclusion

$$t_0(p_0) - \varepsilon < p_0^n < t_0(p_0) + \varepsilon$$

is not fulfilled for any natural n. It follows that the neighbourhood of the point t_0 defined by the set $U(t_0; p_0; \varepsilon)$ is disjoint with S.

(2) there exists a natural n such that $t_0(p_0) = p_0^n$. It follows that $p_0 > 0$ since, in the contrary case, we should have $0 = p_0^n = t_0(p_0)$. Since t_0 non $\in S$, there exists a point $p_1 \in P$ such that $t_0(p_1) \neq p_1^n$. Hence $p_1 \neq p_0$. Now there exists a positive number ε such that the interval $(t_0(p_0) - \varepsilon, t_0(p_0) + \varepsilon)$ does not contain any point of the form p_0^m for $m \neq n$. Clearly ε may be chosen so small that p_1^n non $\in (t_0(p_1) - \varepsilon, t_0(p_1) + \varepsilon)$. It follows that the neighbourhood $U(t_0; p_0, p_1; \varepsilon)$ is disjoint with S.

The topology of S being discrete, we see at once that is not countably compact. Now let x be an arbitrary continuous function defined on T. Suppose there exist points $t_m \in T$ such that $|x(t_m)| > m$. For every natural m and every natural n there exists a finite set $P_{mn} \subset P$ and a number $0 < \sigma_{mn} \leqq \frac{1}{n}$ such that the following implication holds

$$t \in U(t_m; P_{mn}; \sigma_{mn}) \Rightarrow |x(t) - x(t_m)| < \frac{1}{n}.$$

Let $P_m = \cup P_{mn}$, let $Q = \cup P_m$.

Now if $t \in T$ is a point such that

$$t(p) = t_m(p)$$

for every $p \in P_m$, we see at once that $x(t) = x(t_m)$. By means of the diagonal process it is possible to define a subsequence v_m of the sequence t_m such that for every $q \in Q$ the sequence $v_m(q)$ converges to a limit $\alpha(q)$, where $0 \leqq \alpha(q) \leqq 1$.

Now let R be an infinite countable subset of P disjoint with Q. Let us order the elements of the set R into a sequence r_n. For every natural m let us define

$$w_m(q) = v_m(q) \qquad \text{for every } q \in Q,$$

$$w_m(r_n) = \frac{1}{n} \qquad \text{for every natural } n,$$

$$w_m(p) = 0 \qquad \text{for every } p \in P - (Q \cup R).$$

Similarly, let

$$w(q) = \alpha(q) \qquad \text{for every } q \in Q,$$

$$w(r_n) = \frac{1}{n} \qquad \text{for every natural } n,$$

$$w(p) = 0 \qquad \text{for every } p \in P - (Q \cup R).$$

It is easy to see that (i) the functions w and w_m are elements of the space T, (ii) for every natural m we have $x(w_m) = x(v_m)$, (iii) we have $\lim_m w_m(p) = w(p)$ for every $p \in P$. The sequence v_m being a subsequence of t_m, we have

$$|x(w_m)| = |x(v_m)| > m$$

which is a contradiction.

It would be interesting to know the solution of the following problem: Does there exist a pseudocompact space T such that every compact subset of T is necessarily finite?

2.

We have thus far considered the algebraic structure of the spaces $C(T)$ only. It turns out that every completely regular topological space T may be imbedded as a dense subset in a certain space hT such that $C(T)$ and $C(hT)$ are algebraically isomorphic. In the present paragraph we are going to study the connection between the spaces $C(T)$ and $C(hT)$ when they are equipped with a topological structure as well. The most natural topology for spaces $C(T)$ seems to be the following.

If $M \subset T$ and ε is an arbitrary positive number, let $U(M, \varepsilon) = E_x[x \in C(T)\ |x(M)| \leq \varepsilon]$. The topology of $C(T)$ shall be defined by the postulate that the system of all sets $U(K, \varepsilon)$ be a complete system of neighbourhoods of zero, K being an arbitrary compact subset of T, ε an arbitrary positive number. From now on, the symbol $C(T)$ will include both the algebraic and the topological structure of the space considered. Clearly, $C(T)$ becomes thus a convex topological linear space. The space of all linear functionals on $C(T)$ will be denoted by $M(T)$. Clearly every point $t \in T$ can be considered as a linear functional on $C(T)$. We shall not distinguish between points of T and the corresponding members of $M(T)$. The subspace of $M(T)$ consisting of all linear combinations of points of T will be denoted by $P(T)$. The weak topology on $C(T)$ corresponding to $P(T)$ will be called the point topology of $C(T)$. It amounts to the same as considering $C(T)$ as a subspace of a cartesian product of real lines, one coordinate for each point of T.

(2,1) *Let T be a completely regular topological space, K a compact subset of hT. Let B be a pointcompact subset of $C(T)$. Let x_n be an arbitrary sequence of points* *of B. Let x be a limit point of the sequence x_n in the point topology. Then there exists a subsequence x'_n such that*

$$\lim x'_n(w) = x(w)$$

for every $w \in K$.

Proof: For the sake of clarity, the proof will be divided into several parts.

1. Let m be a given natural number. For every $s \in K$ let us take

$$K(m, s) = \underset{w}{E}\left[w \in K,\ |x_i(w - s)| < \frac{1}{m},\quad 1 \leqq i \leqq m\right].$$

The sets $K(m, s)$ form a covering of the compact space K. It follows that this covering contains a finite subcovering consisting of sets $K(m, z)$ where z runs over a suitable finite set $Z_m \subset K$.

Now, let W denote the set of all points $w \in hT$ such that a point $s \in K$ can be found which fulfills $x_i(w) = x_i(s)$ for all i. For every $z \in Z_m$ it is possible to find a $h \in T$ such that $x_i(h) = x_i(z)$ for all i. Clearly we shall have $h \in W$. In this manner we obtain a finite set $H_m \subset W \cap T$. The union of all H_m is a countable subset of $W \cap T$ and will be denoted by H.

2. Let us denote by S the closure (in the point topology) of the set consisting of all x_n. In this section, we shall prove the following assertion.

Let $u \in hT$, $v \in hT$ and suppose that

$$x_i u = x_i v$$

for every i. Then $xu = xv$ for all $x \in S$. To prove this, let us take an arbitrary $x_0 \in S$. Then there exist two points $u_0 \in T$, $v_0 \in T$ such that

$$x_i u_0 = x_i u\ ,$$

$$x_i v_0 = x_i v$$

for $i = 0, 1, 2, \ldots$. Let ε be an arbitrary positive number. Since x_0 lies in the point closure of the set consisting of all points x_n, a natural number j can be found so that

$$|(x_0 - x_j)(u_0 - v_0)| < \varepsilon\ .$$

Now

$$|x_0 u - x_0 v| = |x_0(u - v)| = |(x_0 - x_j)(u - v)| = |(x_0 - x_j)(u_0 - v_0)| < \varepsilon\ .$$

Since ε is arbitrary, our assertion is thus proved.

3. This section is devoted to the proof of the following result.

Let $w_n \in W$, $z \in hT$ and suppose that, for every i,

$$\lim_m x_i w_m = x_i z\ .$$

Then $z \in W$ and the relation

$$\lim_m xw_m = xz$$

holds for every $x \in S$.

To see that, let us take first points $t_m \in K$ such that

$$x_i t_m = x_i w_m$$

for every i. Let x be an arbitrary member of S. It follows from the above considerations that $xt_m = xw_m$ for every m. Since all t_m lie in the compact set K, the sequence $xw_m = xt_m$ is bounded. Suppose now that the relation $\lim_m xw_m = xz$ is not fulfilled. It follows that a subsequence t'_m can be defined such that

$$\lim_m xt'_m = \lambda \neq xz\ .$$

The set K being compact, the sequence t'_m has at least one limit point $s \in K$. Now if a natural i is given, $x_i s$ is a limit point of the sequence $x_i t_m$. Since $x_i t_m = x_i w_m$, the number $x_i s$ is also a limit point of the sequence $x_i w_m$; this sequence, however, has a unique limit point, viz. $x_i z$. Hence

$$x_i s = \lim_m x_i w_m = x_i z$$

for all i. Since $s \in K$ we have thus proved that $z \in W$. According to our preceding result we have also $xs = xz$. This, however, is a contradiction, since

$$xs = \lim xt'_m = \lambda \neq xz\ .$$

This completes the proof.

4. Now we are able to prove our theorem. Let x be an arbitrary limit point of the sequence x_n in the point topology. In the first section of the proof we have constructed a countable set $H \subset W \cap T$. By means of the diagonal process, it is possible to form a subsequence x'_n such that

$$\lim_n x'_n(h) = x(h)$$

for every $h \in H$. We are going to show that

$$\lim x'_n(s) = x(s)$$

for every $s \in K$.

Let s be a fixed point of K. If a natural m is given, let z_m be an element of Z_m for which $s \in K(m, z_m)$. For every z_m let us take the corresponding $h_m \in H_m$. We have thus constructed a sequence $h_m \in H$ such that

$$\lim_m x_i h_m = x_i s$$

for all i.

Now a point $t \in T$ can be found so that

$$xt = xs$$

and at the same time

$$x_i t = x_i s$$

for all i. It follows that $t \in W \cap T$ and that

$$\lim_m x_i h_m = x_i t$$

for all i. Since both h_m and t are contained in W, the result of the preceding section can be applied. We conclude that

$$\lim_m b h_m = bt$$

for all $b \in S$.

Now let $\varepsilon > 0$ be given. Then there exists a natural m_0 such that $m \geqq m_0$ implies $|xh_m - xt| < \varepsilon$. For every natural m we have $\lim_n x'_n h_m = xh_m$. It follows that a number $n(m)$ can be found that such $n \geqq n(m)$ implies

$$|x'_n h_m - xh_m| < \varepsilon .$$

For every $m \geqq m_0$ let

$$S(m, \varepsilon) = \underset{b}{E}[b \in S, |bh_m - bt| < \varepsilon] .$$

The sets $S(m, \varepsilon)$ are open subsets of S (in the point topology). Since $\lim_m bh_m = bt$ for every $b \in S$, the system of all $S(m, \varepsilon)$ forms a covering of S. It follows that there exist natural numbers $m_1, m_2, \ldots, m_r$ so that

$$S(m_1, \varepsilon) \cup \ldots \cup S(m_r, \varepsilon) = S .$$

We shall not forget that all m_i are $\geqq m_0$. Now let $n \geqq \max_{1 \leqq i \leqq r} n(m_i)$. We are going to show that $|x'_n t - xt| < 3\varepsilon$.

First of all, let us take an m_i such that $x'_n \in S(m_i, \varepsilon)$. Now

$$|x'_n t - xt| \leqq |(x'_n - x)\, h_{m_i}| + |x'_n(t - h_{m_i})| + |x(t - h_{m_i})| .$$

Since $n \geqq n(m_i)$, we have $|(x'_n - x)\, h_{m_i}| < \varepsilon$. Since $x'_n \in S(m_i, \varepsilon)$, we have $|x'_n(h_{m_i} - t)| < \varepsilon$. Since $m_i \geqq m_0$ we have $|x(h_{m_i} - t)| < \varepsilon$. This concludes the proof.

(2,2) *Let T be a completely regular topological space. Let B be a pointcompact subset of $C(T)$. Let B be equibounded on every compact $K \subset hT$. Then B is countably compact in the weak topology corresponding to $M(hT)$.*

Proof: This is an immediate consequence of the preceding theorem if we take into account the welknown fact that functionals belonging to $M(hT)$ may be represented as integrals over compact subsets $K \subset hT$.

In the general case, equiboundedness of a $B \subset C(T)$ does not follow from its compactness in the point topology, not even when T is compact. We have shown in [5] for the case of a pseudocompact space T that this implication is true, however, if B is convex. This result, together with its proof, may be extended without essential modifications to the more general case considered here.

(2,3) *Let T be a completely regular topological space. Let B be a symmetrical convex and pointcompact subset of $C(T)$. Let $K \subset hT$ be compact. Then there exists a positive number σ such that*

$$|b(t)| \leqq \sigma$$

for every $b \in B$ and every $t \in K$.

Proof: The proof relies on the fact that the set B is countably compact in the weak topology of $C(T)$ corresponding to the space $P(hT)$. For the sake of brevity, this topology will be called the h-point topology.

If $x \in C(T)$, let $|x|_+ = \max_{t \in K} x(t)$. To prove our theorem, it is sufficient to show that

$$\sup_{b \in B} |b|_+ < \infty$$

suppose that $\sup_{b \in B} |b|_+ = \infty$. Then there exists a $b_1 \in B$ and a $t_1 \in K$ such that $b_1 t_1 > 1$. We shall denote by B_1 the set

$$B_1 = \underset{b}{E}[b \in B, bt_1 > 1]$$

so that $B_1 \neq 0$. Suppose now that the points $t_1, \ldots, t_n \in K$ have been already constructed so that the set

$$B_n = \underset{b}{E}[b \in B, bt_i > i, 1 \leqq i \leqq n]$$

contains at least one point b_n.

The set B being countably compact in the h-point topology, there are non-negative numbers β_i such that $\beta_i = \max_{b \in B} b(t_i)$.

Suppose now that $|B_n|_+ \leqq n + 1$. Choose a real number λ so that

$$1 > \lambda > \max \frac{i + \beta_i}{b_n t_i + \beta_i} .$$

Take an arbitrary $b \in B$. We have then

$$\lambda b_n + (1 - \lambda)\, b \in B$$

and, at the same time,

$$\lambda b_n t_i + (1 - \lambda)\, bt_i \geqq \lambda b_n t_i - (1 - \lambda)\, \beta_i > i$$

so that $\lambda b_n + (1 - \lambda)\, b \in B_n$. It follows that

$$|\lambda b_n + (1 - \lambda)\, b|_+ \leqq n + 1 .$$

On the other hand, we have

$$b = \frac{1}{1-\lambda}\,((\lambda b_n + (1-\lambda)\,b) - \lambda b_n)$$

whence

$$|b|_+ \leqq \frac{1}{1-\lambda}\,(n + 1 + |b|_+)\,.$$

This is a contradition, since b was an arbitrary point of B. We have thus shown that the inequality $|B_n|_+ \leqq n + 1$ is impossible. This assures the existence of a point $b_{n+1} \in B_n$ and a point $t_{n+1} \in K$ such that $b_{n+1}t_{n+1} > n + 1$. We have then

$$b_{n+1} \in B_{n+1} = \underset{b}{E}[b \in B, bt_i > i, 1 \leqq i \leqq n + 1]$$

which completes the induction.

Let us put

$$C_n = \underset{b}{E}[b \in B, bt_i \geqq i, 1 \leqq i \leqq n]\,.$$

The sets C_n are closed in the h-point topology and form a decreasing sequence. They are not empty since $C_n \supset B_n$. The set B being countably compact in the h-point topology, there exists a point $b \in B$ which lies in every C_n. We have then $bt_i \geqq i$ for every i, which is a contradition, since all t_i are contained in the compact set K.

This completes the proof.

In [6], we have introduced the following definition.

Let T be a completely regular topological space. Let f be a function defined on T. The function f is said to be countably continuous on T, if the following condition is fulfilled.

Let $t_0 \in T$ be a limit point of the sequence $t_n \in T$. Then $f(t_0)$ is a limit point of the sequence $f(t_n)$.

(2,4) *Let B be a symmetrical convex and pointcompact subset of $C(T)$. Let r be an arbitrary member of $M(hT)$. Then r is countably continuous on B.*

Proof: Let b_0 be a limit point of the sequence $b_n \in B$ in the point topology. There exists a compact $K \subset hT$ and a number $\lambda > 0$ such that we have $|xr| \leqq \lambda$ for every $x \in C(T)$ which fulfills $|x(K)| \leqq 1$. According to (2,1), a subsequence b'_n can be found such that $b'_n(t) \to b_0(t)$ for every $t \in K$. According to (2,3), the sequence b'_n is equibounded on K. It follows that $b'_n r \to b_0 r$ which completes the proof.

(2,5) *Let B be a symmetrical convex and pointcompact subset of $C(T)$. Let b_n be a sequence of points of B. Let $b \in B$. Then the following properties of b_n are equivalent:*

(1) *the point b is a limit point of the sequence b_n in the point topology,*

(2) *the point b is a limit point of the sequence b_n in the weak topology corresponding to $M(hT)$.*

Proof: Let b be a limit point of the sequence b_n in the point topology. Let $r_1, \ldots, r_p$ be given elements of $M(hT)$, let ε be a given positive number. There exist compact $K_i \subset hT$ and a number $\lambda > 0$ such that we have $|xr_i| \leqq \lambda$ for every $x \in C(T)$ which fulfills $|x(K_i)| \leqq 1$. Put $K = \cup K_i$. According to (2,1), a subsequence b_n' can be found such that $b_n'(t) \to b(t)$ for every $t \in K$. According to (2,3), the sequence b_n' is equibounded on K. It follows that $b_n' r_i \to br_i$ for $i = 1, 2, \ldots, p$, so that there exists a natural n_0 with the following property

$$n \geqq n_0,\ 1 \leqq i \leqq p \ \Rightarrow\ |b_n' r_i - br_i| < \varepsilon .$$

We see thus that condition (2) is a consequence of (1). The other implication being trivial, the proof is complete.

We shall need further a trivial remark concerning convex topological linear spaces. Let X and Y be two dual convex topological linear spaces. Let $A \subset X$ be symmetrical convex and closed. We shall say that A is generated by a set $M \subset X$ if

$$A = M^{rx}.$$

If M is countable, let us form the set W consisting of all linear combinations

$$\lambda_1 m_1 + \ldots + \lambda_r m_r$$

where $m_i \in M$, $\Sigma|\lambda_i| \leqq 1$. If λ_i are restricted to rational numbers only, we obtain a countable set H. It is easy to show that H is dense in A. First of all clearly H is dense in W. Since W is symmetrical and convex, the closure of W (and therefore of H) is equal to W^{rx}. We have, however $M \subset W \subset A$, so that

$$A = M^{rx} \subset W^{rx} \subset A^{rx} = A .$$

It follows that the propositions „A is generated by a countable" set „and A contains a countable dense subset" are equivalent.

(2,6) *Let B be a symmetrical convex and pointcompact subset of $C(T)$. Let B generated by a countable subset of $C(T)$. Let $r \in M(hT)$. Then r is continuous on B, taken in the point topology.*

Proof: According to our assumption and the preceding remark, there exists a countable set W such that B coincides with the point closure of W. Let x_i be a sequence of points of W which contains all points of W.

Since $r \in M(hT)$, there exists a number $\lambda > 0$ and a compact $K \subset hT$ such that

$$x \in C(T),\ |x(K)| \leqq 1 \ \Rightarrow\ |xr| \leqq \lambda .$$

Especially, we have

$$x \in C(T),\ \ x(K) = 0 \ \Rightarrow\ xr = 0 .$$

Let m be a given natural number. For every $s \epsilon K$ let us take

$$K(m, s) = \underset{w}{E}\left[w \epsilon K,\ |x_i(w - s)| < \frac{1}{m},\ 1 \leqq i \leqq m\right].$$

The sets $K(m, s)$ form a covering of the compact set K. This covering contains a finite subcovering consisting of sets $K(m, z)$, where z runs over a suitable finite set $Z_m \subset K$. Let us denote by Z the union of all sets Z_m. Let x be a point of B such that $x(Z) = 0$. We are going to show that $x(K) = 0$.

Let t be an arbitrary point of K. For every natural m, let t_m be a point of Z_m such that $t \epsilon K(m, t_m)$. It follows that, for $m \geqq i$, the following inequality holds

$$|x_i(t_m - t)| < \frac{1}{m},$$

so that we have

$$\lim_m x_i t_m = x_i t$$

for every i. Our point x lies in the point closure of the sequence x_i. We have shown during the proof of theorem (2,1) that these facts imply the relation

$$\lim_m x t_m = xt\ .$$

Now we have $xt_m = 0$ for every m, the points t_m being contained in Z. It follows that $xt = 0$. Here, however, t was an arbitrary point of K. This proves that $x(K) = 0$.

Let v_i be a sequence which contains all points of Z. We are going to show that, for every $\varepsilon > 0$, a natural number n can be found such that

$$b \epsilon B,\ |bv_i| \leqq \frac{1}{n} \quad \text{for} \quad i = 1, 2, \ldots, n \quad \text{implies} \quad |br| < \varepsilon\ .$$

This, clearly, is sufficient to prove our theorem.

Suppose that, for every natural n, a point $b_n \epsilon B$ can be found such that

$$|b_n v_i| \leqq \frac{1}{n} \quad \text{for} \quad i = 1, 2, \ldots, n$$

and, at the same time, $|b_n r| \geqq \varepsilon$. The sequence b_n has at least one limit point $b_0 \epsilon B$ in the point topology. The functional r being countably continuous on B, we have $|b_0 r| \geqq \varepsilon$. Now let i be a fixed natural number, σ an arbitrary positive number. The point b_0 being a limit of the sequence b_n, a natural $n \geqq \max\left(i, \frac{1}{\sigma}\right)$ can be found such that

$$|(b_n - b_0)\, v_i| < \sigma\ .$$

It follows that $|b_0 v_i| \leqq \sigma + |b_n v_i| \leqq \sigma + \frac{1}{n} \leqq 2\sigma$. Since both i and σ were arbitrary, we have $b_0 v_i = 0$ for every v_i, so that $b_0(K) = 0$. This is a contradiction, since $|b_0 r| \geqq \varepsilon$. This completes the proof.

(2,7) *Let B be a symmetrical convex and pointcompact subset of $C(T)$ which contains a countable dense subset. In such a case, the point topology on B coincides with the weak topology corresponding to $M(hT)$.*

Proof: Let $M \subset B$ be closed in the weak topology corresponding to the space $M(hT)$. Let $b_0 \in B$, b_0 non $\in M$. Then there exist $r_i \in M(hT)$, $i = 1, 2, \ldots$ $\ldots, n$ and a positive ε such that

$$b \in B, \quad |(b - b_0)\, r_i| < \varepsilon \quad \Rightarrow \quad b \text{ non } \in M\,.$$

According to (2,6) all r_i are continuous functions on B taken in the point topology. It follows that the point b_0 cannot belong to the point closure of the set M. The set M is therefore closed in the point topology which completes the proof.

3.

In this section we are going to prove the converse result.

(3,1) *Let T be a completely regular topological space. Let r be a linear function on $C(T)$ such that r is countably continuous in the point topology on every symmetrical convex and pointcompact $B \subset C(T)$ which is generated by a countable subset of $C(T)$. Then $r \in M(hT)$.*

Proof: Let us denote by $\mathbf{A}$ the system of all sets $A \in \mathbf{F}^*(hT)$ such that

$$x \in C(T), \quad x(A) = 0 \quad \Rightarrow \quad xr = 0\,.$$

We may limit ourselves to the case $r \neq 0$, so that all sets $A \in \mathbf{A}$ are nonempty. The family $\mathbf{A}$ itself is nonempty, since clearly $hT \in \mathbf{A}$. First of all, let us show that the intersection of an arbitrary finite subfamily of $\mathbf{A}$ is nonvoid. In fact, let $A_1, \ldots, A_n$ be a system of sets $A_i \in \mathbf{A}$ with intersection void. Let $g_i \in C(T)$ be such that

$$0 \leqq g_i(t) \leqq 1, \quad A_i = \underset{t}{E}[t \in hT, g_i(t) = 0]\,.$$

It is easy to see that the function g defined by the relation $g(t) = \Sigma g_i(t)$ is continuous on hT and positive for every $t \in hT$. Let us take

$$e_i(t) = \frac{g_i(t)}{g(t)} \qquad \text{for} \quad i = 1, 2, \ldots, n\,.$$

Let x be an arbitrary member of $C(T)$. Put $x_i(t) = x(t)\, e_i(t)$. It follows that $x = x_1 + \ldots + x_n$ and $x_i(A_i) = 0$ so that $x_i r = 0$. Hence $xr = 0$ which is a contradiction since x was arbitrary and $r \neq 0$.

Let x be an arbitrary member of $C(T)$. We are going to show that there exists an $A \in \mathbf{A}$ such that $x(A)$ is bounded. Suppose this were not true. Let us put

$$P_n = \underset{t}{E}[t \in hT, |x(t)| < n]$$

so that the sets P_n form an ascending sequence of open sets the union of which is the whole space hT. There exist functions $z_n \in C(T)$ such that $z_n(P_n) = 0$ and $z_n r = n$. Let us denote by Z the subset of $C(T)$ consisting of all z_n. Let $M = Z^{P(T)C(T)}$. We are going to show that the set M is pointcompact. Let us denote by S the cartesian product of real lines, one coordinate for each point $t \in T$. It follows that the set $Z^{P(T)S}$ is compact and that $M = C(T) \cap Z^{P(T)S}$. Our assertion will be proved if we show that $Z^{P(T)S} \subset C(T)$.

For that purpose, it is sufficient to show that every function $p \in Z^{P(T)S}$ is continuous on every P_n. This, however, is clear since on every P_n almost all functions z_n are zero. The set M is therefore compact in the point topology, so that r is point continuous on M. This is impossible since $z_n \in M$ and $z_n r = n$. According to (1,3) the intersection of the family $\mathbf{A}$ is a compact subset $K \subset hT$.

First of all, we are going to show that there is a number $\alpha > 0$ such that $|x(T)| \leqq 1$ implies $|xr| \leqq \alpha$. Suppose that such a number does not exist. Then there are $x_n \in C(T)$ such that $|x_n| \leqq 1$ and $|x_n r| > n$. The functions $\frac{1}{\sqrt{n}} x_n$ (extended to βT) form a sequence which converges to zero in the normed space $C(\beta T)$. Let us denote by B the closed symmetrical convex envelope of these functions in $C(\beta T)$. This set is compact in the norm topology of $C(\beta T)$ and therefore compact in the point topology corresponding to T. It follows that there exists a positive β such that

$$\left|\frac{1}{\sqrt{n}} x_n r\right| \leqq \beta$$

whence

$$n < |x_n r| \leqq \sqrt{n}\beta$$

which is a contradition. The existence of a number α with the required property is thus established.

We are going to show now that we have $xr = 0$ for every $x \in C(T)$ which fulfills $x(K) = 0$. To see that, suppose there is a function $x \in C(T)$ such that $x(K) = 0$, $xr \neq 0$.

Let

$$W = \underset{t}{E}\left[t \in hT, \quad |x(t)| \geqq \frac{1}{2\alpha}|xr|\right].$$

We are going to show that W has a nonvoid intersection with every $A \in \mathbf{A}$. In fact, suppose there is an $A \in \mathbf{A}$ such that $W \cap A = 0$. It follows that

$$|x(A)| < \frac{1}{2\alpha}|xr| = \beta .$$

Let us take now the function z defined by

$$z(t) = \tfrac{1}{2}|x(t) + \beta| - \tfrac{1}{2}|x(t) - \beta| .$$

If $t \in A$, we have $-\beta < x(t) < \beta$, so that $x(t) + \beta > 0$, $x(t) - \beta < 0$. It follows that $z(t) = x(t)$, so that $xr = zr$. Now $|z(T)| \leqq \beta$ so that we obtain the following estimate

$$|xr| = |zr| \leqq \alpha \sup |z(T)| \leqq \tfrac{1}{2}|xr| .$$

This contradiction shows that $W \cap A \neq 0$ for every $A \in \mathbf{A}$. It follows from lemma (1,3) that the system $\mathbf{A} \cap W$ has a nonvoid intersection. This is impossible, however, the set W being disjoint with K.

Now it is easy to show that $x \in C(T)$, $|x(K)| \leqq 1$ implies $|xr| \leqq \alpha$. In fact, if $x \in C(T)$, $|x(K)| \leqq 1$, the function z defined by

$$z(t) = \tfrac{1}{2}|x(r) + 1| - \tfrac{1}{2}|x(t) - 1|$$

coincides on K with x, so that $xr = zr$. Since $|z(T)| \leqq 1$, we have $|zr| \leqq \alpha$. It follows that $r \in M(hT)$ and the proof is concluded.

4.

The results of the preceding sections may be resumed in the following manner.

(4,1) *Let T be a completely regular topological space. Let r be a linear form defined on $C(T)$. Then the following conditions are equivalent*

(1) *r is a functional belonging to $M(hT)$*

(2) *r is pointcontinuous on every symmetrical convex and pointcompact subset of $C(T)$ which contains a dense countable subset*

(3) *r is weakly continuous on every symmetrical convex and pointcompact subset of $C(T)$ which contains a dense countable subset*

(4) *r is pointcontinuous on every symmetrical convex and weakly compact subset of $C(T)$ which contains a dense countable subset*

(5) *r is weakly continuous on every symmetrical convex and weakly compact subset of $C(T)$ which contains a dense countable subset.*

Proof: The fact that condition (2) is a consequence of (1) forms the contents of theorem (2,6). The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (5) and (2) $\Rightarrow$ (4) $\Rightarrow$ (5) being trivial it is sufficient to show that (1) follows from (5).

Suppose that r fulfills condition (5). Using lemma (2,7) it is easy to see that r fulfills (2) as well. The implication (2) $\Rightarrow$ (1) is contained in theorem (3,1). This completes the proof.

In a similar manner, we obtain the following series of implications.

(4,2) *Let T be a completely regular topological space. Let r be a linear form defined on $C(T)$. Then the following conditions are equivalent.*

(1) *r belongs to $M(hT)$*

(2) *r is countably continuous in the point topology on every symmetrical convex and pointcompact subset of $C(T)$*

(3) *r is weakly countably continuous on every symmetrical convex and point-compact subset of $C(T)$*

(4) *r is countably continuous in the point topology on every symmetrical convex and weakly compact subset of $C(T)$*

(5) *r is weakly countably continuous on every symmetrical convex and weakly compact subset of $C(T)$.*

Proof: The fact that condition (2) is a consequence of (1) forms the contents of theorem (2,4). The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (5) and (2) $\Rightarrow$ (4) $\Rightarrow$ (5) being trivial, it is sufficient to show that (1) follows from (5). Suppose that r fulfills condition (5). Using lemma (2,5) it is easy to see that r fulfills condition (2) as well. The implication (2) $\Rightarrow$ (1) is contained in theorem (3,1). This completes the proof.

It is easy to see that the essential point of the proof of our result is the theorem (2,1). It is thus natural to ask whether the assumptions made are really necessary. A simple example which we exhibit below shows that the assumption that K be compact cannot be omitted. When the manuscript of the present paper was nearly complete, the author has perceived that the proof of (2,1) may be considerably shortened. The connection between these two methods of proof is, however, not quite clear, so that the present proof does not seem to lose its interest. Anyhow, though much longer, it is much more geometrically intuitive. Besides, the simplified method mentioned may be used to prove further results, which we intend to treat summarily in another paper.

(4,3) Let T consist of all continuous functions defined on the interval $0 \leqq p \leqq 1$. If $t_1 \in T$, $t_2 \in T$, let us define

$$\varrho(t_1, t_2) = \max_{0 \leqq p \leqq 1} |t_1(p) - t_2(p)| .$$

In this way, the space T becomes a metric space, so that T is completely regular.

Let r_n be a sequence of rational numbers $0 < r_n < 1$ and such that r_n is dense in the whole interval $0 \leqq p \leqq 1$.

For every natural i, let us define a function b_i on T by means of the following relation

$$b_i(t) = t(r_i) .$$

It is easy to see that all $b_i \in C(T)$. Let us denote by B the subset of $C(T)$ consisting of all functions of the form

$$x(t) = t(p)$$

where p is an arbitrary point $0 \leqq p \leqq 1$. Clearly, the sequence b_i is dense in B, taken in the point topology. At the same time, the set B is compact in the point topology.

For every natural m, let t_m be defined by

$$t_m(p) = p^m .$$

The point t_0 will be defined by

$$t_0(p) = 0 \quad \text{for} \quad 0 \leqq p \leqq 1 .$$

Clearly we have

$$\lim_m b_i(t_m) = \lim_m r_i^m = 0 = b_i(t_0)$$

for every i. The function $b \in C(T)$ defined by $b(t) = t(1)$ clearly belongs to the point closure of the sequence b_i. We have, however

$$b(t_m) = 1$$

for every m, while $b(t_0) = 0$. The relation

$$\lim_m b(t_m) = b(t_0)$$

is therefore not fulfilled.

Applications of the present results will be collected in another paper.

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Резюме

О ПРОСТРАНСТВАХ НЕПРЕРЫВНЫХ ФУНКЦИЙ

ВЛАСТИМИЛ ПТАК (Vlastimil Pták), Прага.

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Настоящая работа является продолжением исследований автора в области псевдокомпактных подмножеств выпуклых топологических линейных пространств. Дело в том, что, как выясняется, методы, использованные в работах [5] и [6] для случая псевдокомпактного пространства, можно применить (несколько их видоизменив) при доказательстве аналогичных результатов, касающихся оболочки Хьюитта (Hewitta) произвольного вполне регулярного топологического пространства. Большинство этих результатов представляет обобщение соответствующих результатов [5].

Если T — вполне регулярное топологическое пространство, мы обозначим через $C(T)$ линейное пространство всех непрерывных функций, определенных на T. Пространство $C(T)$ топологизировано при помощи семейства псевдонорм $|x|_K = \max_{t \in K} |x(t)|$, где K пробегает все компактные подмножества пространства T. Таким образом, $C(T)$ становится выпуклым топологическим линейным пространством. Обозначим через $M(T)$ пространство всех линейных функционалов определенных на $C(T)$. Известно, что каждое вполне регулярное топологическое пространство может быть погружено в пространство hT (оболочка Хьюитта пространства T), так, что T плотно в hT и каждое $x \in C(T)$ допускает непрерывное продолжение на все пространство hT. Отсюда следует, что с точки зрения алгебраической структуры пространства $C(T)$ и $C(hT)$ тождественны. Если же принимается во внимание и их топологическая структура, то эти пространства нетрудно различить, так как, очевидно, пространство $M(T)$ можно рассматривать как подпространство пространства $M(hT)$. Предположим теперь, что дано какое-либо пространство $C(T)$. Так как $C(T)$ и $C(hT)$ алгебраически изоморфны, то каждый элемент пространства $M(hT)$ представляет линейную форму на $C(T)$. В общем случае эта форма не будет непрерывной на $C(T)$. Напрашивается, однако, вопрос, не сохраняет ли она все же некоторую непрерывность, хотя бы и в ослабленной форме, которая позволила бы нам охарактеризовать члены $M(hT)$ только при помощи $C(T)$. С другой стороны предположим, что в $M(hT)$ введена произвольная топология такая, что $C(hT)$ и $M(hT)$ двойственны друг другу. Тогда $M(T)$ плотно в $M(hT)$. Возникает вопрос, нельзя ли рассматривать $M(hT)$ в некотором смысле как пополнение $M(T)$. Эти два вопроса тесно связаны один с другим и их обсуждение составляет предмет настоящей работы.

Первий параграф содержит важнейшие теоремы теории оболочки Хьюитта. Результаты этого раздела не являются по существу новыми. Большая часть их была доказана (в явной или неявной форме) уже Хьюиттом; однако в настоящей заметке мы даем единую теорию оболочки Хьюитта, опирающуюся на весьма простую лемму, доказанную в предыдущей работе. Оказывается, что при использовании этой леммы доказательства весьма упрощаются, что и послужило поводом для включения этого параграфа в настоящую работу.

Второй и третий параграфы посвящаются доказательству главного результата. Само доказательство подразделяется на несколько частей, из которых самая важная является непосредственным обобщением теоремы (1,2) работы [5]. Полученные результаты сформулированы в теоремах (4,1) и (4,2).

Эти результаты допускают интересные применения, которые будут опубликованы позднее.