

## Concerning the order and the semi-order of $n$ -dimensional Euclidean space \*

by

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In the Proceedings of International Colloquium held in Potsdam in 1973, a paper of Wanda Szmielew appeared (see [1]). The author announced the results and promised the proofs to be published in another paper. Wanda Szmielew died a year ago. The present paper is a complement of [1], based on the author's notes.

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**Abstract.** The  $n$ -dimensional Euclidean geometry is understood as the elementary theory of the equidistance and betweenness relations in the  $n$ -dimensional Cartesian vector space over a formally real and Pythagorean field. The following two questions are answered: which properties of the semi-betweenness have to be postulated to obtain the semi-ordered Euclidean geometry, and then, what sentences have to be added to obtain the ordered Euclidean geometry.

1. We start by recalling the terminology and notation (see [1]).

Let  $\mathbf{F}$  be the class of all formally real and Pythagorean commutative fields

$$\mathfrak{F} = (F, 0, 1, +, \cdot).$$

Given a field  $\mathfrak{F} \in \mathbf{F}$ , the set  $P \subseteq F$  is a *semi-positive cone* of  $\mathfrak{F}$  iff

$$(i) \quad P \cup -P = F,$$

$$(ii) \quad P \cap -P = \{0\},$$

$$(iii) \quad P + P \subseteq P.$$

If moreover

$$(iv) \quad P \cdot P \subseteq P,$$

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then  $P$  is a positive cone of  $\mathfrak{F}$ . Then

$$(v) \quad 1 \in P.$$

A normal semi-positive cone  $P$  besides (i)–(iii) satisfies the condition (v) as well.

Let

$$\mathbf{SOF} = \{(\mathfrak{F}P): \mathfrak{F} \in \mathbf{F} \text{ and } P \text{ is a normal semi-positive cone of } \mathfrak{F}\},$$

$$\mathbf{OF} = \{(\mathfrak{F}P): \mathfrak{F} \in \mathbf{F} \text{ and } P \text{ is a positive cone of } \mathfrak{F}\}.$$

We refer to the couple  $(\mathfrak{F}P) \in \mathbf{SOF}$  as a *semi-ordered (formally real and Pythagorean) field* and to the  $(\mathfrak{F}P) \in \mathbf{OF}$  as an *ordered (formally real and Pythagorean) field*.

Given a semi-ordered field  $(\mathfrak{F}P) \in \mathbf{SOF}$ , we define a *semi-norm*  $\| \cdot \|$  ( $\| \cdot \|_P$ ) in the  $n$ -dimensional Cartesian vector space  $\mathfrak{B}_C^n(\mathfrak{F})$ , ( $n \geq 2$ ):

$$\|a\| \in P \quad \text{and} \quad \|a\|^2 = a \cdot a.$$

In this way we get the  $n$ -dimensional Cartesian semi-metric vector space  $\mathfrak{B}_C^n(\mathfrak{F}P)$ .

In terms of  $\mathfrak{B}_C^n(\mathfrak{F})$  we define in the usual way the collinearity and the equidistance relations,  $L_{\mathfrak{F}}$  and  $D_{\mathfrak{F}}$ . In terms of  $\mathfrak{B}_C^n(\mathfrak{F}P)$  we define the *semi-betweenness relation*:

$$B_{\mathfrak{F}P}(abc) \Leftrightarrow \|a-b\| + \|b-c\| = \|a-c\|.$$

Thus any field  $\mathfrak{F} \in \mathbf{F}$  generates the  $n$ -dimensional Euclidean space over  $\mathfrak{F}$ ,

$$\mathfrak{E}^n(\mathfrak{F}) = (F^n, L_{\mathfrak{F}}, D_{\mathfrak{F}}),$$

and any semi-ordered field  $(\mathfrak{F}P) \in \mathbf{SOF}$  generates the  $n$ -dimensional semi-ordered Euclidean space over  $(\mathfrak{F}P)$ ,

$$\mathfrak{E}^n(\mathfrak{F}P) = (F^n, L_{\mathfrak{F}}, D_{\mathfrak{F}}, B_{\mathfrak{F}P}).$$

If, in particular,  $(\mathfrak{F}P) \in \mathbf{OF}$ , then  $\mathfrak{E}^n(\mathfrak{F}P)$  is the  $n$ -dimensional ordered Euclidean space over  $(\mathfrak{F}P)$ . Let

$$\mathbf{E}^n = \{\mathfrak{E}^n(\mathfrak{F}): \mathfrak{F} \in \mathbf{F}\},$$

$$\mathbf{SOE}^n = \{\mathfrak{E}^n(\mathfrak{F}P): (\mathfrak{F}P) \in \mathbf{SOF}\},$$

$$\mathbf{OE}^n = \{\mathfrak{E}^n(\mathfrak{F}P): (\mathfrak{F}P) \in \mathbf{OF}\}$$

and let  $\mathcal{E}^n$ ,  $\mathcal{SOE}^n$  and  $\mathcal{OE}^n$  be elementary theories of the classes  $\mathbf{E}^n$ ,  $\mathbf{SOE}^n$  and  $\mathbf{OE}^n$  respectively. We refer to the theory  $\mathcal{E}^n$  as the ( $n$ -dimensional) *Euclidean geometry*, and to the theories  $\mathcal{SOE}^n$  and  $\mathcal{OE}^n$  as to ( $n$ -dimensional) respectively *semi-ordered* and *ordered Euclidean geometries*.

2. Consider the following four sentences (<sup>1</sup>):

$$A1. B(abc) \Rightarrow B(cba),$$

$$A2. B(abd) \wedge B(bcd) \Rightarrow B(abc),$$

$$A3. L(abc) \Leftrightarrow B(abc) \vee B(bca) \vee B(cab),$$

$$A4. abc \equiv ab'c' \wedge B(abc) \Rightarrow B(ab'c') \text{ (}^2\text{)},$$

and the weak Pasch axiom:

$$\text{WP. } B(pbc) \wedge B(aqb) \Rightarrow \exists r(L(pqr) \wedge L(arc)).$$

Let

$$\mathcal{K} = \{A1, A2, A3, A4\} \quad \text{and} \quad \mathcal{U} = \{\text{WP}\}.$$

We are going to prove the following two theorems:

**THEOREM 1.**  $\mathbf{SOE}^n$  coincides (up to isomorphism) with the class of models  $\mathfrak{M}(\mathcal{E}^n \cup \mathcal{K})$ .

**THEOREM 2.**  $\mathbf{OE}^n$  coincides (up to isomorphism) with the class  $\mathfrak{M}(\mathcal{SOE}^n \cup \mathcal{U})$ .

Let us notice first that the following seven sentences are derivable from  $\mathcal{E}^n \cup \{A1, A2, A3\}$  (<sup>3</sup>).

$$T1. B(aaa) \text{ (from } A3 \text{ and } \mathcal{E}^n\text{)},$$

$$T2. B(aab) \text{ (from } A1, A3, \text{ and } T1\text{)},$$

$$T3. B(aba) \Rightarrow a = b$$

suppose  $B(aba) \wedge a \neq b$ ; by  $\mathcal{E}^n$ ,  $a \neq b \models \exists c \sim L(abc)$ ; by T2, A2, A1, A3,  $B(aba) \models B(caa) \wedge B(aba) \models B(cab) \models L(abc)$ ,

$$T4. B(abc) \wedge B(acb) \Rightarrow b = c \text{ (by } A1, A2, \text{ and } T3\text{)},$$

$$T5. B(apb) \wedge B(arb) \Rightarrow B(apr) \vee B(arp) \text{ (by } \mathcal{E}^n \text{ and } A3\text{)},$$

$$T6. B(abp) \wedge B(abr) \wedge a \neq b \Rightarrow B(apr) \vee B(arp) \text{ (by } \mathcal{E}^n, A3, \text{ and } T4\text{)},$$

$$T7. B(abc) \wedge B(bcd) \wedge b \neq c \Rightarrow B(abd) \text{ (by } \mathcal{E}^n \text{ and } A1\text{--}A3\text{)}.$$

In turn, using A1–A3 and T4–T7, we can easily prove two lemmas concerning halflines of an arbitrary line  $K$ . The *halfline from  $a$  through  $b$*  ( $a \neq b$ ), is defined as usually:

$$\mathbf{HL}(ab) = \{p: B(apb) \vee B(abp)\}.$$

L1. Let  $a, b \in K$ ,  $a \neq b$ ,  $A = \mathbf{HL}(ab)$ . Then

$$(1) \quad p, r \in A \wedge B(pqr) \Rightarrow q \in A$$

and

$$(2) \quad q \in A - \{a\} \wedge B(aqr) \Rightarrow r \in A.$$

L2. Let  $a, b, c \in K$ ,  $a \neq b \neq c$ ,  $B(abc)$ ,  $A = \mathbf{HL}(ba)$ ,  $C = \mathbf{HL}(bc)$ . Then

$$(1) \quad A \cup C = K, \quad A \cap C = \{b\},$$

$$(2) \quad p, r \in K - \{b\} \wedge B(pbr) \Rightarrow [p \in A \Leftrightarrow r \in C].$$

(<sup>1</sup>) We use the formula  $pq \equiv p'q'$  instead of  $D(pp'q'q')$  and the abbreviation  $pqr \equiv p'q'r'$  for the conjunction  $pq \equiv p'q' \wedge pr \equiv p'r' \wedge qr \equiv q'r'$ .

(<sup>2</sup>) We use two different symbols: the implication symbol  $\Rightarrow$  and the inference symbol  $\models$ .

(<sup>1</sup>) Throughout the whole paper we omit the universal quantifiers which should be placed in front of a formula to bind all the free variables occurring in it.

and

$$(3) \quad p \in A \wedge r \in C \Rightarrow B(pbr).$$

Among the consequences of  $\mathcal{E}^n \cup \{A1-A4\}$  (till now we have not used A4) let us distinguish the following two:

$$A5. B(abc) \wedge ab \equiv ac \Rightarrow b = c$$

and (stronger)

$$A5'. B(abd) \wedge B(bcd) \wedge bc \equiv ad \Rightarrow c = d$$

(comp. [1] and [2]). By  $\mathcal{E}^n$ , A1, A4, A1, A2, and T3

$$\begin{aligned} B(abc) \wedge ab \equiv ac &\Rightarrow B(abc) \wedge abc \equiv acb \Rightarrow B(cba) \wedge B(acb) \\ &\Rightarrow B(cbc) \Rightarrow b = c. \end{aligned}$$

Thus  $A5 \in \text{Cn}(\mathcal{E}^n \cup \{A1-A4\})$ .

It is easy to see that on the base of  $\mathcal{E}^n \cup \{A1-A4\}$  A5 is equivalent to the statement

$$T8. B(b \oplus c \oplus c),$$

which says that the midpoint  $b \oplus c$  of the segment  $bc$  lies between its ends.

Let us show that on the base of  $\mathcal{E}^n \cup \{A1, A2, A3, A5\}$  A5' is equivalent to the following linear case of A4:

$$T9. B(abc) \Rightarrow B(a \sigma_a(b) \sigma_a(c)),$$

where  $\sigma_a$  is the symmetry with respect to  $a$ . Assume first A5'. Let  $B(abc)$  and suppose that  $\sim B(a \sigma_a(b) \sigma_a(c))$ . Then, by  $\mathcal{E}^n \cup \{A1-A3\}$  and A5,  $B(\sigma_a(b) \sigma_a(c) a)$ , and thus, taking in A5'  $\sigma_a(b)$ ,  $\sigma_a(c)$ ,  $b$ ,  $c$  for  $a$ ,  $b$ ,  $c$ ,  $d$ , we get  $b = c$ , which implies  $B(a \sigma_a(b) \sigma_a(c))$ . Assume now T9 and let  $B(abd) \wedge B(acd) \wedge bc \equiv ad$ ; since  $bc \parallel ad$ , we get  $a \oplus b = c \oplus d \vee a \oplus c = b \oplus d$ . If  $a \oplus b = c \oplus d$ , then  $a = b = c = d$ ; if  $a \oplus c = b \oplus d = p$ , then  $c = \sigma_p(a) \wedge d = \sigma_p(b) \wedge B(pba)$ , whence, by T9,  $B(pdc)$ , which together with  $B(pcd)$  implies  $c = d$ . Thus  $(T9 \Leftrightarrow A5') \in \text{Cn}(\mathcal{E}^n \cup \{A1, A2, A3, A5\})$ .

As a result, A5 and A5', as well as T8 and T9, are derivable from  $\mathcal{E}^n \cup \{A1-A4\}$ .

In turn, the outer invariance law T9 is equivalent to the following invariance law:

$$T10. a, b, c, p \in K \wedge B(abc) \Rightarrow B(\sigma_p(a) \sigma_p(b) \sigma_p(c))$$

(see [3], Th. 7.6.2), and thus implies the inner invariance law:

$$T11. B(abc) \Rightarrow B(\sigma_b(a) b \sigma_b(c)).$$

Let us prove

$$T12. a, b, c, a', b', c' \in K \wedge abc \equiv a'b'c' \wedge B(abc) \Rightarrow B(a'b'c').$$

Indeed, let  $p = b \oplus b'$ ; then  $a'b'c' \equiv abc \equiv \sigma_p(a)b'\sigma_p(c)$ , and thus (since either  $\sigma_p(a) = \sigma_b(a)$  and  $\sigma_p(c) = \sigma_b(c)$  or  $\sigma_p(a) = a'$  and  $\sigma_p(c) = c'$ ), applying T10 and T11, we get

$$B(abc) \Rightarrow B(\sigma_p(a)b'\sigma_p(c)) \Rightarrow B(\sigma_b(a')b'\sigma_b(c')) \Rightarrow B(a'b'c'). \blacksquare$$

The statement T12 may be generalized to

$$T13. abc \equiv a'b'c' \wedge B(abc) \Rightarrow B(a'b'c').$$

In fact, let  $a \neq a'$  and let  $K$  be the line through  $a$  and  $a'$ ; there are  $b_1, c_1, b_2, c_2 \in K$  such that  $abc \equiv ab_1c_1 \wedge a'b'c' \equiv a'b_2c_2$ . Applying in turn A4, T12, A4, we get

$$B(abc) \Rightarrow B(ab_1c_1) \Rightarrow B(a'b_2c_2) \Rightarrow B(a'b'c'). \blacksquare$$

By T13, T6, A2 and A5, we obtain

$$T14. ab \equiv a'b' \wedge bc \equiv b'c' \wedge B(abc) \wedge B(a'b'c') \Rightarrow ac \equiv a'c'.$$

Indeed, take  $abc$  and let  $ab \equiv a'b'$ . If  $a = b$  then  $a' = b'$ . Let  $a \neq b$ ; then  $a' \neq b'$  and there is a point  $c''$  such that  $abc \equiv a'b'c''$ . Assume  $B(abc) \wedge bc \equiv b'c' \wedge B(a'b'c')$ . By T13

$$B(abc) \Rightarrow B(a'b'c''),$$

by T6

$$B(a'b'c') \wedge B(a'b'c'') \wedge a' \neq b' \Rightarrow B(b'c'c'') \vee B(b'c''c').$$

By A5

$$(B(b'c'c'') \vee B(b'c''c')) \wedge b'c'' \equiv bc \equiv b'c' \Rightarrow c' = c''.$$

Thus  $ac \equiv a'c'$ .  $\blacksquare$

The last two statements may be expressed together as

$$T15. ab \equiv a'b' \wedge bc \equiv b'c' \wedge B(abc) \Rightarrow [B(a'b'c') \Leftrightarrow ac \equiv a'c'].$$

3. Let us prove now

THEOREM 1.  $\text{SOE}^n \stackrel{\text{iso}}{=} \mathfrak{M}(\mathcal{E}^n \cup \mathcal{X})$ .

Proof.  $\Leftarrow$ : Let

$$\mathfrak{E}^n(\mathfrak{F}P) = (F^n, L_{\mathfrak{F}}, D_{\mathfrak{F}}, B_{\mathfrak{F}P}) \in \text{SOE}^n.$$

Then  $(\mathfrak{F}P) \in \text{SOF}$  and thus the reduct  $(F^n, L_{\mathfrak{F}}, D_{\mathfrak{F}})$  is a model of  $\mathcal{E}^n$ . Hence it suffices to verify A1-A4. By the remarks in [1] pp. 72<sub>8</sub>-73<sub>8</sub>, it follows that the semi-betweenness relation  $B_{\mathfrak{F}P}$  restricted to an arbitrary line  $K$  of the space  $\mathfrak{E}^n(\mathfrak{F}P)$  coincides with the betweenness relation restricted to  $K$ . Thus  $B_{\mathfrak{F}P}$  satisfies the axioms A1-A3. Since every congruence preserves the relation  $B_{\mathfrak{F}P}$  (see [1], p. 73<sub>9</sub>-2<sub>1</sub>), the axiom A4 is satisfied as well.

$\Rightarrow$ : Let  $\mathfrak{S} = (S, L, D, B)$  be a model of  $\mathcal{E}^n \cup \mathcal{X}$ . We are going to find  $(\mathfrak{F}P) \in \text{SOF}$  such that  $\mathfrak{S} \simeq \mathfrak{E}^n(\mathfrak{F}P)$ . Take a reduct  $\mathfrak{S}_0 = (S, L, D)$  of  $\mathfrak{S}$ . Since  $\mathbf{E}^n$  is an elementary class (see [1], p. 73<sub>1</sub>),  $\mathfrak{M}(\mathcal{E}^n) \stackrel{\text{iso}}{=} \mathbf{E}^n$  and thus, for some  $\mathfrak{F} \in \mathbf{F}$ ,

$$\mathfrak{S}_0 \simeq \mathfrak{E}^n(\mathfrak{F}),$$

i.e.  $\mathfrak{S}_0 = (S, L, D) \simeq (F^n, L_{\mathfrak{F}}, D_{\mathfrak{F}}) = \mathfrak{E}^n(\mathfrak{F})$ . Let

$$\Phi: \mathfrak{E}^n(\mathfrak{F}) \rightarrow \mathfrak{S}_0 \quad \text{and} \quad \Psi: \mathfrak{S}_0 \rightarrow \mathfrak{E}^n(\mathfrak{F})$$

be mutually inverse isomorphisms. It now suffices to extend  $\Psi$  over the whole  $\mathfrak{S}$ . Thus we construct a semi-positive cone  $P$  of  $\mathfrak{F}$  such that  $\Psi(B) = B_{\mathfrak{F}P}$ .

For every  $x \in F$ , let  $e(x) = (x, 0, \dots, 0) \in F^n$ . Then

$$e(x) + e(y) = e(x+y) \quad \text{and} \quad -e(x) = e(-x).$$

The set  $K = \Phi e(F)$  is a line in  $S$ . Since

$$e(0) = e(x) \oplus_{\mathfrak{F}} e(-x),$$

we get

$$\Phi e(0) = \Phi e(x) \oplus \Phi e(-x)$$

(because  $\oplus$  is definable in terms of  $\mathfrak{S}_0$ ). Thus, by T8,

$$B(\Phi e(x) \Phi e(0) \Phi e(-x)) \quad \text{for every } x \in F;$$

in particular  $B(\Phi e(1) \Phi e(0) \Phi e(-1))$ .

Define two halflines

$$A = \mathbf{HL}(\Phi e(0) \Phi e(1)) \quad \text{and} \quad C = \mathbf{HL}(\Phi e(0) \Phi e(-1)).$$

By L2

$$A \cup C = K, \quad A \cap C = \{\Phi e(0)\},$$

and

$$\Phi e(-x) \in A \Leftrightarrow \Phi e(x) \in C.$$

Let  $P = \{x: \Phi e(x) \in A\}$ , i.e.  $P = e^{-1}\Psi(A)$ . Then

$$-P = e^{-1}\Psi(C),$$

and therefore

$$P \cup -P = F, \quad P \cap -P = \{0\},$$

i.e.  $(\mathfrak{F}P)$  satisfies (i) and (ii). In turn, by T8,

$$B\left(\Phi e(x) \Phi e\left(\frac{x+y}{2}\right) \Phi e(y)\right) \quad \text{and} \quad B\left(\Phi e(0) \Phi e\left(\frac{x+y}{2}\right) \Phi e(x+y)\right),$$

whence

$$x, y \in P \Rightarrow \Phi e(x), \Phi e(y) \in A \stackrel{\text{L1}}{\Rightarrow} \Phi e\left(\frac{x+y}{2}\right) \in A$$

$$\stackrel{\text{L1}}{\Rightarrow} \Phi e(x+y) \in A \vee \Phi e\left(\frac{x+y}{2}\right) = \Phi e(0)$$

$$\stackrel{\text{(ii)}}{\Rightarrow} x+y \in P \vee y = -x \stackrel{\text{(ii)}}{\Rightarrow} x+y \in P,$$

i.e.  $(\mathfrak{F}P)$  satisfies (iii). Evidently  $1 \in P$ , i.e. (v) is satisfied as well. It remains to show that  $\Psi(B) = B_{\mathfrak{F}P}$  (just now A4 is needed). Take  $p, q, r \in F^n$  and let  $a = \Phi(p)$ ,  $b = \Phi(q)$ ,  $c = \Phi(r)$ . Let

$$(1) \quad \|p-q\| = x \quad \text{and} \quad \|q-r\| = z;$$

then

$$(2) \quad B_{\mathfrak{F}P}(pqr) \Leftrightarrow \|p-r\| = x+z.$$

Since

$$\|e(x) - e(0)\| = x, \quad \|e(0) - e(-z)\| = z,$$

and

$$\|e(x) - e(-z)\| = x+z,$$

by (1) and (2) we get

$$(1') \quad e(x)e(0) \equiv_{\mathfrak{F}} pq \wedge e(0)e(-z) \equiv_{\mathfrak{F}} qr$$

and

$$(2') \quad B_{\mathfrak{F}P}(pqr) \Leftrightarrow e(x)e(-z) \equiv_{\mathfrak{F}} pr.$$

The conditions (1') and (2') imply

$$(1'') \quad \Phi e(x) \Phi e(0) \equiv ab \wedge \Phi e(0) \Phi e(-z) \equiv bc$$

and

$$(2'') \quad B_{\mathfrak{F}P}(pqr) \Leftrightarrow \Phi e(x) \Phi e(-z) \equiv ac.$$

By (1),  $x, z, x+z \in P$ , whence

$$\Phi e(x), \Phi e(z) \in A, \quad \Phi e(-z) \in C$$

and therefore, by L2,  $B(\Phi e(x) \Phi e(0) \Phi e(-z))$ . Thus, applying T15, by (1'') and (2'') we get

$$B_{\mathfrak{F}P}(pqr) \Leftrightarrow B(abc),$$

which completes the proof. ■

**THEOREM 2.**  $\mathbf{OE}^n = \mathfrak{M}(\mathcal{S}\mathcal{O}\mathcal{E}^n \cup \{\mathbf{WP}\})$ .

**Proof.**  $\subseteq$ : If  $(\mathfrak{F}P) \in \mathbf{OF}$  then  $\mathfrak{E}^n(\mathfrak{F}P) \subseteq \mathbf{OE}^n \subseteq \mathbf{SOE}^n$  and  $\mathfrak{E}^n(\mathfrak{F}P)$  satisfies **WP**.

$\supseteq$ : Let  $\mathfrak{S}$  be a model of  $\mathcal{S}\mathcal{O}\mathcal{E}^n \cup \{\mathbf{WP}\}$ . We are going to find  $(\mathfrak{F}P) \in \mathbf{OF}$  such that  $\mathfrak{S} \simeq \mathfrak{E}^n(\mathfrak{F}P)$ . By Theorem 1,  $\mathbf{SOE}^n$  is an elementary class, thus

$$\mathbf{SOE}^n \stackrel{\text{iso}}{=} \mathfrak{M}(\mathcal{S}\mathcal{O}\mathcal{E}^n).$$

Since  $\mathfrak{S} \in \mathfrak{M}(\mathcal{S}\mathcal{O}\mathcal{E}^n \cup \{\mathbf{WP}\}) \subseteq \mathfrak{M}(\mathcal{S}\mathcal{O}\mathcal{E}^n)$ , hence  $\mathfrak{S} \simeq \mathfrak{E}^n(\mathfrak{F}P)$  for some  $(\mathfrak{F}P) \in \mathbf{SOF}$ . So, it suffices to show that the semi-positive cone  $P$  is closed under the multiplication:

$$(iv) \quad P \cdot P \subseteq P.$$

For any  $x \in F$ , let

$$e_1(x) = (x, 0, \dots, 0) \in F^n, \quad e_2(x) = (0, x, 0, \dots, 0) \in F^n.$$

Suppose that

$$x, y \in P \quad \text{and} \quad x \cdot y \notin P.$$

Then

$$B_{\mathbb{R}^n}(e_1(x)e_1(0)e_1(x \cdot y))$$

and

$$B_{\mathbb{R}^n}(e_2(0)e_2(1)e_2(y)) \vee B_{\mathbb{R}^n}(e_2(0)e_2(j)e_2(1))$$

(see Fig. 1). By WP, there is a point  $p \in F^n$  such that

$$L_{\mathbb{R}^n}(e_1(x \cdot y)pe_2(y)) \wedge L_{\mathbb{R}^n}(e_1(x)e_2(1)p).$$

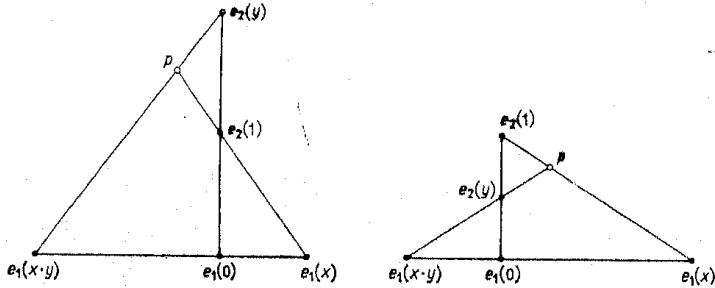


Fig. 1

Thus  $p$  is of the form

$$p = (z_1, z_2, 0, \dots, 0),$$

where

$$z_1 + x \cdot z_2 = x \quad \text{and} \quad (z_1 + x \cdot z_2) \cdot y = x \cdot y^2.$$

Hence  $x \cdot y = x \cdot y^2$ , i.e.  $x \cdot y \cdot (1 - y) = 0$ . Since  $x \cdot y \neq 0$ , we get  $y = 1$  and thus  $x \cdot y = x \in P$  in contrary to the assumption. ■

As corollaries of Theorems 1 and 2, we get

THEOREM 1'.  $\mathcal{S}\mathcal{O}\mathcal{E}^n = \text{Cn}(\mathcal{E}^n \cup \mathcal{X})$ .

THEOREM 2'.  $\mathcal{O}\mathcal{E}^n = \text{Cn}(\mathcal{S}\mathcal{O}\mathcal{E}^n \cup \mathcal{Y})$ .

4. We pass to independence models. As was stated in [1] p. 76, each of the axioms A1–A4 is independent of the remaining three together with the whole theory  $\mathcal{E}^n$ . If, moreover, A3 is replaced by the conjunction of

A3.1.  $B(abc) \Rightarrow L(abc)$

and

A3.2.  $L(abc) \Rightarrow B(abc) \vee B(bca) \vee B(cab)$ ,

then each of the five sentences A1, A2, A3.1, A3.2, A4 is independent of the remaining four together with the whole  $\mathcal{E}^n$ . The independence models are of the form

$$\mathbb{E}_i^n(\mathbb{R}^n) = (F^n, L_{\mathbb{R}^n}, D_{\mathbb{R}^n}, B_i) \quad \text{for} \quad i = 1, 2, 4$$

and

$$\mathbb{E}_j^n(\mathbb{R}^n) = (F^n, L_{\mathbb{R}^n}, D_{\mathbb{R}^n}, B_j) \quad \text{for} \quad j = 1, 2$$

for some  $(\mathbb{R}^n) \in \text{SOF}$ . Thus, we have to define relations  $B_1, B_2, B_4, B_{31}$ , and  $B_{32}$ , such that  $B_i$  ( $B_{3j}$ ) satisfies all the axioms except  $A_i$  ( $A_{3.j}$ ). Let

$$B_1 = B_{\mathbb{R}^n} - \{(aab) : a \neq b\}, \quad B_{31} = (F^n)^3, \\ B_2 = L_{\mathbb{R}^n}, \quad B_{32} = \emptyset.$$

It remains to define  $B_4$ . Consider first an arbitrary line  $K$  in the space  $\mathbb{E}^n(\mathbb{R}^n) = (F^n, L_{\mathbb{R}^n}, D_{\mathbb{R}^n})$  and let  $f: K \rightarrow K$  be a bijection. Let

$$B^{Kf}(abc) \Leftrightarrow \begin{cases} B_{\mathbb{R}^n}(abc) & \text{if } \{abc\} \not\subset K, \\ B_{\mathbb{R}^n}(f(a)f(b)f(c)) & \text{if } \{abc\} \subset K. \end{cases}$$

It is easy to prove the following

LEMMA.  $(F^n, L_{\mathbb{R}^n}, D_{\mathbb{R}^n}, B^{Kf})$  is a model of  $\mathcal{E}^n \cup \{A1, A2, A3\}$ .

Let us choose now a bijection  $f$  as follows. Fix three points  $a, b, c \in F^n$  such that

$$\neq (abc) \quad \text{and} \quad B_{\mathbb{R}^n}(abc),$$

and let  $f$  satisfy the conditions:

$$f(a) = a, \quad f(b) = c, \quad f(c) = b.$$

Then  $B^{Kf}(abc)$  and  $\sim B^{Kf}(ab'c')$  if  $b', c' \notin K$ . But there exist  $b', c' \notin K$  such that  $abc \equiv_{\mathbb{R}^n} ab'c'$ , whence  $B^{Kf}$  does not satisfy A4. Thus, by Lemma, setting

$$B_4 = B^{Kf},$$

we get a desired relation.

Finally, let us turn to the axiom A5. It was stated in [1] p. 76 that A4 is independent of  $\mathcal{E}^n \cup \{A1, A2, A3, A5\}$ . We shall prove even more (comp. [2]): A4 is independent of  $\mathcal{E}^n \cup \{A1, A2, A3, A5'\}$ . To obtain a suitable model, we choose particular  $(\mathbb{R}^n) \in \text{SOF}$  and  $f$  in  $\mathbb{E}_4^n(\mathbb{R}^n)$ . Let  $\mathbb{R} = \mathbb{R}^+$ ; let  $e_1(x) = (x, 0, \dots, 0) \in \mathbb{R}^n$  and  $K = e_1(\mathbb{R})$ . There is a bijection

$$f_0: \mathbb{R} \rightarrow \mathbb{R}$$

satisfying the conditions:

$$f_0(0) = 0, \quad f_0(\sqrt{2}) = \sqrt{3}, \quad f_0(\sqrt{3}) = \sqrt{2}$$

and

$$f_0(x+y) = f_0(x) + f_0(y).$$

Let  $f: K \rightarrow K$  be defined by the formula

$$f(e_1(x)) = e_1(f_0(x)),$$

and let

$$B_5 = B^{Kf}.$$

The structure  $\mathcal{E}_5^n(\mathfrak{R} R^+) = (R^n, L_{\mathfrak{R}}, D_{\mathfrak{R}}, B_5)$  does not satisfy A4, while it is a model of  $\mathcal{E}^n \cup \{A1, A2, A3, A5'\}$ . Indeed, it suffices to verify A5'. If  $\{abcd\} \notin K$  then is obviously satisfied. Let  $a, b, c, d \in K$ . Then  $a = e_1(x)$ ,  $b = e_1(y)$ ,  $c = e_1(d = e_1(v))$  for some  $x, y, z, v \in R$ . Assume

$$B^{Kf}(abd) \wedge B^{Kf}(bcd) \wedge bc \equiv ad;$$

then  $B_{\mathfrak{R}R^+}(f(a)f(b)f(d)) \wedge B_{\mathfrak{R}R^+}(f(b)f(c)f(d)) \wedge ||b-c|| = ||a-d||$ , thus

$$[f_0(x) \leq f_0(y) \leq f_0(z) \leq f_0(v) \vee f_0(x) \geq f_0(y) \geq f_0(z) \geq f_0(v)] \wedge |y-z| = |x-v|.$$

In turn

$$|y-z| = |x-v| \Rightarrow x+z = y+v \vee x+y = z+v$$

$$\Rightarrow f_0(x) + f_0(z) = f_0(y) + f_0(v) \vee f_0(x) + f_0(y) = f_0(z) + f_0(v),$$

whence  $f_0(z) = f_0(v)$  and therefore  $c = d$ .

Thus  $A4 \notin \text{Cn}(A1, A2, A3, A5')$  and so A4 cannot be replaced by A5'.

It is not difficult to check that A4 becomes dependent in the presence of WP.

In conclusion

$$\mathcal{O}\mathcal{E}^n = \text{Cn}(\mathcal{E}^n \cup \{A1, A2, A3, \text{WP}\}),$$

and so the weak Pasch axiom is the only plane axiom of ordered Euclidean geometry, concerning the betweenness relation.

#### References

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## Pointed and unpointed shape and pro-homotopy

by

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**Abstract.** In the paper we consider whether every unpointed shape morphism can be realized as a pointed shape morphism and whether every pointed shape morphism being an unpointed shape equivalence is also a pointed shape equivalence.

**1. Introduction.** The main pointed shape invariants, i.e., pointed 1-movability, pointed movability, being pointed FANR are at the same time invariants of the unpointed shape theory (see [10] and [12]). However it is not known whether they are hereditary shape invariants. On the way to attack this problem arise the following questions:

**QUESTION 1.** Let  $(X, x)$  and  $(Y, y)$  be pointed continua and let  $f: X \rightarrow Y$  be a shape morphism. Does there exist a morphism  $g: (X, x) \rightarrow (Y, y)$  such that the induced morphism  $g': X \rightarrow Y$  is equal to  $f$ ?

**QUESTION 2.** Let  $(X, x)$  and  $(Y, y)$  be pointed continua and let  $f: (X, x) \rightarrow (Y, y)$  be a shape morphism such that the induced morphism  $f': X \rightarrow Y$  is an isomorphism. Is  $f$  an isomorphism?

The analogous questions may be considered in pro-homotopy.

In this paper we consider the above questions. We show that in general the answers to Question 1 and Question 2 (in pro-homotopy) are negative. However they can be positively answered in some special cases.

Specially interesting is Question 2 because the negative answer to it would give a weak proper homotopy equivalence not being a proper homotopy equivalence which existence has been asked by T. A. Chapman and L. C. Siebenmann [7].

**2. Notations and terminology.** By  $H(H_0)$  we denoted the homotopy category of (pointed) connected CW complexes.

For any category  $C$  we denote by  $\text{pro-}C$  its pro-category (see [1] and [19]) and by  $\text{tow}(C)$  we denote a full subcategory of  $\text{pro-}C$  whose objects are towers i.e. inverse sequences in  $C$ . (see [11]).

By  $F: \text{pro-}H_0 \rightarrow \text{pro-}H$  we denote the forgetful functor obtained by suppressing base points.