PAPERS COMMUNICATED

47. *Concircular Geometry I. Concircular Transformations.*

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§1. Let $C: u^\lambda(s)$ be a curve in a Riemannian space V_n whose fundamental quadratic form is

$$ds^2 = g_{\mu\nu} du^\mu du^\nu , \qquad (\lambda, \mu, \nu, \dots = 1, 2, 3, \dots, n). \tag{1.1}$$

Denoting the unit tangent, and unit normals of order $1, 2, \dots, n-1$ and the first, second, ... $(n-1)$-st curvatures of C by $\underset{1}{\xi}^\lambda, \underset{2}{\xi}^\lambda, \dots, \underset{n}{\xi}^\lambda$ and $\overset{1}{\varkappa}, \overset{2}{\varkappa}, \dots, \overset{n-1}{\varkappa}$ respectively, the Frenet equations of C may be written as

$$\frac{\delta}{\delta s} \underset{a}{\xi}^\lambda = -\overset{a-1}{\varkappa} \underset{a-1}{\xi}^\lambda + \overset{a}{\varkappa} \underset{a+1}{\xi}^\lambda , \qquad (a = 1, 2, \dots, n;\ \overset{0}{\varkappa} = \overset{n}{\varkappa} = 0), \tag{1.2}$$

where $\delta/\delta s$ denotes covariant differentiation with respect to arc length s along C.

A geodesic circle[1] is defined as a curve whose first curvature is constant and whose second curvature is identically zero. For such a geodesic circle, we have, from (1.2),

$$\frac{\delta}{\delta s} \underset{1}{\xi}^\lambda = \overset{1}{\varkappa} \underset{2}{\xi}^\lambda , \tag{1.3}$$

$$\frac{\delta}{\delta s} \underset{2}{\xi}^\lambda = -\overset{1}{\varkappa} \underset{1}{\xi}^\lambda , \tag{1.4}$$

where $\overset{1}{\varkappa}$ is a constant. Differentiating (1.3) covariantly and then substituting (1.4) in the obtained equation, we have

$$\frac{\delta^2}{\delta s^2} \underset{1}{\xi}^\lambda = -(\overset{1}{\varkappa})^2 \underset{1}{\xi}^\lambda . \tag{1.5}$$

The $\underset{1}{\xi}^\lambda$ denoting the unit tangent, we may put

$$\underset{1}{\xi}^\lambda = \frac{\delta u^\lambda}{\delta s} ,$$

so that we have, from (1.3),

$$(\overset{1}{\varkappa})^2 = g_{\mu\nu} \frac{\delta^2 u^\mu}{\delta s^2} \frac{\delta^2 u^\nu}{\delta s^2} .$$

The equation (1.5) then becomes

$$\frac{\delta^3 u^\lambda}{\delta s^3} + g_{\mu\nu} \frac{\delta^2 u^\mu}{\delta s^2} \frac{\delta^2 u^\nu}{\delta s^2} \frac{\delta u^\lambda}{\delta s} = 0 . \tag{1.6}$$

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1) A. Fialkow: Conformal geodesics, Trans. Amer. Math. Soc. **45** (1939), 443–473.

Conversely, if the equation (1.6) is satisfied, we have

$$\frac{\delta}{\delta s}\left(g_{\mu\nu}\frac{\delta^2 u^\mu}{\delta s^2}\frac{\delta^2 u^\nu}{\delta s^2}\right)=2g_{\mu\nu}\frac{\delta^3 u^\mu}{\delta s^3}\frac{\delta^2 u^\nu}{\delta s^2}$$

$$=-2\left(g_{\alpha\beta}\frac{\delta^2 u^\alpha}{\delta s^2}\frac{\delta^2 u^\beta}{\delta s^2}\right)g_{\mu\nu}\frac{\delta u^\mu}{\delta s}\frac{\delta^2 u^\nu}{\delta s^2}=0\,,$$

consequently, the first curvature $\overset{1}{\varkappa}$ which appears in

$$\frac{\delta}{\delta s}\underset{1}{\xi}{}^\nu=\quad\overset{1}{\varkappa}\underset{2}{\xi}{}^\lambda$$

is a constant. The first curvature $\overset{1}{\varkappa}$ being a constant, the differentiation of

$$\underset{2}{\xi}{}^\lambda=\frac{1}{\overset{1}{\varkappa}}\frac{\delta}{\delta s}\underset{1}{\xi}{}^\lambda$$

gives us

$$\frac{\delta}{\delta s}\underset{2}{\xi}{}^\lambda=\frac{1}{\overset{1}{\varkappa}}\frac{\delta^2}{\delta s^2}\underset{1}{\xi}{}^\lambda=-\frac{1}{\overset{1}{\varkappa}}(\overset{1}{\varkappa})^2\underset{1}{\xi}{}^\lambda=-\overset{1}{\varkappa}\underset{1}{\xi}{}^\lambda\,.$$

We can, consequently, see that the second curvature $\overset{2}{\varkappa}$ is identically zero. We can then conclude that the equations (1.6) are differential equations of geodesic circles.

§ 2. We shall now consider a conformal transformation

$$\bar{g}_{\mu\nu}=\rho^2 g_{\mu\nu} \tag{2.1}$$

of the fundamental tensor $g_{\mu\nu}$. A geodesic circle is not in general transformed into a geodesic circle by this conformal transformation. The arc length s and the Christoffel symbols $\{{}^{\lambda}_{\mu\nu}\}$ being transformed by

$$\frac{d\bar{s}}{ds}=\rho\,, \tag{2.2}$$

$$\overline{\{{}^{\lambda}_{\mu\nu}\}}=\{{}^{\lambda}_{\mu\nu}\}+\rho_\mu\delta^\lambda_\nu+\rho_\nu\delta^\lambda_\mu-g^{\lambda\alpha}\rho_\alpha g_{\mu\nu}\,, \tag{2.3}$$

respectively, where

$$\rho_\mu=\frac{\partial\log\rho}{\partial u^\mu}\,, \tag{2.4}$$

we have

$$\frac{\delta u^\lambda}{\delta\bar{s}}=\frac{1}{\rho}\frac{\delta u^\lambda}{\delta s}\,, \tag{2.5}$$

$$\frac{\delta^2 u^\lambda}{\delta\bar{s}^2}=\frac{1}{\rho^2}\left[\frac{\delta^2 u^\lambda}{\delta s^2}+\rho_\mu\frac{\delta u^\mu}{\delta s}\frac{\delta u^\lambda}{\delta s}-g^{\lambda\alpha}\rho_\alpha\right], \tag{2.6}$$

$$\begin{aligned}\frac{\delta^3 u^\lambda}{\delta\bar{s}^3}=\frac{1}{\rho^3}\Bigg[&\frac{\delta^3 u^\lambda}{\delta s^3}+\rho_{\mu;\nu}\frac{\delta u^\mu}{\delta s}\frac{\delta u^\nu}{\delta s}\frac{\delta u^\lambda}{\delta s}-\rho^\lambda{}_{;\nu}\frac{\delta u^\nu}{\delta s}\\&+\rho^\lambda\rho_\nu\frac{\delta u^\nu}{\delta s}-g^{\alpha\beta}\rho_\alpha\rho_\beta\frac{\delta u^\lambda}{\delta s}+2\rho_\mu\frac{\delta^2 u^\mu}{\delta s^2}\frac{\delta u^\lambda}{\delta s}\Bigg],\end{aligned} \tag{2.7}$$

where

$$\rho^\lambda = g^{\lambda\alpha}\rho_\alpha\,, \qquad \rho_{\mu;\,\nu} = \frac{\partial \rho_\mu}{\partial u^\nu} - \rho_\lambda \left\{ {\lambda \atop \mu\nu} \right\},$$

and

$$\rho^\lambda{}_{;\,\nu} = g^{\lambda\alpha}\rho_{\alpha;\,\nu}\,.$$

These successive derivatives being calculated, we have, from (2.6),

$$\text{(2.8)} \qquad \bar{g}_{\mu\nu}\frac{\delta^2 u^\mu}{\delta \bar{s}^2}\frac{\delta^2 u^\nu}{\delta \bar{s}^2}\frac{\delta u^\lambda}{\delta \bar{s}} = \frac{1}{\rho^3}\Big[g_{\mu\nu}\frac{\delta^2 u^\mu}{\delta s^2}\frac{\delta^2 u^\nu}{\delta s^2}\frac{\delta u^\lambda}{\delta s} - \rho_\mu\rho_\nu\frac{\delta u^\mu}{\delta s}\frac{\delta u^\nu}{\delta s}\frac{\delta u^{\lambda'}}{\delta s} + g^{\alpha\beta}\rho_\alpha\rho_\beta\frac{\delta u^\lambda}{\delta s} - 2\rho_\mu\frac{\delta^2 u^\mu}{\delta s^2}\frac{\delta u^\lambda}{\delta s}\Big].$$

The equations (2.7) and (2.8) give us

$$\text{(2.9)} \qquad \frac{\delta^3 u^\lambda}{\delta \bar{s}^3} + \bar{g}_{\mu\nu}\frac{\delta^2 u^\mu}{\delta \bar{s}^2}\frac{\delta^2 u^\nu}{\delta \bar{s}^2}\frac{\delta u^\lambda}{\delta \bar{s}} = \frac{1}{\rho^3}\Big[\frac{\delta^3 u^\lambda}{\delta s^3} + g_{\mu\nu}\frac{\delta^2 u^\mu}{\delta s^2}\frac{\delta^2 u^\nu}{\delta s^2}\frac{\delta u^\lambda}{\delta s} + \rho_{\mu\nu}\frac{\delta u^\mu}{\delta s}\frac{\delta u^\nu}{\delta s}\frac{\delta u^\lambda}{\delta s} - \rho^\lambda{}_\nu\frac{\delta u^\nu}{\delta s}\Big],$$

where

$$\text{(2.10)} \qquad \rho_{\mu\nu} = \rho_{\mu;\,\nu} - \rho_\mu\rho_\nu + \frac{1}{2}g^{\alpha\beta}\rho_\alpha\rho_\beta g_{\mu\nu} \quad \text{and} \quad \rho^\lambda{}_\nu = g^{\lambda\rho}\rho_{\alpha\nu}\,.$$

Then we can see that a curve whose conformal transform is a geodesic circle may be defined as a solution of the differential equations

$$\text{(2.11)} \qquad \frac{\delta^3 u^\lambda}{\delta s^3} + g_{\mu\nu}\frac{\delta^2 u^\mu}{\delta s^2}\frac{\delta^2 u^\nu}{\delta s^2}\frac{\delta u^\lambda}{\delta s} + \rho_{\mu\nu}\frac{\delta u^\mu}{\delta s}\frac{\delta u^\nu}{\delta s}\frac{\delta u^\lambda}{\delta s} - \rho^\lambda{}_\nu\frac{\delta u^\nu}{\delta s} = 0\,.$$

We may call such a curve conformal geodesic circle. It may be noticed that the so-called conformal geodesic is a conformal geodesic circle.

If a conformal transformation (2.1) transforms every geodesic circle into a geodesic circle, then the function ρ must satisfy the partial differential equations

$$\text{(2.12)} \qquad \rho_{\mu\nu} = \phi g_{\mu\nu}\,.$$ [1]

It has been shown by A. Fialkow[2] that there exists actually a very large class of V_n's which admit such transformations.

Since a conformal transformation with ρ satisfying (2.12) changes a geodesic circle into a geodesic circle, we shall call it concircular transformation and concircular geometry the geometry in which we concern only with the concircular transformation (2.12) and with the spaces admitting such transformations.

§ 3. Denoting by $R^\lambda_{\mu\nu\omega}$ the curvature tensor of our Riemannian space V_n, we can show by a straight-forword culculation that the cur-

1) See H. W. Brinkmann: Einstein spaces which are mapped conformally on each other. Math. Ann. **94** (1925), 119–145.

2) A. Fialkow, loc. cit. § 12, p. 470.

vature tensor $R^{\lambda}_{\mu\nu\omega}$ is tranformed into $\bar{R}^{\lambda}_{\mu\nu\omega}$ by a conformal transformation (2.1) where

$$\bar{R}^{\lambda}_{\mu\nu\omega}=R^{\lambda}_{\mu\nu\omega}-\rho_{\mu\nu}\delta^{\lambda}_{\omega}+\rho_{\mu\omega}\delta^{\lambda}_{\nu}-g_{\mu\nu}\rho^{\lambda}{}_{\omega}+g_{\mu\omega}\rho^{\lambda}{}_{\nu}\,. \tag{3.1}$$

If the conformal transformation (2.1) is a concircular one, the equation (3.1) becomes

$$\bar{R}^{\lambda}_{\mu\nu\omega}=R^{\lambda}_{\mu\nu\omega}-2\phi(g_{\mu\nu}\delta^{\lambda}_{\omega}-g_{\mu\omega}\delta^{\lambda}_{\nu})\,. \tag{3.2}$$

Contracting, in this equation, with respect to the indices λ and ω, we obtain

$$\bar{R}_{\mu\nu}=R_{\mu\nu}-2(n-1)\phi g_{\mu\nu}\,, \tag{3.3}$$

where

$$\bar{R}_{\mu\nu}=\bar{R}^{\lambda}_{\mu\nu\lambda}\,,\qquad R_{\mu\nu}=R^{\lambda}_{\mu\nu\lambda}\,.$$

Contracting $\bar{g}^{\mu\nu}=\dfrac{1}{\rho^2}g^{\mu\nu}$, we can obtain ϕ from (3.3), say,

$$\bar{R}=\frac{1}{\rho^2}[R-2n(n-1)\phi]\,,$$

$$2\phi=-\frac{\rho^2\bar{R}-R}{n(n-1)}\,, \tag{3.4}$$

where

$$\bar{R}=\bar{g}^{\mu\nu}\bar{R}_{\mu\nu}\,,\qquad R=g^{\mu\nu}R_{\mu\nu}\,.$$

Substituting the value of ϕ into (3.2), we find

$$\bar{R}^{\lambda}_{\mu\nu\omega}=R^{\lambda}_{\mu\nu\omega}+\frac{\rho^2\bar{R}-R}{n(n-1)}(g_{\mu\nu}\delta^{\lambda}_{\omega}-g_{\mu\omega}\delta^{\lambda}_{\nu})\,,$$

or

$$\bar{R}^{\lambda}_{\mu\nu\omega}-\frac{\bar{R}}{n(n-1)}(\bar{g}_{\mu\nu}\delta^{\lambda}_{\omega}-\bar{g}_{\mu\omega}\delta^{\lambda}_{\nu})=R^{\lambda}_{\mu\nu\omega}-\frac{R}{n(n-1)}(g_{\mu\nu}\delta^{\lambda}_{\omega}-g_{\mu\omega}\delta^{\lambda}_{\nu})\,, \tag{3.5}$$

which shows that the tensor

$$Z^{\lambda}_{\mu\nu\omega}=R^{\lambda}_{\mu\nu\omega}-\frac{R}{n(n-1)}(g_{\mu\nu}\delta^{\lambda}_{\omega}-g_{\mu\omega}\delta^{\lambda}_{\nu}) \tag{3.6}$$

is invariant under a concircular transformation.

Contracting with respect to the indices λ and ω, we have from (3.6)

$$Z_{\mu\nu}=Z^{\lambda}_{\mu\nu\lambda}=R_{\mu\nu}-\frac{R}{n}g_{\mu\nu}\,, \tag{3.7}$$

which is also invariant under a concircular transformation. It is easily seen that the contracted tensor $g^{\mu\nu}Z_{\mu\nu}$ vanishes identically.

§ 4. We shall, in this Paragraph, prove the following

Theorem I. The necessary and sufficient condition that a Riemannian *space V_n may be reduced to a Euclidean space by a suitable concircular transformation is that the concircularly invariant tensor $Z^{\lambda}_{\mu\nu\omega}$ should vanish identically.*

Proof: Suppose that we can reduce the curvature tensor $\bar{R}^{\lambda}_{\mu\nu\omega}$ to zero, then we have from (3.5)

$$Z^{\lambda}_{\mu\nu\omega}=R^{\lambda}_{\mu\nu\omega}-\frac{R}{n(n-1)}(g_{\mu\nu}\delta^{\lambda}_{\omega}-g_{\mu\omega}\delta^{\lambda}_{\nu})=0\,. \tag{4.1}$$

Conversely, if the concircularly invariant tensor $Z^{\lambda}_{\mu\nu\omega}$ vanishes identically, we have

$$R^{\lambda}_{\mu\nu\omega}=\frac{R}{n(n-1)}(g_{\mu\nu}\delta^{\lambda}_{\omega}-g_{\mu\omega}\delta^{\lambda}_{\nu})\,, \tag{4.2}$$

then we can see that the scalar curvature R is a constant.

Substituting the equation (4.2) into (3.2), we find

$$\bar{R}^{\lambda}_{\mu\nu\omega}=\left[\frac{R}{n(n-1)}-2\phi\right](g_{\mu\nu}\delta^{\lambda}_{\omega}-g_{\mu\omega}\delta^{\lambda}_{\nu})\,.$$

To reduce $\bar{R}^{\lambda}_{\mu\nu\omega}$ to zero, we must have

$$2\phi=\frac{R}{n(n-1)}\,,$$

which is a constant, consequently, if we choose a concircular transformation such that

$$\rho_{\mu\nu}=\frac{R}{2n(n-1)}g_{\mu\nu}\,, \tag{4.3}$$

the curvature tensor $\bar{R}^{\lambda}_{\mu\nu\omega}$ may be reduced to zero. We shall now show that the partial differential equations (4.3) are completely integrable. The equations (4.3) may be written as

$$\rho_{\mu;\,\nu}=\rho_{\mu}\rho_{\nu}-\left[\frac{1}{2}g^{\alpha\beta}\rho_{\alpha}\rho_{\beta}-\frac{R}{2n(n-1)}\right]g_{\mu\nu}\,. \tag{4.4}$$

Differentiating these equations covariantly, we have

$$\rho_{\mu;\,\nu;\,\omega}=\rho_{\mu;\,\omega}\rho_{\nu}+\rho_{\mu}\rho_{\nu;\,\omega}-g^{\alpha\beta}\rho_{\alpha;\,\omega}\rho_{\beta}g_{\mu\nu}\,. \tag{4.5}$$

Substituting (4.4) into (4.5), we obtain

$$\begin{aligned}\rho_{\mu;\,\nu;\,\omega}=&\left[\rho_{\mu}\rho_{\omega}-\frac{1}{2}\left\{g^{\alpha\beta}\rho_{\alpha}\rho_{\beta}-\frac{R}{n(n-1)}\right\}g_{\mu\omega}\right]\rho_{\nu}\\&+\left[\rho_{\nu}\rho_{\omega}-\frac{1}{2}\left\{g^{\alpha\beta}\rho_{\alpha}\rho_{\beta}-\frac{R}{n(n-1)}\right\}g_{\nu\omega}\right]\rho_{\mu}\\&-g^{\alpha\beta}\left[\rho_{\alpha}\rho_{\omega}-\frac{1}{2}\left\{g^{\gamma\delta}\rho_{\gamma}\rho_{\delta}-\frac{R}{n(n-1)}\right\}g_{\alpha\omega}\right]\rho_{\beta}g_{\mu\nu}\,,\end{aligned} \tag{4.6}$$

from which we have

$$
\begin{aligned}
(4.7) \qquad -\rho_a R^a_{\mu\nu\omega} &= \rho_{\mu;\,\nu;\,\omega} - \rho_{\mu;\,\omega;\,\nu} \\
&= -\rho_a \frac{R}{n(n-1)} (g_{\mu\nu}\delta^a_\omega - g_{\mu\omega}\delta^a_\nu)
\end{aligned}
$$

which is identically satisfied. Then the theorem is proved.

We shall call concircularly flat space a space whose concircular curvature tensor $Z^\lambda_{\mu\nu\omega}$ vanishes identically. A concircularly flat space being a space of constant curvature, we have

Theorem II. A space of constant curvature is transformed into a space of constant curvature by a concircular transformation.

If the concircular tensor $Z_{\mu\nu}$ vanishes identically, then the space is an Einstein space, consequently we have

Theorem III. An Einstein space is tranformed into an Einstein space by a concircular transformation.