

CONCERNING THE LIMITS OF A MEASURE OF SKEWNESS

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In a recent note in the *Annals of Mathematical Statistics*,* Hotelling and Solomons devised an ingenious method of showing that the measure of skewness s defined by the equation

$$s = \frac{\text{mean} - \text{median}}{\text{standard deviation}}$$

cannot be greater than unity in absolute value. I am venturing to offer another proof of the same fact, which seems to me to be of interest because it employs an important and well-known algebraic inequality.

With Hotelling and Solomons, I shall assume that we are concerned with n readings, or x 's, with median zero and mean \bar{x} , where \bar{x} of course is $\Sigma x/n$. We may show that the absolute value of s cannot be greater than one by showing that $1/s^2$ is not less than one. Making obvious substitutions, we must then show that

$$\frac{n \Sigma x^2}{(\Sigma x)^2} \geq 2.$$

Now according to a known theorem if a, b, \dots, k are n positive numbers, and if m is a number not lying between zero and one, then

$$\frac{a^m + b^m + \dots + k^m}{n} \geq \left(\frac{a + b + \dots + k}{n} \right)^m.$$

*Vol. 3, no. 2, May, 1932, 141-2.

While the proof of this theorem is given in Chrystal, we shall outline a (simplified) proof for the case $m=2$, to make this note self-contained. For any number r we obviously have $(r-1)^2 \geq 0$. Now let r equal $na/a+b+\dots+k$, $nb/a+b+\dots+k$, ..., $nk/a+b+\dots+k$ in turn. The first of these gives

$$\frac{n^2 a^2}{(a+b+\dots+k)^2} - \frac{2na}{(a+b+\dots+k)} + 1 \geq 0,$$

while the others give similar inequalities. Summing these inequalities we have

$$\frac{n^2(a^2+b^2+\dots+k^2)}{(a+b+\dots+k)^2} - 2n+n \geq 0,$$

which is Chrystal's theorem,* for $m=2$. The proof shows that some of the numbers a, b, \dots, k can be zero; in fact, some can be negative, provided $a+b+\dots+k$ is not zero.

Now, suppose we have an odd number of readings, say $n=2s+1$. Since the median reading is zero, there are s non-negative readings, which we shall now call y 's, and s non-positive readings, which we shall call z 's. We have at once, by the above.

$$\frac{s \sum y^2}{(\sum y)^2} \geq 1,$$

$$\frac{s \sum z^2}{(\sum z)^2} \geq 1,$$

It follows immediately that

$$s \left(\frac{\sum y^2 + \sum z^2}{(\sum y)^2 + (\sum z)^2} \right) \geq 1$$

*Chrystal, Algebra. Part II, 2nd ed., 1922, p. 49.

and, since $n = 2s + 1$ that

$$n \left(\frac{\sum y^2 + \sum z^2}{(\sum y)^2 + (\sum z)^2} \right) > 2.$$

Finally,

$$n \frac{\sum x^2}{(\sum x)^2} = n \frac{\sum y^2 + \sum z^2}{(\sum y + \sum z)^2} = n \frac{\sum y^2 + \sum z^2}{(\sum y)^2 + (\sum z)^2 + 2(\sum y)(\sum z)} > 2,$$

since $2(\sum y)(\sum z)$ is certainly not positive.

This proof is valid unless all the y 's are zero or all the z 's are zero. Suppose the latter is the case. Then

$$\frac{n \sum x^2}{(\sum x)^2} = \frac{n \sum y^2}{(\sum y)^2} > 2 \left(\frac{s \sum y^2}{(\sum y)^2} \right) > 2.$$

If all the readings are zero our definition of s does not give a definite value.

If n is even, not odd, the proof may be modified by properly defining the median. In this case we can show again that

$$\frac{n \sum x^2}{(\sum x)^2} \geq 2,$$

but the possibility of the equality cannot be ruled out.

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