# CONCERNING THE GEOMETRY OF STOCHASTIC DIFFERENTIAL EQUATIONS AND STOCHASTIC FLOWS

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**Abstract**

Le Jan and Watanabe showed that a non-degenerate stochastic flow $\{\xi_t : t \geq 0\}$ on a manifold $M$ determines a connection on $M$. This connection is characterized here and shown to be the Levi-Civita connection for gradient systems. This both explains why such systems have useful properties and allows us to extend these properties to more general systems. Topics described here include: moment estimates for $T\xi_t$, a Weitzenböck formula for the generator of the semigroup on p-forms induced by the flow, a Bismut type formula for $d \log p_t$ in terms of an arbitrary metric connection, and a generalized Bochner vanishing theorem.

## 1 Introduction and Notations

**A.** Consider a Stratonovich stochastic differential equation

$$dx_t = X(x_t) \circ dB_t + A(x_t)dt \tag{1}$$

on an n-dimensional $C^\infty$ manifold $M$, e.g. $M = \mathbb{R}^n$. Here $A$ is a $C^\infty$ vector field on $M$, so $A(x)$ lies in the tangent space $T_xM$ to $M$ at $x$ for each $x \in M$, while $X(x) \in \mathcal{L}(\mathbb{R}^m; T_xM)$, the space of linear maps of $\mathbb{R}^m$ to $T_xM$, for $x \in M$, and is $C^\infty$ in $x$. The noise $B.$ is a Brownian motion on $\mathbb{R}^m$ defined on a probability space $\{\Omega, \mathcal{F}, P\}$.

For each $e \in \mathbb{R}^m$ let $X^e$ be the vector field given by $X^e(x) = X(x)(e)$. Recall that for each given $x_0 \in M$ equation (1) has a maximal solution $\{\xi_t(x_0) : 0 \leq t < \zeta(x_0)\}$, defined up to an explosion time $\zeta(x_0)$, and unique up to equivalence. The solutions form a Markov process on $M$. Let $\{P_t^0 : t \geq 0\}$

be the associated (sub)-Markovian semigroup, and let $\mathcal{A}$ be the infinitesimal generator. In this article we shall assume that (1) is *non-degenerate*, i.e. $X(x) : \mathbb{R}^m \to T_xM$ is surjective for each $x$, or equivalently that $\mathcal{A}$ is elliptic. Then a Riemannian metric is induced on $M$ with inner product $<,>_x$ on $T_xM$ given by $< X(x)e_1, X(x)e_2 >_x = < e_1, e_2 >_{\mathbb{R}^m}$ provided that $e_1$, $e_2$ are orthogonal to $N(x)$, the kernel of $X(x)$ in $\mathbb{R}^m$. The generator has the form

$$\mathcal{A}(f)(x) = \frac{1}{2}\Delta^0 f(x) + \left\langle \frac{1}{2}\sum_1^m \nabla X^{e_i}(X^{e_i}(x)) + A(x),\ \text{grad}\ f(x) \right\rangle_x, \quad (2)$$

where $e_1, \ldots, e_m$ is an orthonormal basis for $\mathbb{R}^m$. Here $\nabla$ denotes covariant differentiation with respect to the Levi-Civita connection, so $\nabla X^{e_i}$ is a linear map of tangent vectors to tangent vectors, $\nabla X^{e_i}(v) \equiv \nabla_v X^{e_i}$, and $\Delta^0$ is the Laplace-Beltrami operator on functions: $\Delta^0 f = \text{trace}\ \nabla(\text{grad} f)$.

**B.** Our motivating examples are the *gradient Brownian systems*. Here we have an immersion: $g : M \to \mathbb{R}^m$, e.g. the inclusion of the space of $S^n$ in $\mathbb{R}^{n+1}$ (with $m = n+1$), and $X(x) : \mathbb{R}^m \to T_xM$ is the orthogonal projection using $T_xg$ to identify $T_xM$ with a subspace of $\mathbb{R}^m$. The Riemannian inner product $<,>_x$ is just that which makes $T_xg$ an isometry. Set $Y(x) = T_xg : T_xM \to \mathbb{R}^m$. Let $Z$ be a vector field then $Y(x)Z(x) \in \mathbb{R}^m$ for each $x$, giving $Y(\cdot)Z(\cdot) : M \to \mathbb{R}^m$, with differential $d(Y(\cdot)Z(\cdot)) : T_xM \to \mathbb{R}^m$, $x \in M$. It is a fundamental result that if we project this differential to $T_xM$ we obtain the Levi-Civita covariant derivative of $Z$ in the direction of $v$, i.e.

$$\nabla Z(v) = X(x)\left[d(Y(\cdot)Z(\cdot))_x(v)\right], \qquad v \in T_xM, \quad (3)$$

e.g. see [KN69a].

Consider the special case $Z(x) = X^e(x)$ some $e \in \mathbb{R}^m$. Then by (3), for any $v \in T_xM$,

$$\nabla X^e(v) = X(x)\left[d\left(Y(\cdot)X(\cdot)e\right)(v)\right].$$

But $Y(x)X(x)e = e - P_N(x) = P_T(x)$ say, where $P_N(x) : \mathbb{R}^m \to \mathbb{R}^m$ is the orthogonal projection onto the normal space $N(x)$ at $x$, and so, e.g. by differentiating the identity $P_T(x)e = P_T(x)P_T(x)e$, we see that if $P_T(x)e = e$, i.e. if $e \in \text{Image}\ Y(x)$, then $\nabla X^e(v) = 0$ all $v \in T_xM$ (for another proof see §2A below). Alternatively this can be seen from the fact that $\nabla X^{\cdot}(\cdot)$ is essentially the shape operator of the immersion. See e.g. [Elw82]. In particular from this we can conclude that the term $\sum_1^m \nabla X^{e_i}(x)(X^{e_i}(x))$ in (2) vanishes so that $\mathcal{A}f(x) = \frac{1}{2}\Delta^0 f(x) + \langle A(x), \text{grad}\ f(x)\rangle_x$. These identities are behind the fact that gradient systems have particularly nice properties from the point of view of their solution flows, see e.g. [Kus88], [EL94], [ER96], [EY93], [Li94a], and from the point of view of their Itô maps [AE95]. Here we shall show that many of these constructions and properties are also true for general non-degenerate systems provided that we use connections with torsion. Our starting point is:

**Theorem 1.1** *For an arbitrary non-degenerate SDE (1) there is a unique affine connection $\tilde{\nabla}$ on $M$ such that*

$$\left(\tilde{\nabla} X^e\right)(v) = 0, \qquad all\ v \in T_xM, e \in [ker X(x)]^\perp. \tag{4}$$

*It is given by $\tilde{\nabla} Z(v) = \breve{\nabla} Z(v)$ for*

$$\breve{\nabla} Z(v) = X(x) d[Y(\cdot)Z(\cdot)](v), \qquad v \in T_xM \tag{5}$$

*for $Y(x) : T_xM \to \mathbb{R}^m$ the adjoint of $X(x)$, and is metric.*

This is in fact the connection defined by LeJan and Watanabe [LW82].

**C.** The scheme of the paper is as follows: Theorem 1 is proved in §2 together with criteria for $\breve{\nabla}$ to be the Levi-Civita connection and to be torsion-skew symmetric; in §3 we extend results of [EY93] on the conditional expectation of the derivative flow $T_{x_0}\xi_t$ given $\{\xi_t(x_0) : 0 \le t \le T\}$, i.e. filtering out the extraneous noise; in §4 the 'spectral positivity' estimates of [Li94a], see also [ER96], for moment exponents are extended to S.D.E. with $\breve{\nabla}$ torsion skew symmetric; and in §5 we give an expression for the generator of the semigroup $P_t^q$ on $q$-forms given by $P_t^q(\phi) = \mathbb{E}\xi_t^*(\phi)$ of the form $P_t^q = -(\bar{\delta} d + d\bar{\delta})$ and a Weitzenböck formula. An expression for the curvature is derived in Appendix I.

**Remark:**

For simplicity in this expository article we mainly treat equations like (1) with finite dimensional noise whereas stochastic flows correspond canonically to Gaussian measures on the space of vector fields of $M$, [Bax84], [Kun90], [LW82], which may have support on an infinite dimensional space. Essentially this means that $\mathbb{R}^m$ should be replaced by a Hilbert space of vector fields with $X(x)$ the evaluation map (the major role is then taken by the reproducing kernel of the Gaussian measure) [Bax76]. See Appendix II. More generally Gaussian measures on Hilbert spaces of sections of a vector bundle determines a connection on that bundle (and all metric connections arise this way, see §2H below). Finally we also restrict ourselves here to non-degenerate SDE, but a degenerate SDE induces in the same way a differential operator, a 'semi-connection'. These aspects and other more detailed results will be treated in a forthcoming article. See also [ELLb].

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## 2 Existence and basic properties

*Proof of Theorem 1.1*

Let $\breve{\nabla}$ be defined by (5). It is easy to see that it has the linearity and derivation properties, which ensures that it is a connection. Let $\tilde{\nabla}$ be any affine connection on $M$, $Z$ a vector field, and let $v \in T_{x_0}M$. Then

$$Z(x) = X(x)Y(x)Z(x), \qquad x \in M, \tag{6}$$

whence $\tilde{\nabla}Z(v) = \tilde{\nabla}X(v)Y(x_0)Z(x_0) + \breve{\nabla}Z(v)$ using (5). Setting $\tilde{e} = Y(x_0)Z(x_0)$ we see

$$\tilde{\nabla}Z(v) = \tilde{\nabla}X^{\tilde{e}}(v) + \breve{\nabla}Z(v). \tag{7}$$

Taking $\tilde{\nabla} = \breve{\nabla}$, since $Z(x_0)$ is arbitrary we see $\breve{\nabla}$ satisfies the defining criterion (4), giving existence. Assuming $\tilde{\nabla}$ satisfies (4) we see $\tilde{\nabla}Z(v) = \breve{\nabla}Z(v)$, giving uniqueness. To check that $\breve{\nabla}$ is metric it is enough to show that

$$d\left(< Z(\cdot), Z(\cdot) >\right)(v) = 2 < \breve{\nabla}Z(v), Z(x_0) >_{x_0} .$$

In fact

$$\begin{aligned} < \breve{\nabla}Z(v),\, Z(x_0) >_{x_0} &= \langle d[Y(\cdot)Z(\cdot)](v),\, Y(x_0)Z(x_0)\rangle_{\mathbb{R}^m} \\ &= \frac{1}{2} d\left\langle Y(\cdot)Z(\cdot),\, Y(\cdot)Z(\cdot)\right\rangle_{\mathbb{R}^m}(v) \\ &= \frac{1}{2} d\left\langle Z(\cdot),\, Z(\cdot)\right\rangle_{\cdot}(v). \end{aligned}$$

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**Remark:**

Note that $Y(x)Z(x) = \sum_1^m < X^{e_i}(x), Z(x) > e_i$ and by (6) and the equation which follows:

$$\begin{aligned} \breve{\nabla}Z(v) &= \textstyle\sum_1^m X^{e_i} d < X^{e_i}, Z > (v) \\ &= \tilde{\nabla}Z(v) - \tilde{\nabla}_v X^{e_i} < X^{e_i}(x_0), Z(x_0) > \end{aligned} \tag{8}$$

for any affine connection $\tilde{\nabla}$ on $M$.

**B.** In a local chart about $x_0 \in M$ we can take $\tilde{\nabla}$ in the above proof to be the usual derivative so that (7) becomes

$$DZ(x_0)(v) = DX^{\tilde{e}}(x_0)(v) + \breve{\nabla}Z(v)$$

where $\tilde{e} = Y(x_0)Z(x_0)$, (using local representations for $Z$, $X^{\tilde{e}}$, and $v$). But for $\breve{\Gamma}$ the Christoffel symbol of $\breve{\nabla}$ in our chart

$$\breve{\nabla}Z(v) = DZ(x_0)(v) + \breve{\Gamma}(x_0)(v, Z(x_0))$$

giving

$$\breve{\Gamma}(x_0)(v,w) = -DX(x_0)(v)\,(Y(x_0)w)\,, \qquad v, w \in R^n. \tag{9}$$

Equivalently

$$\breve{\Gamma}^i_{jk} = -\sum_{r=1}^{m}\sum_{l=1}^{n} \frac{\partial X(x_0)^{r,i}}{\partial x^j} X(x_0)^{r,\ell} g_{k\ell}, \tag{10}$$

where $\{X(x)^{r,i}\}$, $\{1 \le i \le n\}$, $\{1 \le r \le m\}$ is the matrix representing $X(x) : \mathbb{R}^m \to \mathbb{R}$, i.e. $X(x)^{r,i} =< X(e_r), f_i >$ for $\{e_i\}$ and $\{f_i\}$ orthonormal bases for $\mathbb{R}^m$ and $T_xM$ respectively, and $\{g_{k\ell}\}$ the metric tensor. This shows that $\breve{\nabla}$ is the LeJan-Watanabe connection defined in [LW82].

**C.** Equivalent definitions and properties.

**Lemma 2.1** *For any orthonormal base $\{e_i\}$ of $\mathbb{R}^m$ and $v \in T_{x_0}M$ we have*

$$(i) \qquad \breve{\nabla}Z(v) = \frac{d}{dt}\sum_{1}^{m} X^{e_i}(x_0)\,\langle Z(\sigma(t)), X^{e_i}(\sigma(t))\rangle_{\sigma(t)}\Big|_{t=0} \tag{11}$$

*where $\sigma : [-\delta, \delta] \to M$ is a $C^1$ curve with $\sigma(0) = x_0$ and $\dot{\sigma}(0) = v$.*

$$(ii) \qquad \breve{\nabla}Z(v) = \sum_{1}^{m}[X^i, V](x_0) < X^i(x_0), Z(x_0) > + [V, Z](x_0)$$

*where $V$ is any smooth vector field with $V(x_0) = v$.*

**Proof.** Since $\breve{\nabla}$ is metric the right hand side of (11) is just

$$\sum_{1}^{m} X^{e_i}(x_0)\left\{\left\langle \breve{\nabla}Z(v),\, X^{e_i}(x_0)\right\rangle_{x_0} + \left\langle Z(x_0),\, \breve{\nabla}X^{e_i}(v)\right\rangle_{x_0}\right\}.$$

This is independent of the choice of basis. Choose $\{e_i\}$ so that $e_1 \dots, e_n$ span $[\ker X(x_0)]^{\perp}$, i.e. are in the image of $Y(x_0)$. Then $X^{e_i}(x_0) = 0$ if $i > n$ while $\breve{\nabla} X^{e_i}(v) = 0$ if $1 \le i \le n$ by definition of $\breve{\nabla}$. Since $X^{e_i}(x_0), 1 \le i \le n$, form an orthonormal base for $T_{x_0}M$ the result (i) follows.

For (ii) write

$$[V, Z] = [V, \sum_1^m < X^i, Z > X^i\,]$$

and expand. The use of (8) yields (ii).

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By a similar proof to that above, we obtain a necessary and sufficient condition for a connection to be a metric connection: for simplicity write $X^i \equiv X^{e_i}$,

**Lemma 2.2** *A connection* $\tilde{\nabla}$ *is a metric connection if and only if*

$$\sum_1^m X^{e_i} < Z, \tilde{\nabla}_v X^{e_i} > + \sum_1^m \tilde{\nabla}_v X^{e_i} < Z, X^{e_i} >= 0, \tag{12}$$

*for all vector fields* $Z$.

**Proof.** Take $v \in T_xM$ . If $\tilde{\nabla}$ is metric then

$$\begin{aligned} d < Z(\cdot), Z(\cdot) > (v) &= \sum_1^m d\left(< Z, X^i >< Z, X^i >\right)(v) \\ = \quad & 2 < \tilde{\nabla} Z(v), Z > +2\sum_1^m < Z, \tilde{\nabla} X^i(v) >< Z, X^i > \end{aligned}$$

giving (12) by polarization. Now suppose (12) holds for a connection $\tilde{\nabla}$, then

$$\sum_1^m < Z, X^i >< Z, \tilde{\nabla} X^i(v) >= 0. \tag{13}$$

On the other hand, by (6)

$$\begin{aligned} & \tilde{\nabla} Z(v) = \tilde{\nabla}_v Y(x) Z(x) + X(x) d[Y(x)Z(x)](v) \\ = \quad & \sum_1^m \tilde{\nabla}_v X^i < Z, X^i > + \sum_1^m X^i d < Z(-), X^i(-) > (v) \end{aligned}$$

giving

$$\sum_1^m X^i d < Z(-), X^i(-) > (v) = \tilde{\nabla} Z(v) - \sum_1^m \tilde{\nabla}_v X^i < Z, X^i > .$$

consequently

$$\begin{aligned} & d < Z(\cdot), Z(\cdot) > (v) = \sum_1^m d < Z(-), X^i(-) >^2 (v) \\ = \quad & 2 \sum_1^m < Z(x), X^i(x) > d < Z(-), X^i(-) > (v) \\ = \quad & 2 < Z, \tilde{\nabla} Z(v) > -2 \sum_1^m < Z, \tilde{\nabla}_v X^i >< Z, X^i > \\ = \quad & 2 < Z, \tilde{\nabla}_v Z >, \end{aligned}$$

using (13), and so $\tilde{\nabla}$ is a metric connection. //

**D.** Recall that for any connection $\tilde{\nabla}$ on $M$ the torsion is a bilinear map from tangent vectors to tangent vectors, $\tilde{T} : TM \oplus TM \to TM$, given by

$$\tilde{T}(U(x_0), V(x_0)) = \tilde{\nabla} V(U(x_0)) - \tilde{\nabla} U(V(x_0)) - [U, V](x_0) \tag{14}$$

for vector fields $U$, $V$.

Let $v_1, v_2 \in T_{x_0}M$. There are the vector fields $Z^{v_1}$, $Z^{v_2}$ given by

$$Z^{v_i} = X(x)Y(x_0)v_i, \qquad i = 1, 2.$$

By definition

$$\breve{\nabla} Z^{v_i}(v) = 0, \qquad \text{any } v \in T_{x_0}M.$$

Thus

$$\breve{T}(v_1, v_2) = -\,[Z^{v_1},\, Z^{v_2}]\,(x_0). \tag{15}$$

Alternatively using the Levi-Civita connection in (5)

$$\breve{\nabla} Z(v) = X(x_0)\nabla Y(v)Z(x_0) + \nabla Z(v) \tag{16}$$

whence by (14)

$$\begin{aligned} \breve{T}(v_1, v_2) \quad = \quad & X(x_0)\,(\nabla Y(v_1)(v_2) - \nabla Y(v_2)(v_1)) \\ & + \nabla Z^{v_2}(v_1) - \nabla Z^{v_1}(v_2) - [Z^{v_1}, Z^{v_2}]. \end{aligned}$$

Thus by (14) and the standard formula for exterior differentiation:

$$\breve{T}(v_1, v_2) = X(x_0)dY(v_1, v_2), \qquad v_1, v_2 \in T_{x_0}M. \tag{17}$$

**E.** For any connection $\tilde{\nabla}$ on $M$, there is an adjoint connection $\tilde{\nabla}'$ on $M$ defined by

$$\begin{aligned} \tilde{\nabla}' Z(v) &= \tilde{\nabla} Z(v) - \tilde{T}(v, Z(x_0)) \\ &= \tilde{\nabla} V(Z(x_0)) - [Z, V](x_0). \end{aligned}$$

Here $V$ is a vector field such that $V(x_0) = v$. In terms of Christoffel symbols ([Dri92]) this is equivalent to $\tilde{\Gamma}'^{i}_{jk} = \tilde{\Gamma}^i_{kj}$. If $\hat{\nabla}$ denotes adjoint of $\breve{\nabla}$ we see that $\hat{\nabla} Z(v) = [Z^v, Z](x_0)$.

A connection $\tilde{\nabla}$ on a Riemannian manifold $M$ is *torsion skew symmetric*, see [Dri92], if $u \to \tilde{T}(u, v)$ is skew symmetric as a map $T_{x_0}M \to T_{x_0}M$ for all $v \in T_{x_0}M$, all $x_0 \in M$. We have:

**Lemma 2.3** *A metric connection $\tilde{\nabla}$ on a Riemannian manifold $M$ is torsion skew symmetric if and only if its adjoint connection is metric. If so the geodesics for $\tilde{\nabla}$ are those of the Levi-Civita connection and the (usual) Laplace-Beltrami operator acting on a function $f$, $\Delta^0 f$, is given by the trace of $\tilde{\nabla}(gradf)$.*

**Proof.** See [Dri92] and also [KN69b] (the last part also comes from the next proposition).

**Proposition 2.4** *The connection $\breve{\nabla}$ is*

1. *the Levi-Civita connection if and only if $\nabla Z^v$ vanishes at $x_0$ for all $v \in T_{x_0}M$.*
2. *torsion skew symmetric if and only if $\nabla Z^v|_{T_{x_0}M} : T_{x_0}M \to T_{x_0}M$ is skew symmetric, all $v \in T_{x_0}M$, or equivalently $\nabla_v Z^w + \nabla_w Z^v = 0$ for any $w, v \in T_xM$, or $\breve{\nabla}_U V + \breve{\nabla}_V U = \nabla_U V + \nabla_V U$ for all vector fields $U$ and $V$.*

*Also it is Levi-Civita if and only if $X(x)dY(u, v) = 0$ for all $u, v \in T_xM$, all $x \in M$.*

**Proof.** The first part comes from the defining property of $\breve{\nabla}$ and the third part comes from (17). For the second part, first observe by the definition of torsion

$$\breve{T}(u, v) = \breve{\nabla}_v Z^u - \nabla_v Z^u - \left[\breve{\nabla}_u Z^v - \nabla_u Z^v\right].$$

and so by (7):

$$\breve{T}(u,v)=\sum_1^m X^i < v, \nabla X^i(u) > - \sum_1^m X^i < u, \nabla X^i(v) > . \tag{18}$$

We have:

$$\begin{aligned} & < \breve{T}(u,v), w > \\ = \quad & \sum_1^m < X^i, w >< v, \nabla X^i(u) > - \sum_1^m < X^i, w >< u, \nabla X^i(v) > . \end{aligned}$$

However the second term is anti-symmetric in $u$ and $w$ by (12). Thus

$$\begin{aligned} & < \breve{T}(u,v), w > + < \breve{T}(w,v), u > \\ = \quad & \sum_1^m < X^i, w >< v, \nabla X^i(u) > + \sum_1^m < X^i, u >< v, \nabla X^i(w) > \\ = \quad & \langle \nabla_u Z^w, v\rangle + \langle \nabla_w Z^u, v\rangle \\ = \quad & - < w, \nabla Z^v(u) > - < u, \nabla Z^v(w) >, \end{aligned}$$

since $d < Z^w, Z^v > (u) = 0$ and $d < Z^u, Z^v > (w) = 0$.

Also if $U$ and $V$ are vector fields, by (8)

$$\breve{\nabla}_V U = \sum_1^m X^i < U, \nabla_V X^i > + \nabla_V U$$

and so

$$\breve{\nabla}_V U + \breve{\nabla}_U V = \nabla_V U + \nabla_U V + A$$

for

$$A = \sum_1^m X^i < U, \nabla_V X^i > + \sum_1^m X^i < V, \nabla_U X^i > .$$

But $\breve{T}$ is skew symmetric if and only if $A \equiv 0$. //

**Corollary 2.5** *If $\breve{\nabla}$ is torsion skew symmetric then*

$$\breve{T}(u,v) = 2\sum_{i=1}^m X^i < u, \nabla X^i(v) >$$

*and the Levi -Civita connection can be expressed in terms of the LeJan-Watanabe connection by:*

$$\nabla Z(v) = \breve{\nabla} Z(v) - \frac{1}{2}\breve{T}(Z(x_0), v). \tag{19}$$

*In particular* $\nabla X^i(X^i) = 0$ *for each* $i$.

**Remark:** Most of the results for gradient Brownian systems carry over to the case when $\breve{\nabla}$ is torsion free and, with some adaptation, to the torsion skew symmetric case or even more generally.

**F.** Let $f : M \to \mathbb{R}$ be $C^2$. Then Itô's formula gives

$$\begin{aligned} f(\xi_t(x)) &= f(x_0) + \int_0^t df\,(X(\xi_s(x_0))dB_s) \\ &\quad + \frac{1}{2}\int_0^t \text{trace}\breve{\nabla}(\text{grad} f)(\xi_s(x_0))ds \\ &\quad + \int_0^t A(\xi_s(x_0))ds, \qquad 0 \le t < \zeta(x_0) \end{aligned}$$

since the Stratonovich term $\sum_1^m \breve{\nabla} X^{e_i}(X^{e_i}(x))$ vanishes. Thus as shown in [LW82], the generator is given by

$$\mathcal{A}^0(f) = \frac{1}{2}\text{trace}\breve{\nabla}(\text{grad} f) + < A(\cdot), \text{grad} f > . \tag{20}$$

Note also that the vanishing of the Stratonovich term means that *(1) can be considered as an Itô equation w.r.t.* $\breve{\nabla}$, e.g. see [Elw82], and *the solutions* $\{\xi_t(x_0) : t \ge 0\}$ *will be* $\breve{\nabla}$*-martingales if* $A \equiv 0$, [Eme89]. Furthermore by Corollary 2.5 if $\breve{\nabla}$ is torsion skew symmetric (1) will be an Itô equation for the Levi-Civita connection and the solution will be a Brownian motion with drift $A$.

**G.** Example: Invariant SDE on Lie groups: c.f. [Dri92]. Let $M$ be a Lie group and suppose (1) is a left invariant SDE, with $A = 0$ for simplicity. For $g \in G$ let $R_g : G \to G$ and $L_g : G \to G$ be right and left translations by $g$. Then

$$L_g X(x)(e) = X(gx)e, \qquad g, x \in G, e \in \mathbb{R}^m.$$

We can suppose $m = n$ since $\text{Ker}X(x)$ is independent of $x$. The metric induced on $G$ will be left invariant. We can treat $X(id) : \mathbb{R}^m \to T_{id}G$ as an identification of $\mathbb{R}^m$ with the Lie algebra $\mathcal{G} = T_{id}G$ of $G$, and then $Y$ becomes the

Maurer-Cartan form. For $v \in T_{x_0}G$ the vector field $Z^v = X(\cdot)Y(x_0)(v)$ of §1D is just the left-invariant vector field through $v$. If $\tilde{\nabla}$ is the flat left invariant connection on $G$ then $\tilde{\nabla} Z^v \equiv 0$, and so by definition $\breve{\nabla} = \tilde{\nabla}$. The torsion

$$\begin{aligned} \breve{T}(v_1, v_2) &= -[Z^{v_1}, Z^{v_2}](x_0) \\ &= X(x)dY(v_1, v_2) \end{aligned}$$

by (15) and (17).

Recall that for $\alpha \in \mathcal{G}$,

$$ad(\alpha) : \mathcal{G} \to \mathcal{G}$$

is given by

$$ad(\alpha)\beta = [\alpha, \beta].$$

Taking $x_0 = id \in G$ we see $\breve{T}(v_1, v_2) = -ad(v_1)(v_2)$ and so $\breve{\nabla}$ is torsion skew symmetric if and only if $ad(v_1)$ is skew symmetric for all $v_1 \in \mathcal{G}$. From Lemma 7.2 of [Mil76] we know this holds if and only if the metric on $G$ is bi-invariant (which is only possible if $G$ is isomorphic to the product of compact group and a commutative group). Indeed from the proof of Lemma 7.2 and 7.1 of [Mil76] we see that the adjoint connection is the flat right invariant connection, which is a metric connection for the right invariant metric

$$< v_1, v_2 >'_{x_0} \equiv \left\langle TR_{x_0}^{-1}(v_1), TR_{x_0}^{-1}(v_2) \right\rangle_{id}.$$

**H.** There is a natural correspondence between S.D.E.'s (1) with $A \equiv 0$ and smooth maps of $M$ into the Grassmanian of n-planes in $\mathbb{R}^m$ classifying $TM$. The connection $\breve{\nabla}$ is the pull back of the universal connection on the Stiefel bundle over $M$, described in [NR61]. From there it follows that *every metric connection on $M$ can be obtained as $\breve{\nabla}$ for some* S.D.E. (1), see [ELLa].

For a diffusion on $M$, with $n = \dim M > 1$, with generator $\frac{1}{2}\Delta + L_Z$, for some smooth vector field $Z$, Ikeda and Watanabe showed how to construct a metric connection $\tilde{\nabla}$ on $M$ such that the diffusion process (from any point $x_0$ of $M$), is a $\tilde{\nabla}$-martingale (it is the stochastic development of an n-dimensional Brownian motion). See [IW89]. This $\tilde{\nabla}$ is not uniquely determined. By the remark above $\tilde{\nabla} = \breve{\nabla}$ for some S.D.E. $dx_t = X(x_t) \circ dB_t$, again not uniquely determined. For this S.D.E. the generator satisfies $\sum_{i=1}^m L_{X^i} L_{X^i} = \frac{1}{2}\Delta + L_Z$. As T. Lyons has pointed out to us such a construction is not in general possible when $\dim M = 1$.

**I.** We summarize here some of the notation being used:

$$\begin{aligned} N(x) &= \mathrm{Ker} X(x), \\ Y(x) &= X(x)^* : T_x M \to \mathbb{R}^m, \\ Z^v &= X(\cdot)Y(x_0)v, \quad v \in T_{x_0}M \end{aligned}$$

$\nabla$, Levi-Civita connection, $R$, Ric, its curvature and Ricci curvature;

$\tilde{\nabla}$, any connection, $\tilde{R}$, $\tilde{Ric}$, its curvature and Ricci curvature, $\tilde{\mathrm{Ric}}^{\#}(v) = \sum_1^m \tilde{\mathrm{Ric}}(v, X^i(x))X^i(x)$, and $\tilde{T}$ its torsion tensor

$\breve{\nabla}$, LeJan-Watanabe connection, $\breve{R}$, $\breve{Ric}$, its curvature and Ricci curvature, and $\breve{T}$ its torsion tensor

$\hat{\nabla}$, the adjoint connection of $\breve{\nabla}$, $\hat{R}$, $\hat{Ric}$, its curvature and, Ricci curvature, and $\hat{T}$ its torsion tensor.

## 3 The Derivative flow

**A.** Let $N = \cup_{x \in M} N(x)$. It forms a Riemannian vector bundle over $M$, (the normal bundle in the gradient case). Take any metric connection on it, with parallel translation along a curve $\{\sigma(s) : 0 \le s \le t\}$ denoted by $\tilde{//}_s : N(\sigma(0)) \to N(\sigma(s))$. Let $\breve{//}_t$ be parallel translation for $\breve{\nabla}$. Using $Y$ this induces a parallel translation operator

$$//_t = Y(\sigma(t))\breve{//}_t X(\sigma(0)) : N(\sigma(0))^\perp \to N(\sigma(t))^\perp,$$

which combines with $\tilde{//}_t$ on $N$ to give a parallel translation in $M \times \mathbb{R}^m$, again written $\tilde{//}_t$, as an isometry

$$\tilde{//}_t : \mathbb{R}^m \to \mathbb{R}^m$$

depending on $\sigma$. Following [EY93], set

$$\breve{B}_t := \int_0^t \breve{//}_s^{-1} X(x_s) dB_s \tag{21}$$

and

$$\beta_t := \int_0^t \tilde{//}_s^{-1} K(x_s) dB_s \tag{22}$$

where $K(x) : \mathbb{R}^m \to \mathbb{R}^m$ is the orthogonal projection onto $N(x)$ and $x_s = \xi_s(x_0)$. Finally set

$$\tilde{B}_t = Y(x_0)\breve{B}_t = \int_0^t \tilde{//}_s^{-1} Y(x_s)X(x_s)dB_s,$$

and $\bar{B}_t = \tilde{B}_t + \beta_t$.

For any process $\{y_s : 0 \le s < \zeta\}$ let $\mathcal{F}^{y_\cdot} = \sigma\{y_s : 0 \le s < \zeta\}$, but write $\mathcal{F}^{\xi_\cdot(x_0)}$ as $\mathcal{F}^{x_0}$. The following decomposition theorem is a direct analogue of the corresponding results in [EY93] with the same proof:

**Theorem 3.1** *1.* $\mathcal{F}^{\breve{B}_\cdot} = \mathcal{F}^{x_0}$,

*2.* $\{\bar{B}_t : 0 \le t < \zeta\}$ *is a Brownian motion on* $\mathbb{R}^m$ *with* $B_t = \int_0^t \tilde{//}_s d\bar{B}_s$.

*In particular* $\{\beta_t : 0 \le t < \zeta\}$, *when conditioned on* $\{\breve{B}_t : 0 \le t < \zeta\}$ *is a Brownian motion killed at time* $\zeta$ *(so when* $\zeta = \infty$, $\beta_\cdot$ *and* $\breve{B}_\cdot$ *are independent Brownian motions).*

**B.** The derivative flow $T\xi_t$ on $TM$ is given by the covariant equation

$$\tilde{D}v_t = \tilde{\nabla}X(v_t) \circ dB_t + \tilde{\nabla}A(v_t)dt - \tilde{T}(v_t, X(x_t) \circ dB_t + A(x_t)dt) \qquad (23)$$

for $v_t = T\xi_t(v_0)$, along the paths of $\{\xi_t : 0 \le t < \zeta\}$, since for a $C^1$ map $\sigma : (-\delta, \delta) \times (-\delta, \delta) \to M$

$$\frac{\tilde{D}}{\partial s}\frac{\partial}{\partial t}\sigma(s,t) = \frac{\tilde{D}}{\partial t}\frac{\partial}{\partial s}\sigma(s,t) + \tilde{T}(\frac{\partial \sigma}{\partial s}, \frac{\partial \sigma}{\partial t})$$

(e.g. see [Mil63]). Such covariant equations are described in [Elw82].

Taking $\tilde{\nabla}$ to be the adjoint connection $\hat{\nabla}$, since

$$\hat{\nabla}_U V = \breve{\nabla}_U V - \breve{T}(U, V), \qquad (24)$$

we see

$$\hat{D}v_t = \breve{\nabla}X(v_t) \circ dB_t + \breve{\nabla}A(v_t)dt. \qquad (25)$$

To rewrite this as an Itô equation (which means apply $\hat{//}_t^{-1}$ to both sides and consider the resulting Itô equation in $T_{x_0}M$), the correction term is

$$\begin{aligned} &\frac{1}{2}\sum_1^m \left[\breve{\nabla}X^i\left(\breve{\nabla}X^i(v_t)\right)dt + \hat{\nabla}_{X^i}\left(\breve{\nabla}X^i\right)(v_t)\,dt\right] \\ = \;&\frac{1}{2}\sum_1^m \left[\breve{\nabla}X^i\left(\breve{\nabla}X^i(v_t)\right)dt + \breve{\nabla}^2X^i\left(X^i, v_t\right)dt + \breve{T}\left(\breve{\nabla}X^i(v_t), X^i\right)\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{i=1}^{m}\left[\breve{\nabla}\left(\breve{\nabla}X^i\left(X^i(\cdot)\right)\right)(v_t) + \breve{\nabla}^2 X^i\left(X^i, v_t\right) - \breve{\nabla}^2 X^i\left(v_t, X^i\right)\right] dt \\
&= \frac{1}{2}\sum_{i=1}^{m}\left[\breve{\nabla}\left(\breve{\nabla}X^i\left(X^i(\cdot)\right)\right)(v_t)dt - \frac{1}{2}\breve{\mathrm{Ric}}^{\#}(v_t)dt\right]
\end{aligned}
$$

as in [Elw88], [EY93], where $\breve{\mathrm{Ric}}^{\#}(v) = \sum_1^m \breve{\mathrm{R}}\left(v, X^i(x)\right)X^i(x)$ so that $<\breve{\mathrm{Ric}}^{\#}(v_1), v_2 >_x = \breve{\mathrm{Ric}}(v_1, v_2)$ for $v_1, v_2 \in T_xM$. The first term vanishes as we saw in §1E from the definition of $\breve{\nabla}$. Thus

$$
\hat{D}v_t = \breve{\nabla}X(v_t)dB_t - \frac{1}{2}\breve{\mathrm{Ric}}^{\#}(v_t)dt + \breve{\nabla}A(v_t)dt. \tag{26}
$$

**C.** We can now extend one of the main results of [EY93]. If $\{u_t : 0 \le t < \zeta\}$ is any process along $\{\xi_t : 0 \le t < \zeta\}$ by $\mathbb{E}\{u_t\, \chi_{t<\zeta(x_0)}|\mathcal{F}^{x_0}\}$ we mean $\tilde{//}_t\mathbb{E}\{\tilde{//}_t^{-1} u_t\, \chi_{t<\zeta(x_0)}|\mathcal{F}^{x_0}\}$. As pointed out by M. Emery this is independent of the connection $\tilde{\nabla}$ used to define $\tilde{//}_t$.

**Theorem 3.2** *Assume $|v_t|$ is integrable for each $t \ge 0$. Set* $v_t^{x_0} = \mathbb{E}\{T\xi_t(v_0)\chi_{t<\zeta(x_0)}|\mathcal{F}^{x_0}\}$. *Then $\{v_t^{x_0}\}$ satisfies the covariant equation*

$$
\hat{D}v_t^{x_0} = -\frac{1}{2}\breve{Ric}^{\#}(v_t^{x_0})dt + \breve{\nabla}A(v_t^{x_0})dt. \tag{27}
$$

*along $\{x_t\}$ on $t < \zeta$.*

**Proof.** First assume non-explosion. Using Theorem 3.1 and rewriting (26) as

$$
\begin{aligned}
\hat{D}v_t &= \breve{\nabla}X(v_t)\tilde{//}_t d\bar{B}_t - \frac{1}{2}\breve{\mathrm{Ric}}^{\#}(v_t)dt + \breve{\nabla}A(v_t)dt \\
&= \breve{\nabla}X(v_t)\tilde{//}_t d\beta_t - \frac{1}{2}\breve{\mathrm{Ric}}^{\#}(v_t)dt + \breve{\nabla}A(v_t)dt.
\end{aligned}
$$

The last step used the fact that

$$
\breve{\nabla}X(v_t)\left(\tilde{//}_t d\tilde{B}_t\right) = \breve{\nabla}X(v_t)\left(Y(x_t)X(x_t)dB_t\right) = 0,
$$

by definition of $\breve{\nabla}$. But by theorem 3.1

$$
\mathbb{E}\left\{\int_0^t \breve{\nabla}X(v_s)\tilde{//}_s d\beta_s \,|\, \mathcal{F}^{x_0}\right\} = 0
$$

and the result follows by the linearity and $\mathcal{F}^{x_0}$-measurability of $\mathrm{Ric}^{\#}_{x_t}$ and $\breve{\nabla} A_{x_t}$. If $\zeta(x_0) < \infty$, let $\tau_D$ be the first exit time of $\xi(x_0)$ from a domain $D$ of $M$ with $D$ compact. The above argument show (27) holds on $t < \tau_D$. Now choose $D^i$ with $\tau_{D^i} \to \zeta$. //

**Remark 2.** The integrability of $|v_t|$ is needed in order for $v_t^{x_0}$ to be defined. It holds if $M$ is compact or with conditions on the growth of $|\nabla X|$, $|\nabla^2 X|$, and $|\nabla A|$ [Li94b], and is close to implying non-explosion of $\{x_t : t \geq 0\}$, [Li94a].

As an illustrative application there is the following extension of Bochner's vanishing theorem (however see Proposition 4.3 below):

**Corollary 3.3** *Suppose $M$ is compact. If $M$ admits a vector field $A$ and a metric connection $\tilde{\nabla}$ whose adjoint connection preserves a metric $< -, - >'$ such that*

$$\left\langle \tilde{Ric}^{\#}(v), v \right\rangle' > 2 \left\langle \tilde{\nabla} A(v), v \right\rangle' \qquad \textit{all } v \in TM, v \neq 0,$$

*then the cohomology group $H^1(M; \mathbb{R})$ vanishes.*

**Proof.** Let $\phi$ be a closed smooth 1-form and $\sigma$ a singular 1-cycle in $M$. By DeRham's theorem it is enough to show $\int_\sigma \phi = 0$. According to §2 H we can find an SDE (1) with $\tilde{\nabla} = \breve{\nabla}$. Since $M$ is compact (1) has a smooth solution flow $\{\xi_t : t \geq 0\}$ of diffeomorphisms of $M$. Then, by the continuity in $(t, x) \in \mathbb{R}(\geq 0) \times M$ of $\xi$,

$$\int_\sigma \phi dx = \int_{\xi_t \sigma} \phi dx = \int_\sigma \xi_t^* \phi dx$$

Treating the case when $\sigma : [a, b] \to M$ this gives

$$\begin{aligned} \int_\sigma \phi dx &= \mathbb{E} \int_a^b \phi_{\xi_t(\sigma(\theta))} \left( T\xi_t(\dot{\sigma}(\theta)) \right) d\theta \\ &= \mathbb{E} \int_a^b \phi_{\xi_t(\sigma(\theta))} \mathbb{E} \left\{ T\xi_t(\dot{\sigma}(\theta)) | \mathcal{F}_t^{\sigma(\theta)} \right\} d\theta \\ &= \int_a^b \mathbb{E} \phi_{\xi_t(\sigma(\theta))} \breve{W}_t^A(\dot{\sigma}(\theta)) d\theta \end{aligned}$$

where

$$\begin{cases} \frac{\hat{D}}{\partial t} \breve{W}_t^A(v_0) &= -\frac{1}{2} \breve{\mathrm{Ric}}^{\#}(\breve{W}_t^A(v_0)) + \breve{\nabla} A(\breve{W}_t^A(v_0)) \\ \breve{W}_0^A(v_0) &= v_0 \in TM. \end{cases}$$

by Theorem 3.2. Thus

$$|\int_\sigma \phi| \le \sup_x |\phi_x|' \int_a^b |\breve{W}_t^A(\dot{\sigma}(\theta)|' d\theta.$$

However, for $v_0 \in T_{x_0}M$,

$$\frac{d}{dt}|\breve{W}_t^A(v_0)|'^{,2} = 2\left\langle \frac{\hat{D}}{\partial t}\breve{W}_t^A(v_0), \breve{W}_t^A(v_0)\right\rangle'.$$

So our assumptions imply that $\breve{W}_t^A(v_0)$ decays exponentially as $t \to \infty$, uniformly in $x_0 \in M$, $v \in T_{x_0}M$ with $|v_0|' = 1$. Thus, letting $t \to \infty$, we see $\int_\sigma \phi = 0$. //

Next we give a version of Bismut's formula in this context, c.f. [Dri92].

**Corollary 3.4** *Let $\tilde{\nabla}$ be a metric connection for a compact Riemannian manifold $M$. Let $p_t(x,y)$ be the fundamental solution to*

$$\frac{\partial u_t}{\partial t} = \frac{1}{2} trace \tilde{\nabla}(grad\ u_t) + L_A u_t.$$

*Then*

$$d \log p_t(\cdot, y)(v_0) = \frac{1}{t}\mathbb{E}\left\{\int_0^t \left\langle \tilde{W}_s^A(v_0), \tilde{//}_s d\tilde{B}_s \right\rangle_{x_0} |x_t = y\right\}, \quad v \in T_{x_0}M,$$

*where $\{x_s\}$ is a diffusion on $M$ with generator $\frac{1}{2} trace \tilde{\nabla} grad$ - $+ L_A$, and $\tilde{B}$ the martingale part of the stochastic anti-development of $\{x_s : 0 \le s \le t\}$ using $\tilde{\nabla}$, a Brownian motion on $T_{x_0}M$, while $\tilde{//}_s$ is parallel translation, and $v_s = \tilde{W}_s^A(v_0)$ satisfies*

$$\frac{\tilde{D}'}{\partial s} v_s = -\frac{1}{2}\tilde{Ric}^{\#}(v_s) + \tilde{\nabla} A(v_s)$$

*both along the paths of $\{x_s : 0 \le s < t\}$ where $\tilde{D}'$ refers to covariant differentiation using the adjoint connection $\tilde{\nabla}'$.*

**Proof.** As described in §2H we can choose an SDE (1) with $\breve{\nabla} = \tilde{\nabla}$ and then the generator is as required by (20). If $\xi_\cdot$ is the flow then by [Elw92],

$$d \log p_t(\cdot, y)(v_0) = \frac{1}{t}\mathbb{E}\left\{\int_0^t \langle T\xi_s(v_0), X(x_s) dB_s\rangle |\, \xi_t(x_0) = y\right\}$$

and the result follows from the theorem and the fact that $\breve{B}_t$ given by $d\breve{B}_t = \breve{//}_t^{-1} X(x_t) dB_t$ is the martingale part of the stochastic anti-development (defined by the corresponding Stratonovich equation). //

**Example 3.** For the flat left invariant connection on a Lie group $G$ the SDE is as described in §2 G. Then $W_t$ and $T_{x_0}\xi_t$ are equal (there is no extraneous noise) and they are just right translation by $\xi_t(x_0)$ while $\tilde{//}_t$ is left translation by $\xi_t(x_0)$.

## 4 Moment Exponents

Let $S(t,x)(e)$ be the flow for the vector field $X^e$, and set $\delta S(t,v)(e) = TS(t,x)(e)(v)$. Let $<,>'$ be a Riemannian metric on $M$, not necessarily the induced one from the SDE. Denote by $|-|'$ the corresponding norm. Let $\tilde{\nabla}'$ be a connection compatible with $<,>'$. Then

$$\frac{d}{dt}|\delta S(t,v)e|'^p = p\,|\delta S(t,v)e|'^{p-2}\left\langle \delta S(t,v)e, \frac{\tilde{D}'}{dt}\delta S(t,v)e\right\rangle' . \tag{28}$$

Also

$$\begin{aligned}\frac{\tilde{D}'}{\partial t}\delta S(t,v)(e) &= \tilde{\nabla}' X^e\left(\delta S(t,v)e\right) + \tilde{T}'\left(X^e(S(t,x)e), \delta S(t,x)e\right) \\ &= \tilde{\nabla} X^e\left(\delta S(t,v)e\right),\end{aligned}$$

as for (23) if $\tilde{\nabla}$ is the adjoint of $\tilde{\nabla}'$. Then

$$\frac{d}{dt}|\delta S(t,v)e|'^p = p\,|\delta S(t,v)e|'^{p-2}\left\langle \delta S(t,v)e, \tilde{\nabla} X^e\left(\delta S(t,v)e\right)\right\rangle' . \tag{29}$$

At $t=0$,

$$\frac{d}{dt}|\delta S(t,v)e|'^p = p|v|'^{p-2} < v, \tilde{\nabla} X^e(v) >' . \tag{30}$$

Furthermore

$$\begin{aligned}&\tfrac{d^2}{dt^2}\ |\delta S(t,v)|'^p\Big|_{t=0} \\ &= p(p-2)|v|'^{p-4} < v, \tilde{\nabla} X^e(v) >'^2 + p|v|'^{p-2}\left[|\tilde{\nabla} X^e(v)|'^2 + \left\langle \tilde{T}\left(\tilde{\nabla}_v X^e, X^e\right), v\right\rangle'\right] \\ &+ p|v|'^{p-2}\left\langle v, \tilde{\nabla}^2 X^e(X^e, v)\right\rangle' + p|v|'^{p-2}\left\langle v, \tilde{\nabla} X^e(\tilde{\nabla} X^e(v))\right\rangle' .\end{aligned}$$

Set

$$\begin{aligned} H_p(x)(v,v) & = 2 < \tilde\nabla A(v), v >' + \textstyle\sum_1^m \left\langle \tilde\nabla^2 X^i(X^i,v), v \right\rangle' \\ & + \textstyle\sum_1^m \left\langle \tilde\nabla X^i(\tilde\nabla X^i(v)), v \right\rangle' + \textstyle\sum_1^m \left| \tilde\nabla X^i(v) \right|'^2 \\ & + \textstyle\sum_1^m \left[ \left\langle \tilde T\left(\tilde\nabla_v X^i, X^i\right), v \right\rangle' + (p-2)\frac{1}{|v|'^2} \left\langle \tilde\nabla X^i(v), v \right\rangle'^2 \right]. \end{aligned}$$

In terms of the Ricci curvature,

$$\begin{aligned} H_p(x)(v,v) & = 2 < \tilde\nabla \left(A + \textstyle\sum_1^m \tilde\nabla_{X^i} X^i\right)(v), v >' - < \tilde{\mathrm{Ric}}^{\#}(v), v >' \\ & + \textstyle\sum_1^m \left[ \left\langle \tilde T\left(\tilde\nabla_v X^i, X^i\right), v \right\rangle' + \left| \tilde\nabla X^i(v) \right|'^2 + (p-2)\frac{1}{|v|'^2} \left\langle \tilde\nabla X^i(v), v \right\rangle'^2 \right]. \end{aligned}$$

From (30) and the equation after it we see that $< v, \tilde\nabla X^i(v) >'$ and $H_p(x)(v,v)$ are independent of the choice of such connections for fixed $< -, - >'$. In particular when $<,>'$ is the metric $<,>$ induced by the S.D.E. the $H_p$ defined here agrees with the one used in [Li94a].

Taking $\tilde\nabla = \breve\nabla$, we see that if $\hat\nabla$ is compatible with $< -, - >'$,

$$\begin{aligned} H_p(x)(v,v) & = 2 < \breve\nabla A(v), v >' - < \breve{\mathrm{Ric}}^{\#}(v), v >' \\ & + \sum_1^m \left| \breve\nabla X^i(v) \right|'^2 + (p-2) \sum_1^m \frac{1}{|v|'^2} \left\langle \breve\nabla X^i(v), v \right\rangle'^2. \end{aligned}$$

By Itô's formula (c.f. [Elw88]), we have

**Lemma 4.1** *Let $\tilde\nabla$ be a connection whose dual connection is metric for some metric $< -, - >'$. Then for $v_0 \in T_{x_0}M$,*

$$\begin{aligned} |T\xi_t(v_0)|'^p = & \quad |v_0|'^p + \int_0^t p|T\xi_s(v_0)|'^{p-2} < T\xi_s(v_0), \tilde\nabla X(T\xi_s(v_0)) dB_s >' \\ & + \frac{p}{2} \int_0^t |T\xi_s(v_0)|'^{p-2} H_p(\xi_s(x_0))(T\xi_s(v_0), T\xi_s(v_0)) ds, \end{aligned}$$

Set

$$\begin{aligned} h_p(x) & = \sup_{|v|=1} H_p(x)(v,v), \\ \underline{h}_p(x) & = \inf_{|v|=1} H_p(x)(v,v). \end{aligned}$$

We can now extend the result proved in [Li94a] for gradient Brownian systems:

**Proposition 4.2** *Suppose $\hat\nabla$ is metric for some Riemannian metric $< -,- >'$. Then*

$$\mathbb{E}e^{\frac{1}{2}\int_0^t \underline{h}_p(\xi_s(x_0))ds} \le \mathbb{E}|T_{x_0}\xi_t|'^{\,p} \le n\mathbb{E}e^{\frac{1}{2}\int_0^t h_p(\xi_s(x_0))ds}. \tag{31}$$

**Proof.** Let $P_N(x) : \mathbb{R}^m \to N(x)$ be the orthogonal projection. Define

$$A_x : T_xM \oplus N(x) \to T_xM, \qquad x \in M$$

by

$$A(u,e) = \breve\nabla X(e)(u).$$

Then $A$ is the shape operator when (1) is a gradient system. For $e \in \mathbb{R}^m$ we have

$$A(u, P_N(x)(e)) = \breve\nabla X(e)(u).$$

Note that we can write

$$|T\xi_t(v_0)|'^{\,p} = |v_0|'^{\,p}\varepsilon(M_t^p)e^{a_t^p}$$

for $\varepsilon(M_t^p)$ the exponential martingale corresponding to $M_t^p$ where

$$M_t^p = \sum_1^m \int_0^t p\frac{< \breve\nabla X^i(T\xi_s(v_0)), T\xi_s(v_0) >'_{\xi_s(x_0)}}{|T\xi_s(v_0)|'^{\,2}} dB_s^i$$

and for

$$a_t^p = \frac{p}{2}\int_0^t \frac{H_p(\xi_s(x_0))(T\xi_s(v_0), T\xi_s(v_0))}{|T\xi_s(v_0)|'^{\,2}}ds.$$

Now we are in the situation of [Li94a] and the same proof, by the Girsanov transformation as used there, leads to (31). //

It is worth mentioning that since (30) and the equation after it is invariant under choice of connections we see that if $\breve\nabla$ is torsion skew symmetric then

$$\begin{aligned}\breve{\mathrm{Ric}}_x(v,v) &= \mathrm{Ric}_x(v,v) - \sum_1^m |\nabla X^i(v)|^2 + \sum_1^m |\breve\nabla X^i(v)|^2 \\ &= \mathrm{Ric}_x(v,v) - \sum_1^n |\nabla X^i(v)|^2\end{aligned}$$

because for such connections $\sum_{i=1}^m \nabla X^i(X^i) = 0$ by Corollary 2.5 and $\nabla X^i(v) = \breve\nabla X^i(v)$ for $i > n$ since $X^i(x_0) = 0$. In particular

**Proposition 4.3** *The Ricci curvature of any torsion skew symmetric connection $\tilde\nabla$ is majorized by that of the corresponding Levi-Civita connection. Equality holds everywhere if and only if $\tilde\nabla$ is Levi-Civita.*

## 5 The generator on differential q-forms

Let $\phi$ be a differential q-form and $\xi_t^*\phi$ its pull back by our flow $\xi_t(-)$. This gives rise to a semigroup on bounded q-forms [Elw92]:

$$P_t\phi = \mathbb{E}\xi_t^*\phi,$$

i.e. if $v=(v_1,\dots,v_q)$ is a q-vector in $\bigoplus^q T_xM$, $P_t\phi(v) = \mathbb{E}\phi(T\xi_t(v_1),\dots,T\xi_t(v_q))$.

Its infinitesimal generator $\mathcal{A}^q$ is given by:

$$\mathcal{A}^q\phi = \left(\frac{1}{2}\sum_1^m L_{X^i}L_{X^i} + L_A\right)\phi,$$

where $L_A$ denotes Lie differentiation in the direction of $A$.

Let $i_A\phi$ be the interior product of $\phi$ by $A$, which is a q-1 form defined by: $i_A\phi(v_1,\dots,v_{q-1}) = \phi(A,v_1,\dots,v_{q-1})$. Set

$$\bar\delta\phi = -\sum_1^m i_{X^i}\hat\nabla\phi(X^i). \tag{32}$$

Then it is easy to see that $\bar\delta\phi = -\sum_1^m i_{X^i}L_{X^i}\phi$ and

$$\sum_1^m L_{X^i}L_{X^i}\phi = -\bar\delta d\phi - d\bar\delta\phi$$

for $d$ the exterior differentiation.

There is also a Weitzenböck formula:

$$\mathcal{A}^q\phi = \frac{1}{2}\text{trace}\hat\nabla^2\phi - \frac{1}{2}\breve{R}^q(\phi) + L_A(\phi) \tag{33}$$

where $\breve{R}^q$ is the zero order operator on q-forms obtained algebraically (e.g. via annihilation and creation operators as in [CFKS87] or see [Elw88]) from the curvature tensor $\breve{R}$ of $\breve\nabla$ just as the usual Weitzenböck terms are obtained from the curvature of the Levi-Civita connection. In particular for a 1-form $\phi$,

$$\breve{R}^1(\phi)(v) = \phi(\breve{\text{Ric}}^\#(v)), \qquad v\in T_xM.$$

The case of 1-form is straightforward, or can be seen from Theorem 3.2. For details of the general case and further discussions see [ELLa].

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## Appendix I The Curvature Tensor

To calculate the curvature tensor $\breve{R}$ we will use the expression in Lemma 2.1 (ii) for $\breve{\nabla}$. Thus if $U, V, W$ are vector fields

$$\begin{aligned}
\breve{\nabla}_U \breve{\nabla}_V W &= [U, \breve{\nabla}_V W] + \sum_1^m [X^i, U] < \breve{\nabla}_V W, X^i > \\
&= [U, [V, W]] + \sum_1^m [U, [X^i, V]] < W, X^i > \\
&\quad + \sum_1^m [X^i, V]\, d < W, X^i > (U(\cdot)) + \sum_1^m [X^i, U] < \breve{\nabla}_V W, X^i > .
\end{aligned}$$

Applying Jacobi's identity twice we see

$$\begin{aligned}
\breve{R}(U, V)W : &= \breve{\nabla}_U \breve{\nabla}_V W - \breve{\nabla}_V \breve{\nabla}_U W - \breve{\nabla}_{[U,V]} W \\
&= \sum_1^m \{ [X^i, V]\, d < W, X^i > (U(\cdot)) \\
&\quad - [X^i, U]\, d < W, X^i > (V(\cdot)) \} \\
&\quad + \sum_1^m \left\{ [X^i, U] < \breve{\nabla}_V W, X^i > - [X^i, V] < \breve{\nabla}_U W, X^i > \right\} .
\end{aligned}$$

Now take $U = Z^u$, $V = Z^v$, $W = Z^w$ for $u, v, w \in T_{x_0} M$. Then

$$\begin{aligned}
\breve{R}(u, v)w &= \sum_1^m \left\{ [X^i, Z^v] < \breve{\nabla}_u X^i, w > - [X^i, Z^u] < \breve{\nabla}_v X^i, w > \right\} \\
&= \sum_1^m \left\{ -\nabla_v X^i < \breve{\nabla}_u X^i, w > + \breve{\nabla}_u X^i < \breve{\nabla}_v X^i, w > \right\}
\end{aligned}$$

since the torsion terms vanishes when summed in conjection with the terms which involve $\breve{\nabla} X^i$. Thus

**Proposition A1** *If* $u, v, w \in T_{x_0}M$ *then*

$$\breve{R}(u,v)(w) = \sum_{i=1}^{m} \breve{\nabla}_u X^i < \breve{\nabla}_v X^i, w > - \sum_{i=1}^{m} \breve{\nabla}_v X^i < \breve{\nabla}_u X^i, w > .$$

**Corollary A2**

$$< \breve{R}(u,v)w, z >= -\sum_{1}^{m} < \breve{\nabla}_u X^i \wedge \breve{\nabla}_v X^i, w \wedge z >_{\Lambda^2 T_x M} .$$

**Remark:** For $A$ the 'shape operator' defined in §4, the proposition gives

$$\breve{R}(u,v)w = \text{ trace } \{A(u,-)\langle A(v,-), w\rangle - A(v,-)\langle A(u,-), w\rangle\}$$

which reduces in the gradient case to Gauss's equation for the curvature of a submanifold in $\mathbb{R}^m$ (e.g. p. 23 [KN69a]).

## Appendix II

There is a direct correspondence between stochastic flows and Gaussian measures $\gamma$ on the space $\Gamma(TM)$ of vector fields on $M$, [Bax84], [LW82], [Kun90]. The latter is determined by its reproducing kernel Hilbert space (Cameron-Martin space), a Hilbert space $H$ of vector fields on $M$, together with its mean $\bar{\gamma}$, a vector field on $M$ [Bax76]. For the flow corresponding to our S.D.E. (1), the measure $\gamma$ is the image measure of the standard Gaussian measure on $\mathbb{R}^m$ by the map $e \mapsto X^e$ from $\mathbb{R}^m$ to vector fields on $M$ shifted by $\bar{\gamma}$, in this case the vector field $A$. The space $H$ is just $\{X^e : e \in \mathbb{R}^m\}$ with quotient inner product. However in general $H$ may be infinite dimensional e.g. for isotropic stochastic flows [LeJ85].

Nevertheless given such $H$ and vector fields $\bar{\gamma}$, if $\gamma_0$ is the corresponding centered Gaussian measure and $\{W_t : t \geq 0\}$ the Wiener process on the space of vector fields with $W_1$ distributed as $\gamma_0$, the corresponding stochastic flow is obtained as the solution flow of

$$dx_t = \rho_{x_t} \circ dW_t + \bar{\gamma}(x_t)dt$$

where $\rho_x : \Gamma(TM) \to T_xM$ is the evaluation map (assuming sufficient regularity), see [Elw92]. This reduces the situation to that discussed above with $\mathbb{R}^m$ replaced by the possibly infinite dimensional Hilbert space $H$. Assume non-degeneracy, so $\rho_x$ is surjective for each $x$, and let $M$ have the induced Riemannian metric. It is worth noting that the adjoint $Y(x) : T_xM \to \mathbb{R}^m$ of $X(x)$ is now replaced by the adjoint of $\rho_x : H \to T_xM$ which is essentially the reproducing kernel of $H$, i.e. the covariance of $\gamma$:

$$\rho_x^*(v) = k(x, \cdot)(v) \in H$$

where

$$< k(x, \cdot)v, h >_H = < h(x), v >_x, \qquad x \in M, v \in T_xM. \tag{34}$$

In particular

$$Z^v = k(x_0, \cdot)v, \qquad v \in T_{x_0}M.$$

and for a vector field $Z$ on $M$ our basic definition (5) becomes

$$\breve{\nabla} Z(v) = d[k(\cdot, x_0)Z(\cdot)](v) \tag{35}$$

treating $y \mapsto k(y, x_0)Z(y)$ as a map from $M$ to $T_{x_0}M$. The defining condition (4) for $\breve{\nabla}$ can be written $\breve{\nabla}(k(x_0, \cdot)v)w = 0$ all $v, w \in T_{x_0}M$ all $x_0 \in M$.

In terms of expectation with respect to our basic Gaussian measure $\gamma_0$, treating vector fields $W$ as a random field, equation (11) for $\breve{\nabla} Z$ becomes

$$\breve{\nabla} Z(v) = \frac{d}{dt}\, \mathbb{E} W(x_0) < Z(\sigma(t), W(\sigma(t)) >_{\sigma(t)} \big|_{t=0} \tag{36}$$

$$k(x, \cdot)v = \mathbb{E} < W(x), v >_x W(\cdot)$$

and in terms of conditional expectations

$$k(x, y)W(x) = \mathbb{E}\left\{W(y) | W(x)\right\} \quad \in T_yM$$

$$k(x, \cdot)v = \mathbb{E}\left\{W | W(x) = v\right\},$$

giving

$$\breve{\nabla} Z(v) = \frac{d}{dt} \mathbb{E}\left\{W(x_0) \,|\, W(\sigma(t)) = Z(\sigma(t))\right\}\big|_{t=0}.$$

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