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CONCHOID OF NICOMEDES AND LIMACON OF PASCAL AS ELECTRODE OF STATIC FIELD AND AS WAVEGUIDE OF HIGH FREQUENCY WAVE

W. Lin and Z. Yu

University of Electronic Science and Technology of China
Chengdu, 610054, China

E. K. N. Yuang and K. M. Luk

City University of Hong Kong
Hong Kong, China

Abstract–Groove guides have sharp corners in the guide proper so are not low loss and high power carrying wave guides. Proposed has been a groove rectangular waveguide with rounded internal angles [1]. However, the latter is not practical in having the infinite arms. Let us try the Conchoid of Nicomedes and Limacon of Pascal to see how can they be boundaries of static fields or high frequency waves. Also has been proposed the pentagon waveguide [2] for the high power transmission in high frequency. Yet we feel that we have very limited forms of waveguides, so we study the existing curves to see whether we can find some better guides such as rounded corner rectangular waveguide and others by starting from the boundary curves of $\rho = a/\cos\varphi + l$ and $\rho = a\cos\varphi + l$.

curves		Xmax	Ymax
1	1.5	-0.31446	0.17288
2	2.0	-0.58031	0.45031
3	3.0	-1.05364	1.12201
4	4.0	-1.54969	1.87326
5	5.0	-1.93892	2.66870
6	7.0	-2.75078	4.33455
7	10.	-3.53989	6.94745

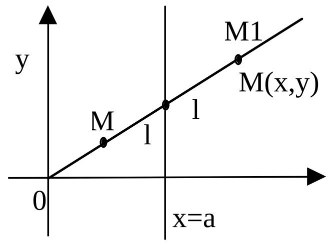

Figure 1. Locus of $M(x, y)$.

1. CONCHOID OF NICOMEDES

We now see how to construct the Conchoid of Nicomedes, given a pole 0 and a straight line, the base line $x = a$, Fig. 1, and a line segment l, $l > 0$, we can write down how to describe $M1$ and $M2$

$$\rho = \frac{a}{\cos\varphi} + l \tag{1a}$$

$$x = a + l\cos\varphi \tag{1b}$$

$$y = a\tan\varphi + l\sin\varphi \tag{1c}$$

$$\cos\varphi = \frac{x}{\sqrt{x^2 + y^2}} \tag{1d}$$

So that

$$(x - a)^2 |z|^2 = l^2 x^2 \tag{2}$$

We can write from (1a), by the multiplication of $e^{j\varphi}$

$$z = x + jy = le^{j\varphi} + a\frac{e^{j\varphi}}{\cos\varphi}$$

now we construct a new equation containing $\omega = u + j\nu$, by replacing φ by $b\omega$, b is a constant to be fixed

$$z = le^{jbw} + a\frac{e^{jbw}}{\cos(bw)}$$

and then take $jb = 1$ for simplicity we have then

$$z = le^{w} + a\frac{e^{w}}{\cosh(w)} = le^{\omega} + a(1 + \tanh w) \tag{3}$$

where $bw = -ju + \varphi$, so for $u = 0,\ \varphi = \nu$.

Equation (2) describes curve in Fig. 2 and Fig. 3, which can be the boundaries of waveguide of groove type and close type of good quality. The tall arms of the former will go up rapidly to meet each other at $x = a$.

2. LIMACON OF PASCAL

Now in Fig. 4, we have a pole o and a base curve of a circle

$$(x - a/2)^2 + y^2 = a^2/4$$

instead of a base line, so the points $M1$ and $M2$ obey the equation in polar coordinates

$$\rho = a\cos\varphi + l \tag{4a}$$

so that

$$x = \frac{a}{2} + \frac{a}{2}\cos 2\varphi + l\cos\varphi \tag{4b}$$

$$y = \frac{a}{2}\sin 2\varphi + l\sin\varphi \tag{4c}$$

now (4b) gives

$$x - \frac{a}{2} = \frac{a}{2}\cos 2\varphi + l\cos\varphi$$

so together with (4c) we have

$$\left(x - \frac{a}{2}\right)^2 + y^2 = \frac{a^2}{4} + al\cos\varphi + l^2$$

that is

$$x^2 + y^2 - ax = l(a\cos\varphi + l) = l\rho$$

by (4a), we finally have then

$$\left(x^2 + y^2 - ax\right)^2 = l^2\left(x^2 + y^2\right) \tag{5}$$

as the equation of curves in Fig. 5, which are then boundaries of a new family of rectangular waveguides of rounded inner corner, so the cut-off wavelength of the first mode will be approximately

$$\lambda_c = 4y_{\max} \tag{6}$$

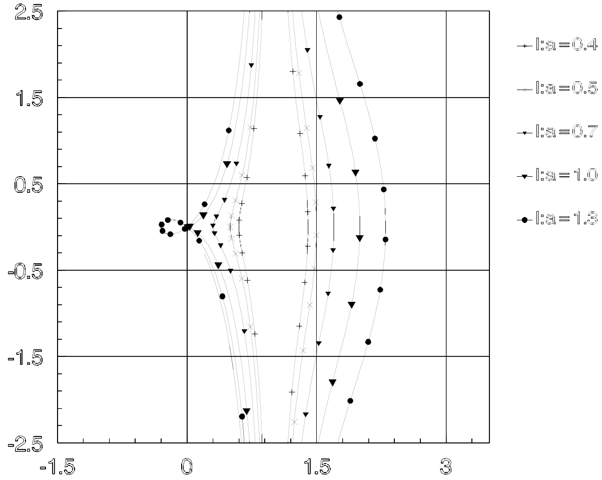

Figure 2. Groove guide the tall arms of which do not open up to infinity but asymptotically meets the base line $x = a$. We may take $a = 1$, they may be electrode of static field or guide of surface waves. The equation of the curves are $x = a + l\cos\varphi$ and $y = a\tan\varphi + l\sin\varphi$.

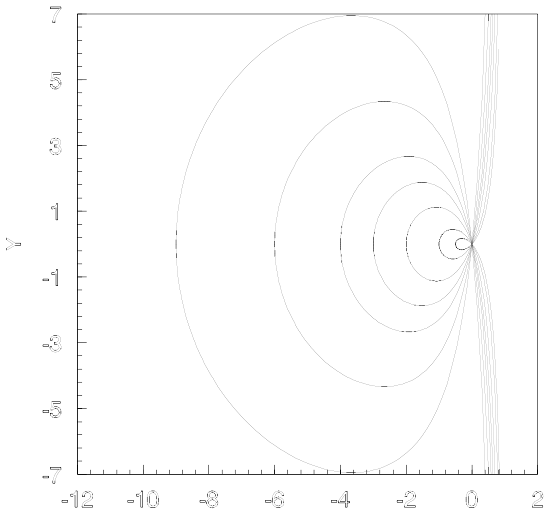

Figure 3. The close waveguides, when l changed from 1.5 to 10 and $a = 1$. The equations of the curves are $x = a + l\cos\varphi$, $y = a\tan\varphi + l\sin\varphi$, $x_{\max} = (l^2a)^{1/3} - l$; $y_{\max} = 2(la^{1/3} - al^{1/3}) : (l^{2/3} + a^{2/3})^{1/2}$.

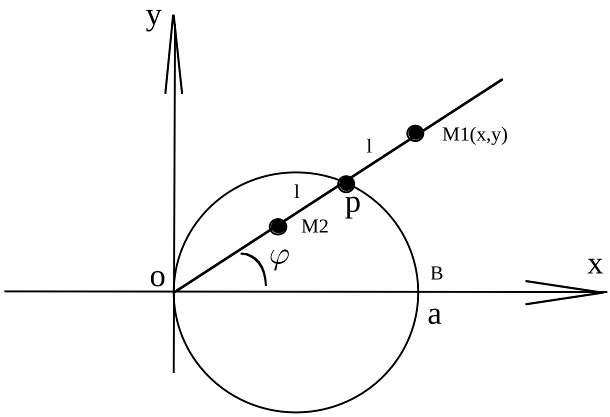

Figure 4. Base Circle, $OB = a$, Locus of $M1$ and $M2$.

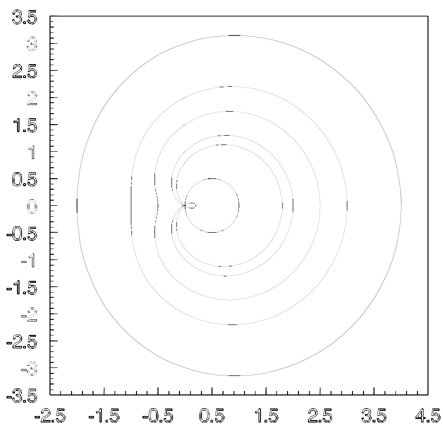

Figure 5. The Limacon Curves $(\rho = a\cos\varphi + l)$ $a = 1$, $l = 0.8$; 1; 1.5; 2; 3. There are two loops, when $l < a$ the inner loop shrinks with increasing l. When $l \geq a$, there is one loop. We can have ridge waveguide and close empty guide. $x = a\cos^2\varphi + l\cos\varphi$, $y = a\sin\varphi\cos\varphi + l\sin\varphi$.

where the $(x_{\max}, y_{\max})$ are the maximums on the Fig. 5 and are on the curve

$$\rho = a\frac{\sin^2\varphi}{\cos\varphi}$$

so together with (5) we find

$$x_{Max} = a\frac{l^2+1}{l^2+2}$$
$$y_{Max} = x_{\max}\sqrt{l^2+1}$$

From (4a), replace φ by bw, $w = u + j\nu$, make $jb = 1$, we find, by timing by $e^{j\varphi}$ first,

$$z = e^w(a\mathrm{cosh}w + l) = \frac{a}{2} + e^w\left(\frac{a}{2}e^{2w} + l\right) \tag{6a}$$

therefore we find, as needed in (5)

$$\begin{aligned} x^2 + y^2 &= e^{2u}|a\mathrm{cosh}u\cos\nu + l + ja\mathrm{sinh}u\sin\nu|^2 \\ &= e^{2u}\left\{(\mathrm{cosh}2u + \cos 2\nu)a^2 + l^2 + 2al\mathrm{cosh}u\cos\nu\right\} \end{aligned} \tag{6b}$$

and

$$\left|z - \frac{a}{2}\right|^2 = e^{2u}\left|\frac{a}{2}e^w + l\right|^2 = e^{2u}\left(\frac{a^2}{4}e^{2u} + l^2 + ale^u\cos\nu\right) \tag{6c}$$

and

$$\left|z - \frac{a}{2}\right|^2 - \frac{a^2}{4}e^{2u}e^{-2u} = e^{2u}\left(\frac{a^2}{2}\mathrm{sinh}2u + l^2 + ale^u\cos\nu\right) = x^2 + y^2 - ax \tag{6d}$$

Now from (6a) we find

$$\frac{a}{2}e^{2w} + le^{2w} - \left(z - \frac{a}{2}\right) = 0$$

So we can solve for e^w,

$$e^w = \frac{1}{a}\left(-l \pm \sqrt{l^2 + 2a\left(z - \frac{a}{2}\right)}\right)$$

and then we have,

$$w = u + jv,$$

$$e^{2u} = \frac{1}{a^2}\left|-l \pm \sqrt{l^2 - a^2 + 2a\left(z - \frac{a}{2}\right)}\right|^2 \tag{7a}$$

and we have transformed (5) to (7), with u as a parameter, let

$$m + jn = \sqrt{l^2 - a^2 + 2a\left(z - \frac{a}{2}\right)},$$

then

$$a^2 e^{2u} = l^2 \pm 2lm + m^2 + n^2$$

so finally

$$\left[a^2 e^{2u} - l^2 - m^2 - n^2\right]^2 = 4l^2 m^2 \tag{7b}$$

where

$$\begin{aligned} m^2 + n^2 &= \pm\sqrt{(l^2 + 2ax)^2 + 4a^2y^2} \\ m^2 &= \frac{1}{2}\left\{l^2 - 2ax \pm \sqrt{(l^2 + 2ax)^2 + 4a^2y^2}\right\} \\ n^2 &= \frac{1}{2}\left\{l^2 - 2ax \pm \sqrt{(l^2 + 2ax)^2 + 4a^2y^2}\right\} \end{aligned}$$

of course the curve (5) in the x-y plane of curve in Fig. 5 can be transform into curve in $w = u + j\nu$ plane by means of (6b) and (6d)

$$\begin{aligned} &e^{2u}\left(\frac{a^2}{4}\text{sinh}2u + l^2 + ale^u \cos\nu\right)^2 \\ &\qquad = l^2\left([\text{cosh}2u + \cos\nu]\, a^2 + l^2 + 2al\text{cosh}u \cos\nu\right) \end{aligned} \tag{8a}$$

so that

$$\left|\frac{dz}{dw}\right|^2 = e^{2u}\left\{a^2 e^{2u} + 2ale^u \cos\nu + l^2\right\} \tag{8b}$$

and for the inner region of a guide, $u < 0$, $\left|\frac{dz}{dw}\right|^2 \approx e^{2u} l^2$, so we may find the solution of the guide problem by starting from this approximation.

3. THE THEORY

Now eq. (2) is the well known concloid of Nicomedes from which we construct a complex equation (3): z in term of ω, so we can interpret $\omega = u + j\nu$ as a complex potential, the real part u and the imaginary part ν of which are both solution of the Laplace's equation, by the

Cauchy Riemahn relations between u and ν. When $u = 0$, $\omega = j\nu$, and (3) goes back to eq. (1), so (1) is the zero potential boundary of the static field defined by (3).

Also we can interpret (3) as a coordinate transformation, transforming the points on the z-plane onto points on the w-plane, so that the two dimensional Helmoholtz equation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + k_c^2\psi = 0 \tag{9}$$

is changed into that in the w-plane as

$$\frac{\partial^2\psi}{\partial u^2} + \frac{\partial^2\psi}{\partial \nu^2} + k_c^2\left|\frac{dz}{dw}\right|^2 = 0 \tag{10}$$

While in the case where u is a static potential, the magnitude $\left|\frac{dw}{dz}\right|$ is the field intensity E,

$$E = \left|\frac{dw}{dz}\right| = \frac{1}{h} \tag{11}$$

From (3), we have

$$\begin{aligned}\frac{dw}{dz} &= le^{\omega} + \left(\frac{1}{\cos u \cos\nu + j\mathrm{sinh}(u)\sin\nu}\right)^2 \\ &= le^{\omega} + \frac{\{\cosh(u)\cos(\nu) - j\mathrm{sinh}u\sin\nu\}^2}{(\cos 2u + \cos 2\nu)^2}\end{aligned} \tag{12}$$

So that (6) becomes, making $a = 1$ without loss of generality

$$\begin{aligned}h^2 = &\left\{le^u\cos\nu + \frac{\cos 2u\cos 2\nu + 1}{\cos 2u\cos 2\nu + \dfrac{\mathrm{cosh}4u + \cos 4\nu + 2}{4}}\right\} \\ &+ \sin^2\nu\left\{le^u - \frac{\mathrm{sinh}2u\cos\nu + 1}{\cos 2u\cos 2\nu + \dfrac{\cos 4u + \cos 4\nu + 2}{4}}\right\}\end{aligned} \tag{13}$$

Now from (3) we find, let $a = 1$,

$$x = le^u\cos\nu + \frac{e^u + \cos 2\nu}{\mathrm{cosh}2u + \cos 2\nu} \tag{14a}$$

$$y = \sin\nu\left(le^u + \frac{2\cos\nu}{\mathrm{cosh}2u + \cos 2\nu}\right) \tag{14b}$$

to observe how do u and ν vary along the real axial $y=0$, we see

(1) $y=0,\ \nu=0,\ x=le^u+1+\tanh u \geq 0$ for $-\infty \leq u \leq \infty$

(2) $y=0,\ \nu=\pi,$ $\begin{array}{l} x=-le^{\omega}+1+\tanh u=0 \text{ for } u=0; \\ x\to 0 \text{ as } u\to-\infty,\ x\to-\infty \text{ as } u\to\infty \end{array}$

(3) $y=0,\ le^u=-\dfrac{2\cos\nu}{\cosh 2u+\cos 2\nu}$ then (14a) gives

$$x=-\frac{e^{2u}-1}{\cosh 2u+\cos 2\nu} \tag{14c}$$

which gives $x=0$ when $u=0$ if $y=0$ only, while (14a) gives $x=l\cos\nu+1$. So we have to go back to (3) to investigate w as function of z, by changing (3) into the following equation, letting $a=1$:

$$e^{3w}+\frac{(2-z)}{l}e^{2\omega}+e^{\omega}+e^{\omega}-z/l=0 \tag{15}$$

we can solve e^{ω} from this equation, first by solving

$$\xi^3+p\xi+q=0 \tag{16}$$

$$e^{\omega}=\xi-\frac{1}{3}\frac{2-z}{l} \tag{17}$$

$$p=1-\frac{1}{3}\left(\frac{2-z}{l}\right)^3$$
$$q=\frac{-z}{l}-\frac{1}{3}\left(\frac{2-z}{l}\right)+\frac{2}{27}\left(\frac{2-z}{l}\right)^3$$

the discriminate Δ is

$$\begin{aligned}\Delta&=\left(\frac{q}{3}\right)^3+\left(\frac{q}{2}\right)^2\\&=\frac{1}{27}\left\{1-\frac{1}{l^2}+\frac{z}{l}\left[\left(\frac{z-2}{l}\right)^2+2\left(\frac{z-2}{l}\right)+\frac{14}{l}\right]\right\}\\&=\frac{1}{27}\left\{1-\frac{1}{l^2}+\frac{z}{l}\eta\right\}\end{aligned}$$

where

$$\eta=\left(\frac{z-2}{l}\right)^2+2\left(\frac{z-2}{l}\right)+\frac{14}{l}$$

Here η is given by prescribed z and l, usually, prescribed is $z = x+jy$. Then (16) is solved, and solved is e^{ω} in (17), so that we find the static field E, (11) and the scale factor h^2, (13), enabling us to study the field ψ in (10). From (13), $e^u = |e^w| = \left|\xi - \frac{1}{2}\frac{2-z}{l}\right| = F(x, y)$, so that (2) is transform into an equation with u of (3) as a parameter.

Of course (2) can be transformed into an equation of u and ν, a curve in u-ν-plane. To do so, we find from (3)

$$x - a = le^u \cos\nu + \frac{a\text{sinh}2u}{\cos 2u + \cos 2\nu} \tag{18a}$$

$$|z|^2 = e^{2u}\left(l^2 + \frac{4al\text{cosh}u \cos\nu}{\text{cosh}2u + \cos 2\nu}\right) \tag{18b}$$

and write (2) in the from

$$(x-a)^2\left|\frac{z}{l}\right|^2 = (x-a+a)^2 = (x-a)^2 + 2a(x-a) + a^2$$

so that we can solve for $(x-a)$:

$$(x-a) = \frac{a \pm a\left|\frac{z}{l}\right|}{\left|\frac{z}{l}\right|^2 - 1}$$

and to remove the $\pm$ sign, we have to write

$$\left(x - a - \frac{a}{\left(\frac{z}{l}\right)^2 - 1}\right)^2 = \frac{\left(a\frac{z}{l}\right)^2}{\left(\left(\frac{z}{l}\right)^2 - 1\right)^2}$$

then putting in (15a) and (15b) we have an equation in u and ν completely, for curves in u-ν plane or w-plane of (2) of x and y variables. Now from (3) we find

$$\frac{dz}{dw} = le^w + (1 - \text{tanh}^2 w)$$

Making use of

$$\frac{d}{dw}\text{tanh}w = \left(\text{tanh}^2 w - 1\right) = \frac{\cos^2 2\nu + 2j\text{cosh}2u \sin 2\nu}{(\text{cosh}2u + \cos 2\nu)^2},$$

so that it is nearly true that $\left|\frac{dz}{dw}\right|^2 \approx l^2 e^{2u}$ and is free of ν so is separable in variable u and ν.

4. APPLICATIONS

A plot eq. (2), with $l > 0$ as a parameter, $x = a$ is the asymptote in all the cases of different l. So the vertical arms will be closed for large y, not open up to infinity. Each curve of Fig. 2 can be made to act as an electrode when charged and/or to act as waveguides, in three difference ways: (a) The part in the left half plane, shown also the maximum y and its positions can be some forms of ridge wave guide to give wide operating frequency ranges; (b) On the right are tall waveguides good for high microwave power and high frequency transmissions which call for high clearance for high microwave field and for low wavelength; (c) The whole structure is a guiding structure to a surface wave, i.e., the whole is artificial dielectric wave guide.

If the configuration of Fig. 2 is charged with a charge Q per meter the field intensity can be found by (14), at the surface $u = 0$, it can be calculated from (14),

$$|E| = \varepsilon \left|\frac{dz}{dw}\right|^{-1} = \frac{Q}{2\pi l}\left\{1 + \frac{2}{x-1} + \left(\frac{2}{x-1}\right)^4\right\}^{-1} \tag{19}$$

We plot Fig. 5 similarly as another family of smooth wave guides, so we have very large choices of guide figure in applications. Detail studies will follow in coming papers.

5. CONCLUDING REMARKS

Nowadays we have triangular, rectangular, pentagon, circular, elliptical and parabolic waveguides. In this paper we propose two families of waveguides whose boundaries are described by simple equations

$$\rho = a/\cos\varphi + l.$$

and

$$\rho = a\cos\varphi + l$$

respectively.

They are shown in Fig. 2, Fig. 3 and Fig. 5. They look like regular rectangular waveguide with smooth, rounded inner corners, so they should be more efficient in transmission of high frequency power. Further study, numerically and experimentally will be carried out and will be reported shortly.

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