CONCISE TENSORS OF MINIMAL BORDER RANK

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Abstract. We determine defining equations for the set of concise tensors of minimal border rank in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ when $m = 5$ and the set of concise minimal border rank 1_*-generic tensors when $m = 5, 6$. We solve the classical problem in algebraic complexity theory of classifying minimal border rank tensors in the special case $m = 5$. Our proofs utilize two recent developments: the 111-equations defined by Buczyńska-Buczyński and results of Jelisiejew-Šivic on the variety of commuting matrices. We introduce a new algebraic invariant of a concise tensor, its 111-algebra, and exploit it to give a strengthening of Friedland's normal form for 1-degenerate tensors satisfying Strassen's equations. We use the 111-algebra to characterize wild minimal border rank tensors and classify them in $\mathbb{C}^5 \otimes \mathbb{C}^5 \otimes \mathbb{C}^5$.

1. Introduction

This paper is motivated by algebraic complexity theory and the study of secant varieties in algebraic geometry. It takes first steps towards overcoming complexity lower bound barriers first identified in [22, 26]. It also provides new "minimal cost" tensors for Strassen's laser method to upper bound the exponent of matrix multiplication that are not known to be subject to the barriers identified in [2] and later refined in numerous works, in particular [10] which shows there are barriers for minimal border rank *binding* tensors (defined below), as our new tensors are not binding.

Let $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m = A \otimes B \otimes C$ be a tensor. One says T has *rank one* if $T = a \otimes b \otimes c$ for some nonzero $a \in A$, $b \in B$, $c \in C$, and the *rank* of T, denoted $\mathbf{R}(T)$, is the smallest r such that T may be written as a sum of r rank one tensors. The *border rank* of T, denoted $\underline{\mathbf{R}}(T)$, is the smallest r such that T may be written as a limit of a sum of r rank one tensors. In geometric language, the border rank is smallest r such that T belongs to the r-th secant variety of the Segre variety, $\sigma_r(Seg(\mathbb{P}^{m-1} \times \mathbb{P}^{m-1} \times \mathbb{P}^{m-1})) \subseteq \mathbb{P}(\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m)$.

Informally, a tensor T is *concise* if it cannot be expressed as a tensor in a smaller ambient space. (See §1.1 for the precise definition.) A concise tensor $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ must have border rank at least m, and if the border rank equals m, one says that T has *minimal border rank*.

As stated in [16], tensors of minimal border rank are important for algebraic complexity theory as they are "an important building stone in the construction of fast matrix multiplication algorithms". More precisely, tensors of minimal border rank have produced the best upper bound on the exponent of matrix multiplication [21, 46, 51, 41, 1] via Strassen's laser method [48]. Their investigation also has a long history in classical algebraic geometry as the study of secant varieties of Segre varieties.

2010 *Mathematics Subject Classification.* 68Q15, 15A69, 14L35.

Key words and phrases. Tensor rank, border rank, secant variety, Segre variety, Quot scheme, spaces of commuting matrices, spaces of bounded rank, smoothable rank, wild tensor, 111-algebra.

Landsberg supported by NSF grants AF-1814254 and AF-2203618. Jelisiejew supported by National Science Centre grant 2018/31/B/ST1/02857.

Problem 15.2 of [16] asks to classify concise tensors of minimal border rank. This is now understood to be an extremely difficult question. The difficulty manifests itself in two substantially different ways:

- *Lack of structure.* Previous to this paper, an important class of tensors (1-*degenerate*, see §1.1) had no or few known structural properties. In other words, little is known about the geometry of singular loci of secant varieties.
- *Complicated geometry.* Under various genericity hypotheses that enable one to avoid the previous difficulty, the classification problem reduces to hard problems in algebraic geometry: for example the classification of minimal border rank *binding* tensors (see §1.1) is equivalent to classifying smoothable zero-dimensional schemes in affine space [35, §5.6.2], a longstanding and generally viewed as impossible problem in algebraic geometry, which is however solved for $m \leq 6$ [42, 44].

The main contributions of this paper are as follows: (i) we give equations for the set of concise minimal border rank tensors for $m \leq 5$ and classify them, (ii) we discuss and consolidate the theory of minimal border rank 1_*-generic tensors, extending their characterization in terms of equations to $m \leq 6$, and (iii) we introduce a new structure associated to a tensor, its *111-algebra*, and investigate new invariants of minimal border rank tensors coming from the 111-algebra.

Our contributions allow one to streamline proofs of earlier results. This results from the power of the 111-equations, and the utilization of the ADHM correspondence discussed below. While the second leads to much shorter proofs and enables one to avoid using the classification results of [50, 37], there is a price to be paid as the language and machinery of modules and the Quot scheme need to be introduced. This language will be essential in future work, as it provides the only proposed path to overcome the lower bound barriers of [22, 26], namely *deformation theory.* We emphasize that this paper is the first direct use of deformation theory in the study of tensors. Existing results from deformation theory were previously used in [9].

Contribution (iii) addresses the *lack of structure* and motivates many new open questions, see §1.4.

1.1. **Results on tensors of minimal border rank.** Given $T \in A \otimes B \otimes C$, we may consider it as a linear map $T_C : C^* \to A \otimes B$. We let $T(C^*) \subseteq A \otimes B$ denote its image, and similarly for permuted statements. A tensor T is A-*concise* if the map T_A is injective, i.e., if it requires all basis vectors in A to write down T in any basis, and T is *concise* if it is A, B, and C concise.

A tensor $T \in \mathbb{C}^{\mathbf{a}} \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ is 1_A-*generic* if $T(A^*) \subseteq B \otimes C$ contains an element of rank m and when $\mathbf{a} = m$, T is 1-*generic* if it is 1_A, 1_B, and 1_C generic. Define a tensor $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ to be 1_*-*generic* if it is at least one of 1_A, 1_B, or 1_C-generic, and *binding* if it is at least two of 1_A, 1_B, or 1_C-generic. We say T is 1-*degenerate* if it is not 1_*-generic. Note that if T is 1_A generic, it is both B and C concise. In particular, binding tensors are concise.

Two classical sets of equations on tensors that vanish on concise tensors of minimal border rank are Strassen's equations and the End-closed equations. These are discussed in §2.1. These equations are sufficient for $m \leq 4$, [27, Prop. 22], [47, 24].

In [13, Thm 1.3] the following polynomials for minimal border rank were introduced: Let $T \in A \otimes B \otimes C = \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$. Consider the map

$$(T(A^*) \otimes A) \oplus (T(B^*) \otimes B) \oplus (T(C^*) \otimes C) \to A \otimes B \otimes C \oplus A \otimes B \otimes C \tag{1.1}$$

that sends (T_1, T_2, T_3) to $(T_1 - T_2, T_2 - T_3)$, where the A, B, C factors of tensors are understood to be in the correct positions, for example $T(A^*)\otimes A$ is more precisely written as $A\otimes T(A^*)$. If T has border rank at most m, then the rank of the above map is at most $3m^2 - m$. The resulting equations are called the *111-equations*.

Consider the space

$$(T(A^*)\otimes A) \cap (T(B^*)\otimes B) \cap (T(C^*)\otimes C). \tag{1.2}$$

We call this space the *triple intersection* or the *111-space*. We say that T is *111-abundant* if the inequality

$$\text{(111–abundance)} \quad \dim\big((T(A^*)\otimes A) \cap (T(B^*)\otimes B) \cap (T(C^*)\otimes C)\big) \geq m \tag{1.3}$$

holds. If equality holds, we say T is *111-sharp*. When T is concise, 111-abundance is equivalent to requiring that the equations of [13, Thm 1.3] are satisfied, i.e., the map (1.1) has rank at most $3m^2 - m$.

Example 1.1. For $T = a_1\otimes b_1\otimes c_2 + a_1\otimes b_2\otimes c_1 + a_2\otimes b_1\otimes c_1 \in \mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$, a tangent vector to the Segre variety, also called the W-state in the quantum literature, the triple intersection is $\langle T, a_1\otimes b_1\otimes c_1\rangle$.

We show that for concise tensors, the 111-equations imply both Strassen's equations and the End-closed equations:

Proposition 1.2. *Let $T \in \mathbb{C}^m\otimes\mathbb{C}^m\otimes\mathbb{C}^m$ be concise. If T satisfies the 111-equations then it also satisfies Strassen's equations and the End-closed equations. If T is 1_A generic, then it satisfies the 111-equations if and only if it satisfies the A-Strassen equations and the A-End-closed equations.*

The first assertion is proved in §3.3. The second assertion is Proposition 3.2.

In [43], and more explicitly in [40], equations generalizing Strassen's equations for minimal border rank, called $p = 1$ *Koszul flattenings* were introduced. (At the time it was not clear they were a generalization, see [39] for a discussion.). The $p = 1$ Koszul flattenings of type 210 are equations that are the size $m(m-1)+1$ minors of the map $T_A^{\wedge 1} : A\otimes B^* \to \Lambda^2 A\otimes C$ given by $a\otimes\beta \mapsto \sum T^{ijk}\beta(b_j)a \wedge a_i\otimes c_k$. Type 201, 120, etc. are defined by permuting A, B and C. Together they are called $p = 1$ Koszul flattenings. These equations reappear in border apolarity as the 210-equations, see [20].

Proposition 1.3. *The $p = 1$ Koszul flattenings for minimal border rank and the 111-equations are independent, in the sense that neither implies the other, even for concise tensors in $\mathbb{C}^m\otimes\mathbb{C}^m\otimes\mathbb{C}^m$.*

Proposition 1.3 follows from Example 3.5 where the 111-equations are nonzero and the $p = 1$ Koszul flattenings are zero and Example 5.9 where the reverse situation holds.

We extend the characterization of minimal border rank tensors under the hypothesis of 1_*-genericity to dimension $m = 6$, giving two different characterizations:

Theorem 1.4. *Let $m \leq 6$ and consider the set of tensors in $\mathbb{C}^m\otimes\mathbb{C}^m\otimes\mathbb{C}^m$ which are 1_*-generic and concise. The following subsets coincide*

1. *the zero set of Strassen's equations and the End-closed equations,*
2. *111-abundant tensors,*

(3) 111-sharp tensors,

(4) minimal border rank tensors.

More precisely, in (1), if the tensor is 1_A-generic, only the A-Strassen and A-End-closed conditions are required.

The equivalence of (1), (2), (3) in Theorem 1.4 is proved by Proposition 3.2. The equivalence of (1) and (4) is proved in §8.

For 1_A-generic tensors, the $p = 1$ Koszul flattenings of type 210 or 201 are equivalent to the A-Strassen equations, hence they are implied by the 111-equations in this case. However, the other types are not implied, see Example 5.9.

The result fails for $m \geq 7$ by [37, Prop. 5.3], see Example 5.9. This is due to the existence of additional components in the *Quot scheme*, which we briefly discuss here.

The proof of Theorem 1.4 introduces new algebraic tools by reducing the study of 1_A-generic tensors satisfying the A-Strassen equations to *deformation theory* in the Quot scheme (a generalization of the Hilbert scheme, see [34]) in two steps. First one reduces to the study of commuting matrices, which implicitly appeared already in [47], and was later spelled out in in [37], see §2. Then one uses the ADHM construction as in [34]. From this perspective, the tensors satisfying (1)-(3) correspond to points of the Quot scheme, while tensors satisfying (4) correspond to points in the *principal component* of the Quot scheme, see §8.1 for explanations; the heart of the theorem is that when $m \leq 6$ there is only the principal component. We expect deformation theory to play an important role in future work on tensors. As discussed in [20], at this time deformation theory is the *only* proposed path to overcoming the lower bound barriers of [22, 26]. As another byproduct of this structure, we obtain the following proposition:

Proposition 1.5. *A 1-generic tensor in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ with $m \leq 13$ satisfying the A-Strassen equations has minimal border rank. A 1_A and 1_B-generic tensor in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ with $m \leq 7$ satisfying the A-Strassen equations has minimal border rank.*

Proposition 1.5 is sharp: the first assertion does not hold for higher m by [31, Lem. 6.21] and the second by [17].

Previously it was known (although not explicitly stated in the literature) that the A-Strassen equations combined with the A-End-closed conditions imply minimal border rank for 1-generic tensors when $m \leq 13$ and binding tensors when $m \leq 7$. This can be extracted from the discussion in [35, §5.6].

While Strassen's equations and the End-closed equations are nearly useless for 1-degenerate tensors, this does not occur for the 111-equations, as the following result illustrates:

Theorem 1.6. *When $m \leq 5$, the set of concise minimal border rank tensors in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ is the zero set of the 111-equations.*

We emphasize that no other equations, such as Strassen's equations, are necessary. Moreover Strassen's equations, or even their generalization to the $p = 1$ Koszul flattenings, and the End-closed equations are not enough to characterize concise minimal border rank tensors in $\mathbb{C}^5 \otimes \mathbb{C}^5 \otimes \mathbb{C}^5$, see Example 3.5 and §1.4.3.

By Theorem 1.4, to prove Theorem 1.6 it remains to prove the 1-degenerate case, which is done in §7. The key difficulty here is the above-mentioned lack of structure. We overcome this problem by providing a new normal form, which follows from the 111-equations, that strengthens Friedland's normal form for corank one 1_A-degenerate tensors satisfying Strassen's equations [24, Thm. 3.1], see Proposition 3.3.

It is possible that Theorem 1.6 also holds for $m = 6$; this will be subject to future work. It is false for $m = 7$, as already Theorem 1.4 fails when $m = 7$.

The 1_*-generic tensors of minimal border rank in $\mathbb{C}^5 \otimes \mathbb{C}^5 \otimes \mathbb{C}^5$ are essentially classified in [37], following the classification of abelian linear spaces in [50]. We write "essentially", as the list has redundancies and it remains to determine the precise list. Using our normal form, we complete (modulo the redundancies in the 1_*-generic case) the classification of concise minimal border rank tensors:

Theorem 1.7. *Up to the action of $\mathrm{GL}_5(\mathbb{C})^{\times 3} \rtimes \mathfrak{S}_3$, there are exactly five concise 1-degenerate, minimal border rank tensors in $\mathbb{C}^5 \otimes \mathbb{C}^5 \otimes \mathbb{C}^5$. Represented as spaces of matrices, the tensors may be presented as:*

$$T_{\mathcal{O}_{58}} = \begin{pmatrix} x_1 & & x_2 & x_3 & x_5 \\ x_5 & x_1 & x_4 & -x_2 & \\ & & x_1 & & \\ & & -x_5 & x_1 & \\ & & & x_5 & \end{pmatrix}, \quad T_{\mathcal{O}_{57}} = \begin{pmatrix} x_1 & & x_2 & x_3 & x_5 \\ & x_1 & x_4 & -x_2 & \\ & & x_1 & & \\ & & & x_1 & \\ & & & x_5 & \end{pmatrix},$$

$$T_{\mathcal{O}_{56}} = \begin{pmatrix} x_1 & & x_2 & x_3 & x_5 \\ & x_1 + x_5 & & x_4 & \\ & & x_1 & & \\ & & & x_1 & \\ & & & x_5 & \end{pmatrix}, \quad T_{\mathcal{O}_{55}} = \begin{pmatrix} x_1 & & x_2 & x_3 & x_5 \\ & x_1 & x_5 & x_4 & \\ & & x_1 & & \\ & & & x_1 & \\ & & & x_5 & \end{pmatrix}, \quad T_{\mathcal{O}_{54}} = \begin{pmatrix} x_1 & & x_2 & x_3 & x_5 \\ & x_1 & & x_4 & \\ & & x_1 & & \\ & & & x_1 & \\ & & & x_5 & \end{pmatrix}.$$

In tensor notation: set

$$T_{\mathrm{M1}} = a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_2 + b_3 \otimes c_3 + b_4 \otimes c_4) + a_2 \otimes b_3 \otimes c_1 + a_3 \otimes b_4 \otimes c_1 + a_4 \otimes b_4 \otimes c_2 + a_5 \otimes (b_5 \otimes c_1 + b_4 \otimes c_5)$$

and

$$T_{\mathrm{M2}} = a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_2 + b_3 \otimes c_3 + b_4 \otimes c_4) + a_2 \otimes (b_3 \otimes c_1 - b_4 \otimes c_2) + a_3 \otimes b_4 \otimes c_1 + a_4 \otimes b_3 \otimes c_2 + a_5 \otimes (b_5 \otimes c_1 + b_4 \otimes c_5).$$

Then

$$\begin{aligned} T_{\mathcal{O}_{58}} &= T_{\mathrm{M2}} + a_5 \otimes (b_1 \otimes c_2 - b_3 \otimes c_4) \\ T_{\mathcal{O}_{57}} &= T_{\mathrm{M2}} \\ T_{\mathcal{O}_{56}} &= T_{\mathrm{M1}} + a_5 \otimes b_2 \otimes c_2 \\ T_{\mathcal{O}_{55}} &= T_{\mathrm{M1}} + a_5 \otimes b_3 \otimes c_2 \\ T_{\mathcal{O}_{54}} &= T_{\mathrm{M1}}. \end{aligned}$$

Moreover, each subsequent tensor lies in the closure of the orbit of previous: $T_{\mathcal{O}_{58}} \trianglerighteq T_{\mathcal{O}_{57}} \trianglerighteq T_{\mathcal{O}_{56}} \trianglerighteq T_{\mathcal{O}_{55}} \trianglerighteq T_{\mathcal{O}_{54}}$.

The subscript in the name of each tensor is the dimension of its $\mathrm{GL}(A) \times \mathrm{GL}(B) \times \mathrm{GL}(C)$ orbit in projective space $\mathbb{P}(A \otimes B \otimes C)$. Recall that $\dim \sigma_5(Seg(\mathbb{P}^4 \times \mathbb{P}^4 \times \mathbb{P}^4)) = 64$ and that it is the orbit closure of the so-called unit tensor $[\sum_{j=1}^5 a_j \otimes b_j \otimes c_j]$.

Among these tensors, $T_{\mathcal{O}_{58}}$ is (after a change of basis) the unique symmetric tensor on the list (see Example 4.6 for its symmetric version). The subgroup of $\mathrm{GL}(A)\times\mathrm{GL}(B)\times\mathrm{GL}(C)$ preserving $T_{\mathcal{O}_{58}}$ contains a copy of $\mathrm{GL}_2\mathbb{C}$ while all other stabilizers are solvable.

The *smoothable rank* of a tensor $T\in A\otimes B\otimes C$ is the minimal degree of a smoothable zero dimensional scheme $\mathrm{Spec}(R)\subseteq\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C$ which satisfies the condition $T\in\langle\mathrm{Spec}(R)\rangle$. See, e.g., [49, 14] for basic definitions regarding zero dimensional schemes.

The smoothable rank of a polynomial with respect to the Veronese variety was introduced in [45] and generalized to points with respect to arbitrary projective varieties in [11]. It arises because the span of the (scheme theoretic) limit of points may be smaller than the limit of the spans. The smoothable rank lies between rank and border rank. Tensors (or polynomials) whose smoothable rank is larger than their border rank are called *wild* in [11]. The first example of a wild tensor occurs in $\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3$, see [11, §2.3] and it has minimal border rank. We characterize wild minimal border rank tensors:

Theorem 1.8. *The concise minimal border rank tensors that are wild are precisely the concise minimal border rank* 1*-degenerate tensors.*

Thus Theorem 1.7 classifies concise wild minimal border rank tensors in $\mathbb{C}^5\otimes\mathbb{C}^5\otimes\mathbb{C}^5$.

The proof of Theorem 1.8 utilizes a new algebraic structure arising from the triple intersection that we discuss next.

1.2. **The 111-algebra and its uses.** We emphasize that 111-abundance, as defined by (1.3), is a necessary condition for border rank m only when T is concise. The condition can be defined for arbitrary tensors and we sometimes allow that.

Remark 1.9. The condition (1.3) is not closed: for example it does not hold for the zero tensor. It is however closed in the set of concise tensors as then $T(A^*)$ varies in the Grassmannian, which is compact.

For $X\in\mathrm{End}(A)=A^*\otimes A$, let $X\circ_A T$ denote the corresponding element of $T(A^*)\otimes A$. Explicitly, if $X=\alpha\otimes a$, then $X\circ_A T:=T(\alpha)\otimes a$ and the map $(-)\circ_A T\colon\mathrm{End}(A)\to A\otimes B\otimes C$ is extended linearly. Put differently, $X\circ_A T=(X\otimes\mathrm{Id}_B\otimes\mathrm{Id}_C)(T)$. Define the analogous actions of $\mathrm{End}(B)$ and $\mathrm{End}(C)$.

Definition 1.10. Let T be a concise tensor. We say that a triple $(X,Y,Z)\in\mathrm{End}(A)\times\mathrm{End}(B)\times\mathrm{End}(C)$ *is compatible with* T if $X\circ_A T=Y\circ_B T=Z\circ_C T$. The *111-algebra* of T is the set of triples compatible with T. We denote this set by $\mathcal{A}^T_{111}$.

The name is justified by the following theorem:

Theorem 1.11. *The 111-algebra of a concise tensor* $T\in A\otimes B\otimes C$ *is a commutative unital subalgebra of* $\mathrm{End}(A)\times\mathrm{End}(B)\times\mathrm{End}(C)$ *and its projection to any factor is injective.*

Theorem 1.11 is proved in §4.

Example 1.12. Let T be as in Example 1.1. Then

$$\mathcal{A}^T_{111}=\langle(\mathrm{Id},\mathrm{Id},\mathrm{Id}),(a_1\otimes\alpha_2,b_1\otimes\beta_2,c_1\otimes\gamma_2)\rangle.$$

In this language, the triple intersection is $\mathcal{A}^T_{111} \cdot T$. Once we have an algebra, we may study its modules. The spaces A, B, C are all $\mathcal{A}^T_{111}$-modules: the algebra $\mathcal{A}^T_{111}$ acts on them as it projects to $\mathrm{End}(A)$, $\mathrm{End}(B)$, and $\mathrm{End}(C)$. We denote these modules by $\underline{A}$, $\underline{B}$, $\underline{C}$ respectively.

Using the 111-algebra, we obtain the following algebraic characterization of *all* 111-abundant tensors as follows: a tensor T is 111-abundant if it comes from a bilinear map $N_1 \times N_2 \to N_3$ between m-dimensional $\mathcal{A}$-modules, where $\dim \mathcal{A} \geq m$, $\mathcal{A}$ is a unital commutative associative algebra and N_1, N_2, N_3 are $\mathcal{A}$-modules, see Theorem 5.5. This enables an algebraic investigation of such tensors and shows how they generalize abelian tensors from [37], see Example 5.6. We emphasize that there are no genericity hypotheses here beyond conciseness, in contrast with the 1_*-generic case. In particular the characterization applies to *all* concise minimal border rank tensors.

In summary, for a concise tensor T we have defined new algebraic invariants: the algebra $\mathcal{A}^T_{111}$ and its modules $\underline{A}$, $\underline{B}$, $\underline{C}$. There are four consecutive obstructions for a concise tensor to be of minimal border rank:

(1) the tensor must be 111-abundant. For simplicity of presentation, for the rest of this list we assume that it is 111-sharp (compare §1.4.1). We also fix a surjection from a polynomial ring $S = \mathbb{C}[y_1, \dots, y_{m-1}]$ onto $\mathcal{A}^T_{111}$ as follows: fix a basis of $\mathcal{A}^T_{111}$ with the first basis element equal to $(\mathrm{Id}, \mathrm{Id}, \mathrm{Id})$ and send $1 \in S$ to this element, and the variables of S to the remaining $m-1$ basis elements. In particular $\underline{A}$, $\underline{B}$, $\underline{C}$ become S-modules (the conditions below do not depend on the choice of surjection).

(2) the algebra $\mathcal{A}^T_{111}$ must be smoothable (Lemma 5.7),

(3) the S-modules $\underline{A}$, $\underline{B}$, $\underline{C}$ must lie in the principal component of the Quot scheme, so there exist a sequence of modules $\underline{A}_\epsilon$ limiting to $\underline{A}$ with general $\underline{A}_\epsilon$ semisimple, and similarly for $\underline{B}$, $\underline{C}$ (Lemma 5.8),

(4) the surjective module homomorphism $\underline{A} \otimes_{\mathcal{A}^T_{111}} \underline{B} \to \underline{C}$ associated to T as in Theorem 5.5 must be a limit of module homomorphisms $\underline{A}_\epsilon \otimes_{\mathcal{A}_\epsilon} \underline{B}_\epsilon \to \underline{C}_\epsilon$ for a choice of smooth algebras $\mathcal{A}_\epsilon$ and semisimple modules $\underline{A}_\epsilon$, $\underline{B}_\epsilon$, $\underline{C}_\epsilon$.

Condition (3) is shown to be nontrivial in Example 5.9.

In the case of 1-generic tensors, by Theorem 1.8 above, they have minimal border rank if and only if they have minimal smoothable rank, that is, they are in the span of some zero-dimensional smoothable scheme $\mathrm{Spec}(R)$. Proposition 9.1 remarkably shows that one has an algebra isomorphism $\mathcal{A}^T_{111} \cong R$. This shows that to determine if a given 1-generic tensor has minimal smoothable rank it is enough to determine smoothability of its 111-algebra, there is no choice for R. This is in contrast with the case of higher smoothable rank, where the choice of R presents the main difficulty.

Remark 1.13. While throughout we work over $\mathbb{C}$, our constructions (except for explicit computations regarding classification of tensors and their symmetries) do not use anything about the base field, even the characteristic zero assumption. The only possible nontrivial applications of the complex numbers are in the cited sources, but we expect that our main results, except for Theorem 1.7, are valid over most fields.

1.3. **Previous work on tensors of minimal border rank in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$.**

When $m = 2$ it is classical that all tensors in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ have border rank at most two.

For $m = 3$ generators of the ideal of $\sigma_3(Seg(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2))$ are given in [38].

For $m = 4$ set theoretic equations for $\sigma_4(Seg(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3))$ are given in [24] and lower degree set-theoretic equations are given in [25, 6] where in the second reference they also give numerical evidence that these equations generate the ideal. It is still an open problem to prove the known equations generate the ideal. (This is the "salmon prize problem" posed by E. Allman in 2007. At the time, not even set-theoretic equations were known).

Regarding the problem of classifying concise tensors of minimal border rank:

For $m = 3$ a complete classification of all tensors of border rank three is given in [15].

For $m = 4$, a classification of all 1_*-generic concise tensors of border rank four in $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ is given in [37].

When $m = 5$, a list of all abelian subspaces of $\mathrm{End}(\mathbb{C}^5)$ up to isomorphism is given in [50].

The equivalence of (1) and (4) in the $m = 5$ case of Theorem 1.4 follows from the results of [37], but is not stated there. The argument proceeds by first using the classification in [32], [50] of spaces of commuting matrices in $\mathrm{End}(\mathbb{C}^5)$. There are 15 isolated examples (up to isomorphism), and examples that potentially depend on parameters. (We write "potentially" as further normalization is possible.) Then each case is tested and the tensors passing the End-closed condition are proven to be of minimal border rank using explicit border rank five expressions. We give a new proof of this result that is significantly shorter, and self-contained. Instead of listing all possible tensors, we analyze the possible Hilbert functions of the associated modules in the Quot scheme living in the unique non-principal component.

1.4. **Open questions and future directions.**

1.4.1. *111-abundant, not 111-sharp tensors.* We do not know any example of a concise tensor T which is 111-abundant and is not 111-sharp, that is, for which the inequality in (1.3) is strict. By Proposition 3.2 such a tensor would have to be 1-degenerate, with $T(A^*), T(B^*), T(C^*)$ of bounded (matrix) rank at most $m - 2$, and by Theorems 1.7 and 1.6 it would have to occur in dimension greater than 5. Does there exist such an example?[1]

1.4.2. *111-abundant* 1*-degenerate tensors.* The 111-abundant tensors of bounded rank $m-1$ have remarkable properties. What properties do 111-abundant tensors with $T(A^*)$, $T(B^*)$, $T(C^*)$ of bounded rank less than $m - 1$ have?

1.4.3. *111-abundance v. classical equations.* A remarkable feature of Theorem 1.6 is that 111-equations are enough: there is no need for more classical ones, like $p = 1$ Koszul flattenings [40]. In fact, the $p = 1$ Koszul flattenings, together with End-closed condition, are almost sufficient, but not quite: the 111-equations are only needed to rule out one case, described in Example 3.5. Other necessary closed conditions for minimal border rank are known, e.g., the higher Koszul flattenings of [40], the flag condition (see, e.g., [37]), and the equations of [36]. We plan to investigate the relations between these and the new conditions introduced in this paper. As mentioned above, the 111-equations in general do not imply the $p = 1$ Koszul flattening equations, see Example 5.9.

[1] After this paper was submitted, A. Conca pointed out an explicit example of a 111-abundant, not 111-sharp tensor when $m = 9$. We do not know if such exist when $m = 6, 7, 8$. The example is a generalization of Example 4.6.

1.4.4. *111-abundance in the symmetric case.* Given a concise symmetric tensor $T \in S^3\mathbb{C}^m \subseteq \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$, one classically studies its apolar algebra $\mathcal{A} = \mathbb{C}[x_1, \dots, x_m]/\mathrm{Ann}(T)$, where $x_1, \dots, x_m$ are coordinates on the dual space $\mathbb{C}^{m*}$ and $\mathrm{Ann}(T)$ are the polynomials that give zero when contracted with T. This is a *Gorenstein* (see §2.4) zero-dimensional graded algebra with Hilbert function $(1, m, m, 1)$ and each such algebra comes from a symmetric tensor. A weaker version of Question 1.4.1 is: does there exist such an algebra with $\mathrm{Ann}(T)$ having at least m minimal cubic generators? There are plenty of examples with $m-1$ cubic generators, for example $T = \sum_{i=1}^m x_i^3$ or the 1-degenerate examples from the series [30, §7].

1.4.5. *The locus of concise, 111-sharp tensors.* There is a natural functor associated to this locus, so we have the machinery of deformation theory and in particular, it is a linear algebra calculation to determine the tangent space to this locus at a given point and, in special cases, even its smoothness. This path will be pursued further and it gives additional motivation for Question 1.4.1.

1.4.6. *111-algebra in the symmetric case.* The 111-algebra is an entirely unexpected invariant in the symmetric case as well. How is it computed and how can it be used?

1.4.7. *The Segre-Veronese variety.* While in this paper we focused on $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$, the 111-algebra can be defined for any tensor in $V_1 \otimes V_2 \otimes V_3 \otimes \dots \otimes V_q$ and the argument from §4 generalizes to show that it is still an algebra whenever $q \geq 3$. It seems worthwhile to investigate it in greater generality.

1.4.8. *Strassen's laser method.* An important motivation for this project was to find new tensors for Strassen's laser method for bounding the exponent of matrix multiplication. This method has barriers to further progress when using the Coppersmith-Winograd tensors that have so far given the best upper bounds on the exponent of matrix multiplication [2]. Are any of the new tensors we found in $\mathbb{C}^5 \otimes \mathbb{C}^5 \otimes \mathbb{C}^5$ better for the laser method than the big Coppersmith-Winograd tensor CW_3? Are any 1-degenerate minimal border rank tensors useful for the laser method? (At this writing there are no known laser method barriers for 1-degenerate tensors.)

1.5. **Overview.** In §2 we review properties of binding and more generally 1_A-generic tensors that satisfy the A-Strassen equations. In particular we establish a dictionary between properties of modules and such tensors. In §3 we show 1_A-generic 111-abundant tensors are exactly the 1_A-generic tensors that satisfy the A-Strassen equations and are A-End-closed. We establish a normal form for 111-abundant tensors with $T(A^*)$ corank one that generalizes Friedland's normal for tensors with $T(A^*)$ corank one that satisfy the A-Strassen equations. In §4 we prove Theorem 1.11 and illustrate it with several examples. In §5 we discuss 111-algebras and their modules, and describe new obstructions for a tensor to be of minimal border rank coming from its 111-algebra. In §6 we show certain classes of tensors are not concise to eliminate them from consideration in this paper. In §7 we prove Theorems 1.6 and 1.7. In §8 we prove Theorem 1.4 using properties of modules, their Hilbert functions and deformations. In §9 we prove Theorem 1.8.

1.6. **Definitions/Notation.** Throughout this paper we adopt the index ranges

$$1 \leq i, j, k \leq \mathbf{a}$$
$$2 \leq s, t, u \leq \mathbf{a} - 1,$$

and A, B, C denote complex vector spaces respectively of dimension $\mathbf{a}, m, m$. Except for §2 we will also have $\mathbf{a} = m$. The general linear group of changes of bases in A is denoted $\mathrm{GL}(A)$ and the subgroup of elements with determinant one by $\mathrm{SL}(A)$ and their Lie algebras by $\mathfrak{gl}(A)$ and $\mathfrak{sl}(A)$. The dual space to A is denoted A^*. For $Z \subseteq A$, $Z^\perp := \{\alpha \in A^* \mid \alpha(x) = 0 \forall x \in Z\}$ is its annihilator, and $\langle Z \rangle \subseteq A$ denotes the span of Z. Projective space is $\mathbb{P}A = (A \backslash \{0\})/\mathbb{C}^*$. When A is equipped with the additional structure of being a module over some ring, we denote it $\underline{A}$ to emphasize its module structure.

Unital commutative algebras are usually denoted $\mathcal{A}$ and polynomial algebras are denoted S.

Vector space homomorphisms (including endomorphisms) between m-dimensional vector spaces will be denoted K_i, X_i, X, Y, Z, and we use the same letters to denote the corresponding matrices when bases have been chosen. Vector space homomorphisms (including endomorphisms) between $(m-1)$-dimensional vector spaces, and the corresponding matrices, will be denoted $\mathbf{x}_i, \mathbf{y}, \mathbf{z}$.

We often write $T(A^*)$ as a space of $m \times m$ matrices (i.e., we choose bases). When we do this, the columns index the B^* basis and the rows the C basis, so the matrices live in $\mathrm{Hom}(B^*, C)$. (This convention disagrees with [37] where the roles of B and C were reversed.)

For $X \in \mathrm{Hom}(A, B)$, the symbol $X^{\mathbf{t}}$ denotes the induced element of $\mathrm{Hom}(B^*, A^*)$, which in bases is just the transpose of the matrix of X.

The A-*Strassen equations* were defined in [47]. The B and C Strassen equations are defined analogously. Together, we call them *Strassen's equations*. Similarly, the A-*End-closed equations* are implicitly defined in [28], we state them explicitly in (3.13). Together with their B and C counterparts they are the End-closed equations. We never work with these equations directly (except proving Proposition 1.2), we only consider the conditions they impose on 1_*-generic tensors.

For a tensor $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$, we say that $T(A^*) \subseteq B \otimes C$ is of *bounded (matrix) rank* r if all matrices in $T(A^*)$ have rank at most r, and we drop reference to "matrix" when the meaning is clear. If rank r is indeed attained, we also say that $T(A^*)$ is of *corank* $m - r$.

1.7. **Acknowledgements.** We thank M. Michałek for numerous useful discussions, in particular leading to Proposition 1.5, M. Michałek and A. Conner for help with writing down explicit border rank decompositions, and J. Buczyński for many suggestions to improve an earlier draft. Macaulay2 and its *VersalDeformation* package [33] was used in computations. We thank the anonymous referee for helpful comments.

2. Dictionaries for 1_*-generic, binding, and 1-generic tensors satisfying Strassen's equations for minimal border rank

2.1. **Strassen's equations and the End-closed equations for 1_*-generic tensors.** A 1_*-generic tensor satisfying Strassen's equations may be reinterpreted in terms of classical objects in matrix theory and then in commutative algebra, which allows one to apply existing results in these areas to their study.

Fix a tensor $T \in A \otimes B \otimes C = \mathbb{C}^{\mathbf{a}} \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ which is A-concise and 1_A-generic with $\alpha \in A^*$ such that $T(\alpha) : B^* \to C$ has full rank. The 1_A-genericity implies that T is B and C-concise.

Consider
$$\mathcal{E}_\alpha(T) := T(A^*)T(\alpha)^{-1} \subseteq \mathrm{End}(C).$$
This space is $T'(A^*)$ where $T' \in A \otimes C^* \otimes C$ is a tensor obtained from T using the isomorphism $\mathrm{Id}_A \otimes (T(\alpha)^{-1})^{\mathbf{t}} \otimes \mathrm{Id}_C$. It follows that T is of rank m if and only if the space $\mathcal{E}_\alpha(T)$ is simultaneously diagonalizable and that T is of border rank m if and only if $\mathcal{E}_\alpha(T)$ is a limit of spaces

of simultaneously diagonalizable endomorphisms [37, Proposition 2.8] also see [36]. Note that $\mathrm{Id}_C = T(\alpha)T(\alpha)^{-1} \in \mathcal{E}_\alpha(T)$.

A necessary condition for a subspace $\widetilde{E} \subseteq \mathrm{End}(C)$ to be a limit of simultaneously diagonalizable spaces of endomorphisms is that the elements of $\widetilde{E}$ pairwise commute. The A-Strassen equations [24, (1.1)] in the 1_A-generic case are the translation of this condition to the language of tensors, see, e.g., [37, §2.1]. For the rest of this section, we additionally assume that T satisfies the A-Strassen equations, i.e., that $\mathcal{E}_\alpha(T)$ is abelian.

Another necessary condition on a space to be a limit of simultaneously diagonalizable spaces has been known since 1962 [28]: the space must be closed under composition of endomorphisms. The corresponding equations on the tensor are the A-End-closed equations.

2.2. **Reinterpretation as modules.** In this subsection we introduce the language of modules and the ADHM correspondence. This extra structure will have several advantages: it provides more invariants for tensors, it enables us to apply theorems in the commutative algebra literature to the study of tensors, and perhaps most importantly, it will enable us to utilize deformation theory.

Let $\widetilde{E} \subseteq \mathrm{End}(C)$ be a space of endomorphisms that contains Id_C and consists of pairwise commuting endomorphisms. Fix a decomposition $\widetilde{E} = \langle \mathrm{Id}_C \rangle \oplus E$. A canonical such decomposition is obtained by requiring that the elements of E are traceless. To eliminate ambiguity, we will use this decomposition, although in the proofs we never make use of the fact that $E \subseteq \mathfrak{sl}(C)$. Let $S = \mathrm{Sym}\, E$ be a polynomial ring in $\dim E = \mathbf{a} - 1$ variables. By the ADHM correspondence [3], as utilized in [34, §3.2] we define the *module associated to* E to be the S-module $\underline{C}$ which is the vector space C with action of S defined as follows: let $e_1, \ldots, e_{\mathbf{a}-1}$ be a basis of E, write $S = \mathbb{C}[y_1, \ldots, y_{\mathbf{a}-1}]$, define $y_j(c) \coloneqq e_j(c)$, and extend to an action of the polynomial ring.

It follows from [34, §3.4] that $\widetilde{E}$ is a limit of simultaneously diagonalizable spaces if and only if $\underline{C}$ is a limit of *semisimple modules*, which, by definition, are S-modules of the form $N_1 \oplus N_2 \oplus \ldots \oplus N_m$ where $\dim N_h = 1$ for every h. The limit is taken in the *Quot scheme*, see [34, §3.2 and Appendix] for an introduction, and [23, §5], [49, §9] for classical sources. The Quot scheme will not be used until §5.2.

Now we give a more explicit description of the construction in the situation relevant for this paper. Let A, B, C be $\mathbb{C}$-vector spaces, with $\dim A = \mathbf{a}$, $\dim B = \dim C = m$, as above. Let $T \in A \otimes B \otimes C$ be a concise 1_A-generic tensor that satisfies Strassen's equations (see §2.1). To such a T we associated the space $\mathcal{E}_\alpha(T) \subseteq \mathrm{End}(C)$. The *module associated to* T is the module $\underline{C}$ associated to the space $\widetilde{E} \coloneqq \mathcal{E}_\alpha(T)$ using the procedure above. The procedure involves a choice of α and a basis of E, so the module associated to T is only defined up to isomorphism.

Example 2.1. Consider a concise tensor $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ of minimal rank, say $T = \sum_{i=1}^m a_i \otimes b_i \otimes c_i$ with $\{a_i\}$, $\{b_i\}$, $\{c_i\}$ bases of A, B, C and $\{\alpha_i\}$ the dual basis of A^* etc.. Set $\alpha = \sum_{i=1}^m \alpha_i$. Then $\mathcal{E}_\alpha(T)$ is the space of diagonal matrices, so $E = \langle E_{ii} - E_{11} \mid i = 2, 3, \ldots, m \rangle$ where $E_{ij} = \gamma_i \otimes c_j$. The module $\underline{C}$ decomposes as an S-module into $\bigoplus_{i=1}^m \mathbb{C} c_i$ and thus is semisimple. Every semisimple module is a limit of such.

If a module $\underline{C}$ is associated to a space $\widetilde{E}$, then the space $\widetilde{E}$ may be recovered from $\underline{C}$ as the set of the linear endomorphisms corresponding to the actions of elements of $S_{\leq 1}$ on $\underline{C}$. If $\underline{C}$ is associated to a tensor T, then the tensor T is recovered from $\underline{C}$ up to isomorphism as the tensor of the bilinear map $S_{\leq 1} \otimes \underline{C} \to \underline{C}$ coming from the action on the module.

Remark 2.2. The restriction to $S_{\le 1}$ may seem unnatural, but observe that if $\widetilde{E}$ is additionally End-closed then for every $s \in S$ there exists an element $s' \in S_{\le 1}$ such that the actions of s and s' on $\underline{C}$ coincide.

Additional conditions on a tensor transform to natural conditions on the associated module. We explain two such additional conditions in the next two subsections.

2.3. **Binding tensors and the Hilbert scheme.**

Proposition 2.3. *Let $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m = A \otimes B \otimes C$ be concise, 1_A-generic, and satisfy the A-Strassen equations. Let $\underline{C}$ be the S-module obtained from T as above. The following conditions are equivalent*

(1) *the tensor T is 1_B-generic (so it is binding),*

(2) *there exists an element $c \in \underline{C}$ such that $S_{\le 1} c = \underline{C}$,*

(3) *the S-module $\underline{C}$ is isomorphic to S/I for some ideal I and the space $\mathcal{E}_\alpha(T)$ is End-closed,*

(4) *the S-module $\underline{C}$ is isomorphic to S/I for some ideal I,*

(5) *the tensor T is isomorphic to a multiplication tensor in a commutative unital rank m algebra $\mathcal{A}$.*

The algebra $\mathcal{A}$ in (5) will be obtained from the module $\underline{C}$ as described in the proof.

The equivalence of (1) and (5) for minimal border rank tensors was first obtained by Bläser and Lysikov [9].

Proof. Suppose (1) holds. Recall that $\mathcal{E}_\alpha(T) = T'(A^*)$ where $T' \in A \otimes C^* \otimes C$ is obtained from $T \in A \otimes B \otimes C$ by means of $(T(\alpha)^{-1})^{\mathbf{t}} : B \to C^*$. Hence T' is 1_{C^*}-generic, so there exists an element $c \in (C^*)^* \simeq C$ such that the induced map $A^* \to C$ is bijective. But this map is exactly the multiplication map by c, $S_{\le 1} \to \underline{C}$, so (2) follows.

Let $\varphi : S \to \underline{C}$ be defined by $\varphi(s) = sc$ and let $I = \ker \varphi$. (Note that φ depends on our choice of c.) Suppose (2) holds; this means that $\varphi|_{S_{\le 1}}$ is surjective. Since $\dim S_{\le 1} = m = \dim C$, this surjectivity implies that we have a vector space direct sum $S = S_{\le 1} \oplus I$. Now $X \in \mathcal{E}_\alpha(T) \subseteq \mathrm{End}(C)$ acts on C in the same way as the corresponding linear polynomial $\underline{X} \in S_{\le 1}$. Thus a product $XY \in \mathrm{End}(C)$ acts as the product of polynomials $\underline{XY} \in S_{\le 2}$. Since $S = I \oplus S_{\le 1}$ we may write $\underline{XY} = U + \underline{Z}$, where $U \in I$ and $\underline{Z} \in S_{\le 1}$. The actions of $XY, Z \in \mathrm{End}(C)$ on C are identical, so $XY = Z$. This proves (3). Property (3) implies (4).

Suppose that (4) holds and take an S-module isomorphism $\varphi' : \underline{C} \to S/I$. Reversing the argument above, we obtain again $S = I \oplus S_{\le 1}$. Let $\mathcal{A} := S/I$. This is a finite algebra of rank $\dim S_{\le 1} = m$. The easy, but key observation is that the multiplication in $\mathcal{A}$ is induced by the multiplication $S \otimes \mathcal{A} \to \mathcal{A}$ on the S-module $\mathcal{A}$. The multiplication maps arising from the S-module structure give the following commutative diagram:

$$
\begin{tikzcd}
S_{\le 1} \arrow[d, hook] \arrow[dd, bend right=60, "\psi"'] & \otimes & \underline{C} \arrow[r] \arrow[d, equal] & \underline{C} \arrow[d, equal] \\
S \arrow[d, two heads] & \otimes & \underline{C} \arrow[r] \arrow[d, equal] & \underline{C} \arrow[d, equal] \\
S/I \arrow[d, equal] & \otimes & \underline{C} \arrow[r] \arrow[d, "\varphi'"] & \underline{C} \arrow[d, "\varphi'"] \\
S/I & \otimes & S/I \arrow[r] & S/I
\end{tikzcd}
$$

The direct sum decomposition implies the map ψ is a bijection. Hence the tensor T, which is isomorphic to the multiplication map from the first row, is also isomorphic to the multiplication map in the last row. This proves (5). Finally, if (5) holds, then T is 1_B-generic, because the multiplication by $1 \in \mathcal{A}$ from the right is bijective. $\square$

The structure tensor of a module first appeared in Wojtala [52]. The statement that binding tensors satisfying Strassen's equations satisfy End-closed conditions was originally proven jointly with M. Michałek. A binding tensor is of minimal border rank if and only if $\underline{C}$ is a limit of semisimple modules if and only if S/I is a *smoothable* algebra. For $m \le 7$ all algebras are smoothable [17].

2.4. **1-generic tensors.** A 1-generic tensor satisfying the A-Strassen equations is isomorphic to a symmetric tensor by [37]. (See [39] for a short proof.). For a commutative unital algebra $\mathcal{A}$, the multiplication tensor of $\mathcal{A}$ is 1-generic if and only if $\mathcal{A}$ is *Gorenstein*, see [35, Prop. 5.6.2.1]. By definition, an algebra $\mathcal{A}$ is Gorenstein if $\mathcal{A}^* = \mathcal{A}\phi$ for some $\phi \in \mathcal{A}^*$, or in tensor language, if its structure tensor $T_{\mathcal{A}}$ is 1-generic with $T_{\mathcal{A}}(\phi) \in \mathcal{A}^* \otimes \mathcal{A}^*$ of full rank. For $m \le 13$ all Gorenstein algebras are smoothable [18], proving Proposition 1.5.

2.5. **Summary.** We obtain the following dictionary for tensors in $\mathbb{C}^{\mathbf{a}} \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ with $\mathbf{a} \le m$:

tensor satisfying A-Strassen eqns.	is isomorphic to	multiplication tensor in
1_A-generic		module
1_A- and 1_B-generic (hence binding and $\mathbf{a} = m$)		unital commutative algebra
1-generic ($\mathbf{a} = m$)		Gorenstein algebra

3. Implications of 111-abundance

For the rest of this article, we restrict to tensors $T \in A \otimes B \otimes C = \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$. Recall the notation $X \circ_A T$ from §1.2 and that $\{a_i\}$ is a basis of A. In what follows we allow $\widetilde{a}_h$ to be arbitrary elements of A.

Lemma 3.1. *Let $T = \sum_{h=1}^{r} \widetilde{a}_h \otimes K_h$, where $\widetilde{a}_h \in A$ and $K_h \in B\otimes C$ are viewed as maps $K_h\colon B^* \to C$. Let $X \in \mathrm{End}(A)$, $Y \in \mathrm{End}(B)$ and $Z \in \mathrm{End}(C)$. Then*

$$X \circ_A T = \sum_{h=1}^{r} X(\widetilde{a}_h) \otimes K_h,$$

$$Y \circ_B T = \sum_{h=1}^{r} \widetilde{a}_h \otimes (K_h Y^{\mathbf{t}}),$$

$$Z \circ_C T = \sum_{h=1}^{r} \widetilde{a}_h \otimes (Z K_h).$$

If T is concise and Ω is an element of the triple intersection (1.2), then the triple (X, Y, Z) such that $\Omega = X \circ_A T = Y \circ_B T = Z \circ_C T$ is uniquely determined. In this case we call X, Y, Z the matrices corresponding to Ω.

Proof. The first assertion is left to the reader. For the second, it suffices to prove it for X. Write $T = \sum_{i=1}^{m} a_i \otimes K_i$. The K_i are linearly independent by conciseness. Suppose $X, X' \in \mathrm{End}(A)$ are such that $X \circ_A T = X' \circ_A T$. Then for $X'' = X - X'$ we have $0 = X'' \circ_A T = \sum_{i=1}^{m} X''(a_i) \otimes K_i$. By linear independence of K_i, we have $X''(a_i) = 0$ for every i. This means that $X'' \in \mathrm{End}(A)$ is zero on a basis of A, hence $X'' = 0$. $\square$

3.1. 1_A-generic case.

Proposition 3.2. *Suppose that $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m = A\otimes B\otimes C$ is 1_A-generic with $\alpha \in A^*$ such that $T(\alpha) \in B\otimes C$ has full rank. Then T is 111-abundant if and only if the space $\mathcal{E}_\alpha(T) = T(A^*)T(\alpha)^{-1} \subseteq \mathrm{End}(C)$ is m-dimensional, abelian, and End-closed. Moreover if these hold, then T is concise and 111-sharp.*

Proof. Assume T is 111-abundant. The map $(T(\alpha)^{-1})^{\mathbf{t}}\colon B \to C^*$ induces an isomorphism of T with a tensor $T' \in A\otimes C^* \otimes C$, so we may assume that $T = T'$, $T(\alpha) = \mathrm{Id}_C$ and $B = C^*$. We explicitly describe the tensors Ω in the triple intersection. We use Lemma 3.1 repeatedly. Fix a basis $a_1, \dots, a_m$ of A and write $T = \sum_{i=1}^{m} a_i \otimes K_i$ where $K_0 = \mathrm{Id}_C$, but we do not assume the K_i are linearly independent, i.e., that T is A-concise. Let $\Omega = \sum_{i=1}^{m} a_i \otimes \omega_i \in A\otimes B\otimes C$. Suppose $\Omega = Y^{\mathbf{t}} \circ_B T = Z \circ_C T$ for some $Y \in \mathrm{End}(C)$ and $Z \in \mathrm{End}(C)$.

The condition $\Omega = Y^{\mathbf{t}} \circ_B T$ means that $\omega_i = K_i Y$ for every i. The condition $\Omega = Z \circ_C T$ means that $\omega_i = Z K_i$. For $i = 1$ we obtain $Y = \mathrm{Id}_C \cdot Y = \omega_1 = Z \cdot \mathrm{Id}_C = Z$, so $Y = Z$. For other i we obtain $Z K_i = K_i Z$, which means that Z is in the joint commutator of $T(A^*)$.

A matrix X such that $\Omega = X \circ_A T$ exists if and only if $\omega_i \in \langle K_1, \dots, K_m \rangle = T(A^*)$ for every i. This yields $Z K_i = K_i Z \in T(A^*)$ and in particular $Z = Z \cdot \mathrm{Id}_C \in T(A^*)$.

By assumption, we have a space of choices for Ω of dimension at least m. Every Ω is determined uniquely by an element $Z \in T(A^*)$. Since $\dim T(A^*) \le m$, we conclude that $\dim T(A^*) = m$, i.e., T is A-concise (and thus concise), and for every $Z \in T(A^*)$, the element $\Omega = Z \circ_C T$ lies in the triple intersection. Thus for every $Z \in T(A^*)$ we have $Z K_i = K_i Z$, which shows that $T(A^*) \subseteq \mathrm{End}(C)$ is abelian and $Z K_i \in T(A^*)$, which implies that $\mathcal{E}_\alpha(T)$ is End-closed. Moreover, the triple intersection is of dimension $\dim T(A^*) = m$, so T is 111-sharp.

Conversely, if $\mathcal{E}_\alpha(T)$ is m-dimensional, abelian and End-closed, then reversing the above argument, we see that $Z \circ_C T$ is in the triple intersection for every $Z \in T(A^*)$. Since $(Z \circ_C T)(\alpha) = Z$, the map from $T(A^*)$ to the triple intersection is injective, so that T is 111-abundant and the above argument applies to it, proving 111-sharpness and conciseness. □

3.2. **Corank one 1_A-degenerate case: statement of the normal form.** We next consider the 1_A-degenerate tensors which are as "nondegenerate" as possible: there exists $\alpha \in A^*$ with $\operatorname{rank}(T(\alpha)) = m-1$.

Proposition 3.3 (characterization of corank one concise tensors that are 111-abundant)**.** *Let $T = \sum_{i=1}^m a_i \otimes K_i$ be a concise tensor which is 111-abundant and not 1_A-generic. Suppose that $K_1 \colon B^* \to C$ has rank $m-1$. Choose decompositions $B^* = B^{*\prime} \oplus \ker(K_1) =: B^{*\prime} \oplus \langle \beta_m \rangle$ and $C = \operatorname{Im}(K_1) \oplus \langle c_m \rangle =: C' \oplus \langle c_m \rangle$ and use K_1 to identify $B^{*\prime}$ with C'. Then there exist bases of A, B, C such that*

$$K_1 = \begin{pmatrix} \operatorname{Id}_{C'} & 0 \\ 0 & 0 \end{pmatrix}, \qquad K_s = \begin{pmatrix} \mathbf{x}_s & 0 \\ 0 & 0 \end{pmatrix} \quad \text{for } 2 \le s \le m-1, \quad \text{and} \quad K_m = \begin{pmatrix} \mathbf{x}_m & w_m \\ u_m & 0 \end{pmatrix}, \tag{3.1}$$

for some $\mathbf{x}_2, \dots, \mathbf{x}_m \in \operatorname{End}(C')$ and $0 \ne u_m \in B' \otimes c_m \cong C'^*$, $0 \ne w_m \in \beta_m \otimes C' \cong C'$ where, setting $\mathbf{x}_1 := \operatorname{Id}_{C'}$,

1. *$u_m x^j w_m = 0$ for every $j \ge 0$ and $x \in \langle \mathbf{x}_1, \dots, \mathbf{x}_m \rangle$, so in particular $u_m w_m = 0$.*
2. *the space $\langle \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{m-1} \rangle \subseteq \operatorname{End}(C')$ is $(m-1)$-dimensional, abelian, and End-closed.*
3. *the space $\langle \mathbf{x}_2, \dots, \mathbf{x}_{m-1} \rangle$ contains the rank one matrix $w_m u_m$.*
4. *For all $2 \le s \le m-1$, $u_m \mathbf{x}_s = 0$ and $\mathbf{x}_s w_m = 0$.*
5. *For every s, there exist vectors $u_s \in C'^*$ and $w_s \in C'$, such that*

$$\mathbf{x}_s \mathbf{x}_m + w_s u_m = \mathbf{x}_m \mathbf{x}_s + w_m u_s \in \langle \mathbf{x}_2, \dots, \mathbf{x}_{m-1} \rangle. \tag{3.2}$$

   The vector $[u_s,\ w_s^{\mathbf{t}}] \in \mathbb{C}^{2(m-1)*}$ is unique up to adding multiples of $[u_m,\ w_m^{\mathbf{t}}]$.

6. *For every $j \ge 1$ and $2 \le s \le m-1$*

$$\mathbf{x}_s \mathbf{x}_m^j w_m = 0 \text{ and } u_m \mathbf{x}_m^j \mathbf{x}_s = 0. \tag{3.3}$$

Moreover, the tensor T is 111-sharp.

Conversely, any tensor satisfying (3.1) and (1)–(5) is 111-sharp, concise and not 1_A-generic, hence satisfies (6) as well.

Additionally, for any vectors $u^* \in C'$ and $w_m^* \in (C')^*$ with $u_m u^* = 1 = w^* w_m$, we may normalize $\mathbf{x}_m$ such that for every $2 \le s \le m-1$

$$\mathbf{x}_m u^* = 0, \ w^* \mathbf{x}_m = 0, \ u_s = w^* \mathbf{x}_s \mathbf{x}_m, \text{ and } w_s = \mathbf{x}_m \mathbf{x}_s u^*. \tag{3.4}$$

Remark 3.4. Atkinson [4] defined a normal form for spaces of corank $m-r$ where one element is $\begin{pmatrix} \operatorname{Id}_r & 0 \\ 0 & 0 \end{pmatrix}$ and all others of the form $\begin{pmatrix} \mathbf{x} & W \\ U & 0 \end{pmatrix}$ and satisfy $U \mathbf{x}^j W = 0$ for every $j \ge 0$. The zero block is clear and the equation follows from expanding out the minors of $\begin{pmatrix} \xi \operatorname{Id}_r + \mathbf{x} & W \\ U & 0 \end{pmatrix}$ with a variable ξ. This already implies (3.1) and (1) except for the zero blocks in the K_s just using bounded rank.

Later, Friedland [24], assuming corank one, showed that the A-Strassen equations are exactly equivalent to having a normal form satisfying (3.1), (1), and (6). In particular, this shows the 111-equations imply Strassen's equations in the corank one case.

Proof. We use Atkinson normal form, in particular we use K_1 to identify $B^{*\prime}$ with C'.

Take $(Y,Z)\in \mathrm{End}(B)\times\mathrm{End}(C)$ with $0\neq Y\circ_B T = Z\circ_C T\in T(A^*){\otimes}A$, which exist by 111-abundance. Write these elements following the decompositions of B^* and C as in the statement:

$$Y^{\mathbf{t}} = \begin{pmatrix} \mathbf{y} & w_Y\\ u_Y & t_Y\end{pmatrix} \qquad Z = \begin{pmatrix} \mathbf{z} & w_Z\\ u_Z & t_Z\end{pmatrix},$$

with $\mathbf{y}\in\mathrm{End}((B^*)')$, $\mathbf{z}\in\mathrm{End}(C')$ etc. The equality $Y\circ_B T = Z\circ_C T\in T(A^*){\otimes}A$ says $K_iY^{\mathbf{t}} = ZK_i\in T(A^*) = \langle K_1,\dots,K_m\rangle$. When $i=1$ this is

$$\begin{pmatrix} \mathbf{y} & w_Y\\ 0 & 0\end{pmatrix} = \begin{pmatrix} \mathbf{z} & 0\\ u_Z & 0\end{pmatrix}\in T(A^*), \tag{3.5}$$

so $w_Y=0$, $u_Z=0$, and $\mathbf{y}=\mathbf{z}$. For future reference, so far we have

$$Y^{\mathbf{t}} = \begin{pmatrix} \mathbf{z} & 0\\ u_Y & t_Y\end{pmatrix} \qquad Z = \begin{pmatrix} \mathbf{z} & w_Z\\ 0 & t_Z\end{pmatrix}. \tag{3.6}$$

By (3.5), for every (Y,Z) above the matrix $\mathbf{z}$ belongs to $B'{\otimes}C'\cap T(A^*)$. By conciseness, the subspace $B'{\otimes}C'\cap T(A^*)$ is proper in $T(A^*)$, so it has dimension less than m. The triple intersection has dimension at least m as T is 111-abundant, so there exists a pair (Y,Z) as in (3.6) with $\mathbf{z}=0$, and $0\neq Y\circ_B T = Z\circ_C T$. Take any such pair (Y_0,Z_0). Consider a matrix $X\in T(A^*)$ with the last row nonzero and write it as

$$X = \begin{pmatrix} \mathbf{x} & w_m\\ u_m & 0\end{pmatrix}$$

where $u_m\neq 0$. The equality

$$XY_0^{\mathbf{t}} = \begin{pmatrix} w_mu_{Y_0} & w_mt_{Y_0}\\ 0 & 0\end{pmatrix} = Z_0X = \begin{pmatrix} w_{Z_0}u_m & 0\\ t_{Z_0}u_m & 0\end{pmatrix} \tag{3.7}$$

implies $w_mt_{Y_0}=0$, $0=t_{Z_0}$ (as $u_m\neq 0$) and $w_{Z_0}u_m = w_mu_{Y_0}$. Observe that $w_{Z_0}\neq 0$ as otherwise $Z_0=0$ while we assumed $Z_0\circ_B T\neq 0$. Since $u_m\neq 0$ and $w_{Z_0}\neq 0$, we have an equality of rank one matrices $w_{Z_0}u_m = w_mu_{Y_0}$. Thus $u_m=\lambda u_{Y_0}$ and $w_m=\lambda w_{Z_0}$ for some nonzero $\lambda\in\mathbb{C}$. It follows that $w_m\neq 0$, so $t_{Y_0}=0$. The matrix X was chosen as an arbitrary matrix with nonzero last row and we have proven that every such matrix yields a vector $[u_m,\ w_m^{\mathbf{t}}]$ proportional to a fixed nonzero vector $[u_{Y_0},\ w_{Z_0}^{\mathbf{t}}]$. It follows that we may choose a basis of A such that there is only one such matrix X. The same holds if we assume instead that X has last column nonzero. This gives (3.1).

Returning to (3.5), from $u_Z=0$ we deduce that $\mathbf{z}\in\langle \mathbf{x}_1,\dots,\mathbf{x}_{m-1}\rangle$.

Now Y_0 and Z_0 are determined up to scale as

$$Y_0^{\mathbf{t}} = \begin{pmatrix} 0 & 0\\ u_m & 0\end{pmatrix} \qquad Z_0 = \begin{pmatrix} 0 & w_m\\ 0 & 0\end{pmatrix}, \tag{3.8}$$

so there is only a one-dimensional space of pairs (Y,Z) with $Y\circ_B T = Z\circ_C T$ and upper left block zero. The space of possible upper left blocks $\mathbf{z}$ is $\langle \mathbf{x}_1,\dots,\mathbf{x}_{m-1}\rangle$ so it is $(m-1)$-dimensional. Since the triple intersection is at least m-dimensional, for any matrix $\mathbf{z}\in\langle \mathbf{x}_1,\dots,\mathbf{x}_{m-1}\rangle$ there exist matrices $Y^{\mathbf{t}}$ and Z as in (3.6) with this $\mathbf{z}$ in the top left corner.

Consider any matrix as in (3.6) corresponding to an element $Y \circ_B T = Z \circ_C T \in T(A^*)\otimes A$. For $2 \le s \le m-1$ we get $\mathbf{z}\mathbf{x}_s = \mathbf{x}_s\mathbf{z} \in \langle \mathbf{x}_1, \dots, \mathbf{x}_{m-1}\rangle$. Since for any matrix $\mathbf{z} \in \langle \mathbf{x}_1, \dots, \mathbf{x}_{m-1}\rangle$ a suitable pair (Y, Z) exists, it follows that $\langle \mathbf{x}_1, \dots, \mathbf{x}_{m-1}\rangle \subseteq \mathrm{End}(C')$ is abelian and closed under composition proving (2). The coefficient of a_m in $Y \circ_B T = Z \circ_C T$ gives

$$(3.9) \qquad \begin{pmatrix} \mathbf{x}_m\mathbf{z} + w_m u_Y & w_m t_Y \\ u_m\mathbf{z} & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{z}\mathbf{x}_m + w_Z u_m & \mathbf{z}w_m \\ t_Z u_m & 0 \end{pmatrix} = \lambda_Y K_m + K_Y,$$

where $\lambda_Y \in \mathbb{C}$ and $K_Y \in \langle K_1, \dots, K_{m-1}\rangle$. It follows that $t_Y = \lambda_Y = t_Z$ and that $\mathbf{z}w_m = \lambda_Y w_m$ as well as $u_m\mathbf{z} = \lambda_Y u_m$.

Iterating over $\mathbf{z} \in \langle \mathbf{x}_1, \dots, \mathbf{x}_{m-1}\rangle$, we see that w_m is a right eigenvector and u_m a left eigenvector of any matrix from this space, and u_m, w_m have the same eigenvalues for each matrix. We make a $\mathrm{GL}(A)$ coordinate change: we subtract this common eigenvalue of $\mathbf{x}_s$ times $\mathbf{x}_1$ from $\mathbf{x}_s$, so that $\mathbf{x}_s w_m = 0$ and $u_m\mathbf{x}_s = 0$ for all $2 \le s \le m-1$ proving (4). Take $\mathbf{z} \in \langle \mathbf{x}_2, \dots, \mathbf{x}_{m-1}\rangle$ so that $\mathbf{z}w_m = 0$ and $u_m\mathbf{z} = 0$. The top left block of (3.9) yields

$$(3.10) \qquad \mathbf{z}\mathbf{x}_m + w_Z u_m = \mathbf{x}_m\mathbf{z} + w_m u_Y = \lambda_Y \mathbf{x}_m + K_Y.$$

Since $\mathbf{z}w_m = 0$, the upper right block of (3.9) implies $\lambda_Y = 0$ and we deduce that

$$(3.11) \qquad \mathbf{z}\mathbf{x}_m + w_Z u_m = \mathbf{x}_m\mathbf{z} + w_m u_Y = K_Y \in \langle \mathbf{x}_2, \dots, \mathbf{x}_{m-1}\rangle.$$

For a pair (Y, Z) with $\mathbf{z} = \mathbf{x}_s$, set $w_s := w_Z$ and $u_s := u_Y$. Such a pair is unique up to adding matrices (3.8), hence $[u_s,\ w_s^{\mathbf{t}}]$ is uniquely determined up to adding multiples of $[u_m,\ w_m^{\mathbf{t}}]$. With these choices (3.11) proves (5). Since $\mathbf{x}_s$ determines u_s, w_s we see that T is 111-sharp.

The matrix (3.7) lies in $T(A^*)$, hence $w_m u_m \in \langle \mathbf{x}_1, \dots, \mathbf{x}_{m-1}\rangle$. Since $0 = (u_m w_m)u_m = u_m(w_m u_m)$ we deduce that $w_m u_m \in \langle \mathbf{x}_2, \dots, \mathbf{x}_{m-1}\rangle$, proving (3).

Conversely, suppose that the space of matrices $K_1, \dots, K_m$ satisfies (3.1) and (1)–(5). Conciseness and 1_A-degeneracy of $K_1, \dots, K_m$ follow by reversing the argument above. That T is 111-sharp follows by constructing the matrices as above.

To prove (6), we fix s and use induction to prove that there exist vectors $v_h \in C'^*$ for $h = 1, 2, \dots$ such that for every $j \ge 1$ we have

$$(3.12) \qquad \mathbf{x}_m^j\mathbf{x}_s + \sum_{h=0}^{j-1} \mathbf{x}_m^h w_m v_{j-h} \in \langle \mathbf{x}_2, \dots, \mathbf{x}_{m-1}\rangle.$$

The base case $j = 1$ follows from (5). To make the step from j to $j+1$ use (5) for the element (3.12) of $\langle \mathbf{x}_2, \dots, \mathbf{x}_{m-1}\rangle$, to obtain

$$\mathbf{x}_m\left(\mathbf{x}_m^j\mathbf{x}_s + \sum_{h=0}^{j-1} \mathbf{x}_m^h w_m v_{j-h}\right) + w_m v_{j+1} \in \langle \mathbf{x}_2, \dots, \mathbf{x}_{m-1}\rangle,$$

for a vector $v_{j+1} \in C'$. This concludes the induction. For every j, by (4), the expression (3.12) is annihilated by u_m:

$$u_m \cdot \left(\mathbf{x}_m^j\mathbf{x}_s + \sum_{h=0}^{j-1} \mathbf{x}_m^h w_m v_{j-h}\right) = 0.$$

By (1) we have $u_m\mathbf{x}_m^h w_m = 0$ for every h, so $u_m\mathbf{x}_m^j\mathbf{x}_s = 0$ for all j. The assertion $\mathbf{x}_s\mathbf{x}_m^j w_m = 0$ is proved similarly. This proves (6).

Finally, we proceed to the "Additionally" part. The main subtlety here is to adjust the bases of B and C. Multiply the tuple from the left and right respectively by the matrices

$$\begin{pmatrix}\mathrm{Id}_{C'} & \gamma\\ 0 & 1\end{pmatrix} \in GL(C) \qquad \begin{pmatrix}\mathrm{Id}_{B'^*} & 0\\ \beta & 1\end{pmatrix} \in GL(B^*)$$

and then add $\alpha w_m u_m$ to $\mathbf{x}_m$. These three coordinate changes do not change the $\mathbf{x}_1$, $\mathbf{x}_s$, u_m, or w_m and they transform $\mathbf{x}_m$ into $\mathbf{x}'_m := \mathbf{x}_m + w_m\beta + \gamma u_m + \alpha w_m u_m$. Take $(\alpha, \beta, \gamma) := (w^*\mathbf{x}_m u^*, -w^*\mathbf{x}_m, -\mathbf{x}_m u^*)$, then $\mathbf{x}'_m$ satisfies $w^*\mathbf{x}'_m = 0$ and $\mathbf{x}'_m u^* = 0$. Multiplying (3.2) from the left by w^* and from the right by u^* we obtain respectively

$$w^*\mathbf{x}_s\mathbf{x}_m + (w^*w_s)u_m = u_s$$
$$w_s = \mathbf{x}_m\mathbf{x}_s u^* + w_m(u_s u^*).$$

Multiply the second line by w^* to obtain $w^*w_s = u_s u^*$, so

$$[u_s,\ w_s^{\mathbf{t}}] - w^*(w_s)[u_m,\ w_m^{\mathbf{t}}] = [w^*\mathbf{x}_s\mathbf{x}_m,\ (\mathbf{x}_m\mathbf{x}_s u^*)^{\mathbf{t}}].$$

Replace $[u_s,\ w_s^{\mathbf{t}}]$ by $[u_s,\ w_s^{\mathbf{t}}] - w^*(w_s)[u_m,\ w_m^{\mathbf{t}}]$ to obtain $u_s = w^*\mathbf{x}_s\mathbf{x}_m$, $w_s = \mathbf{x}_m\mathbf{x}_s u^*$, proving (3.4). □

Example 3.5. Consider the space of 4×4 matrices $\mathbf{x}_1 = \mathrm{Id}_4, \mathbf{x}_2 = E_{14}, \mathbf{x}_3 = E_{13}, \mathbf{x}_4 = E_{34}$. Take $\mathbf{x}_5 = 0$, $u_m = (0,0,0,1)$ and $w_m = (1,0,0,0)^{\mathbf{t}}$. The tensor built from this data as in Proposition 3.3 does *not* satisfy the 111-condition, since $\mathbf{x}_3$ and $\mathbf{x}_4$ do not commute. Hence, it is not of minimal border rank. However, this tensor does satisfy the A-End-closed equations (described in §2.1) and Strassen's equations (in all directions), and even the $p = 1$ Koszul flattenings. This shows that 111-equations are indispensable in Theorem 1.6; they cannot be replaced by these more classical equations.

3.3. **Proof of Proposition 1.2.** The 1_A-generic case is covered by Proposition 3.2 together with the description of the A-Strassen and A-End-closed equations for 1_A-generic tensors which was given in §2.1.

In the corank one case, Remark 3.4 observed that the 111-equations imply Strassen's equations. The End-closed equations are: Let $\alpha_1, \dots, \alpha_m$ be a basis of A^*. Then for all $\alpha', \alpha'' \in A^*$,

$$(T(\alpha')T(\alpha_1)^{\wedge m-1}T(\alpha'')) \wedge T(\alpha_1) \wedge \cdots \wedge T(\alpha_m) = 0 \in \Lambda^{m+1}(B\otimes C). \tag{3.13}$$

Here, for $Z \in B\otimes C$, $Z^{\wedge m-1}$ denotes the induced element of $\Lambda^{m-1}B\otimes\Lambda^{m-1}C$, which, up to choice of volume forms (which does not effect the space of equations), is isomorphic to $C^*\otimes B^*$, so $(T(\alpha')T(\alpha_1)^{\wedge m-1}T(\alpha'')) \in B\otimes C$. In bases $Z^{\wedge m-1}$ is just the cofactor matrix of Z. (Aside: when T is 1_A-generic these correspond to $\mathcal{E}_\alpha(T)$ being closed under composition of endomorphisms.) When $T(\alpha_1)$ is of corank one, using the normal form (3.1) we see $T(\alpha')T(\alpha_1)^{\wedge m-1}T(\alpha'')$ equals zero unless $\alpha' = \alpha'' = \alpha_m$ in which case it equals $w_m u_m$ so the vanishing of (3.13) is implied by Proposition 3.3(3).

Finally if the corank is greater than one, both Strassen's equations and the End-closed equations are trivial. □

4. Proof of Theorem 1.11

We prove Theorem 1.11 that $\mathcal{A}^T_{111}$ is indeed a unital subalgebra of $\mathrm{End}(A) \times \mathrm{End}(B) \times \mathrm{End}(C)$ which is commutative for T concise. The key point is that the actions are linear with respect to A, B, and C. We have $(\mathrm{Id}, \mathrm{Id}, \mathrm{Id}) \in \mathcal{A}^T_{111}$ for any T.

Lemma 4.1 (composition and independence of actions)**.** *Let $T \in A\otimes B\otimes C$. For all $X, X' \in \operatorname{End}(A)$ and $Y \in \operatorname{End}(B)$,*

$$X \circ_A (X' \circ_A T) = (XX') \circ_A T, \text{ and} \tag{4.1}$$

$$X \circ_A (Y \circ_B T) = Y \circ_B (X \circ_A T). \tag{4.2}$$

The same holds for (A, B) replaced by (B, C) or (C, A).

Proof. Directly from the description in Lemma 3.1. □

Lemma 4.2 (commutativity)**.** *Let $T \in A\otimes B\otimes C$ and suppose $(X, Y, Z), (X', Y', Z') \in \mathcal{A}^T_{111}$. Then $XX' \circ_A T = X'X \circ_A T$ and similarly for the other components. If T is concise, then $XX' = X'X$, $YY' = Y'Y$ and $ZZ' = Z'Z$.*

Proof. We will make use of compatibility to move the actions to independent positions and (4.2) to conclude the commutativity, much like one proves that π_2 in topology is commutative. Concretely, Lemma 4.1 implies

$$XX' \circ_A T = X \circ_A (X' \circ_A T) = X \circ_A (Y' \circ_B T) = Y' \circ_B (X \circ_A T) = Y' \circ_B (Z \circ_C T), \text{ and}$$
$$X'X \circ_A T = X' \circ_A (X \circ_A T) = X' \circ_A (Z \circ_C T) = Z \circ_C (X' \circ_A T) = Z \circ_C (Y' \circ_B T).$$

Finally $Y' \circ_B (Z \circ_C T) = Z \circ_C (Y' \circ_B T)$ by (4.2). If T is concise, then the equation $(XX' - X'X) \circ_A T = 0$ implies $XX' - X'X = 0$ by the description in Lemma 3.1, so X and X' commute. The commutativity of other factors follows similarly. □

Lemma 4.3 (closure under composition)**.** *Let $T \in A\otimes B\otimes C$ and suppose $(X, Y, Z), (X', Y', Z') \in \mathcal{A}^T_{111}$. Then $(XX', YY', ZZ') \in \mathcal{A}^T_{111}$.*

Proof. By Lemma 4.1

$$XX' \circ_A T = X \circ_A (X' \circ_A T) = X \circ_A (Y' \circ_B T) = Y' \circ_B (X \circ_A T) = Y' \circ_B (Y \circ_B T) = Y'Y \circ_B T.$$

We conclude by applying Proposition 4.2 and obtain equality with $Z'Z \circ_C T$ similarly. □

Proof of Theorem 1.11. Commutativity follows from Lemma 4.2, the subalgebra assertion is Lemma 4.3, and injectivity of projections follows from Lemma 3.1 and conciseness. □

Remark 4.4. Theorem 1.11 without the commutativity conclusion still holds for a non-concise tensor T. An example with a noncommutative 111-algebra is $\sum_{i=1}^r a_i\otimes b_i\otimes c_i$, where $r \le m-2$. In this case the 111-algebra contains a copy of $\operatorname{End}(\mathbb{C}^{m-r})$.

Example 4.5. If T is a 1_A-generic 111-abundant tensor, then by Proposition 3.2 its 111-algebra is isomorphic to $\mathcal{E}_\alpha(T)$. In particular, if T is the structure tensor of an algebra $\mathcal{A}$, then $\mathcal{A}^T_{111}$ is isomorphic to $\mathcal{A}$.

Example 4.6. Consider the symmetric tensor $F \in S^3\mathbb{C}^5 \subseteq \mathbb{C}^5\otimes\mathbb{C}^5\otimes\mathbb{C}^5$ corresponding to the cubic form $x_3x_1^2 + x_4x_1x_2 + x_5x_2^2$, where, e.g., $x_3x_1^2 = 2(x_3\otimes x_1\otimes x_1 + x_1\otimes x_3\otimes x_1 + x_1\otimes x_1\otimes x_3)$. This cubic has vanishing Hessian, hence F is 1-degenerate. The triple intersection of the corresponding tensor is $\langle F, x_1^3, x_1^2x_2, x_1x_2^2, x_2^3\rangle$ and its 111-algebra is given by the triples (x, x, x) where

$$x \in \langle \mathrm{Id}, x_1\otimes\alpha_3, x_2\otimes\alpha_3 + x_1\otimes\alpha_4, x_2\otimes\alpha_4 + x_1\otimes\alpha_5, x_2\otimes\alpha_5\rangle,$$

where α_j is the basis vector dual to x_j. Since all compositions of basis elements other than Id are zero, this 111-algebra is isomorphic to $\mathbb{C}[\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4]/(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)^2$.

Example 4.7. Consider a tensor in the normal form of Proposition 3.3. The projection of the 111-algebra to $\mathrm{End}(B) \times \mathrm{End}(C)$ can be extracted from the proof. In addition to $(\mathrm{Id}, \mathrm{Id})$ we have:

$$Y_0 = \begin{pmatrix} 0 & 0 \\ u_m & 0 \end{pmatrix}, \; Z_0 = \begin{pmatrix} 0 & w_m \\ 0 & 0 \end{pmatrix},$$
$$Y_s = \begin{pmatrix} \mathbf{x}_s & 0 \\ u_s & 0 \end{pmatrix}, \; Z_s = \begin{pmatrix} \mathbf{x}_s & w_s \\ 0 & 0 \end{pmatrix}.$$

Theorem 1.11 implies for matrices in $\mathrm{End}(C)$ that

$$\begin{pmatrix} \mathbf{x}_s\mathbf{x}_t & \mathbf{x}_s w_t \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{x}_s & w_s \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{x}_t & w_t \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{x}_t & w_t \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{x}_s & w_s \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{x}_t\mathbf{x}_s & \mathbf{x}_t w_s \\ 0 & 0 \end{pmatrix}$$

which gives $\mathbf{x}_s w_t = \mathbf{x}_t w_s$ for any $2 \le s, t \le m-1$. Considering matrices in $\mathrm{End}(B)$ we obtain $u_t \mathbf{x}_s = u_s \mathbf{x}_t$ for any $2 \le s, t \le m-1$. (Of course, these identities are also a consequence of Proposition 3.3, but it is difficult to extract them directly from the Proposition.)

5. New obstructions to minimal border rank via the 111-algebra

In this section we characterize 111-abundant tensors in terms of an algebra equipped with a triple of modules and a module map. We then exploit this extra structure to obtain new obstructions to minimal border rank via deformation theory.

5.1. Characterization of tensors that are 111-abundant.

Definition 5.1. A *tri-presented algebra* is a commutative unital subalgebra $\mathcal{A} \subseteq \mathrm{End}(A) \times \mathrm{End}(B) \times \mathrm{End}(C)$.

For any concise tensor T its 111-algebra $\mathcal{A}_{111}^T$ is a tri-presented algebra. A tri-presented algebra $\mathcal{A}$ naturally gives an $\mathcal{A}$-module structure on A, B, C. For every $\mathcal{A}$-module N the space N^* is also an $\mathcal{A}$-module via, for any $r \in \mathcal{A}$, $n \in N$, and $f \in N^*$, $(r \cdot f)(n) := f(rn)$. (This indeed satisfies $r_2 \cdot (r_1 \cdot f) = (r_2 r_1) \cdot f$ because $\mathcal{A}$ is commutative.) In particular, the spaces A^*, B^*, C^* are $\mathcal{A}$-modules. Explicitly, if $r = (X, Y, Z) \in \mathcal{A}$ and $\alpha \in A^*$, then $r\alpha = X^{\mathbf{t}}(\alpha)$.

There is a canonical surjective map $\pi \colon A^* \otimes B^* \to \underline{A}^* \otimes_{\mathcal{A}} \underline{B}^*$, defined by $\pi(\alpha \otimes \beta) = \alpha \otimes_{\mathcal{A}} \beta$ and extended linearly. For any homomorphism $\varphi \colon \underline{A}^* \otimes_{\mathcal{A}} \underline{B}^* \to \underline{C}$ of $\mathcal{A}$-modules, we obtain a linear map $\varphi \circ \pi \colon A^* \otimes B^* \to C$ hence a tensor in $A \otimes B \otimes C$ which we denote by T_φ.

We need the following lemma, whose proof is left to the reader.

Lemma 5.2 (compatibility with flattenings)**.** *Let $T \in A \otimes B \otimes C$, $X \in \mathrm{End}(A)$, $Z \in \mathrm{End}(C)$ and $\alpha \in A^*$. Consider $T(\alpha) : B^* \to C$. Then*

$$(Z \circ_C T)(\alpha) = Z \cdot T(\alpha), \tag{5.1}$$
$$T\left(X^{\mathbf{t}}(\alpha)\right) = (X \circ_A T)(\alpha), \tag{5.2}$$

and analogously for the other factors. □

Proposition 5.3. *Let T be a concise 111-abundant tensor. Then T is 1_A-generic if and only if the $\mathcal{A}^T_{111}$-module $\underline{A}^*$ is generated by a single element, i.e., is a cyclic module. More precisely, an element $\alpha \in A^*$ generates the $\mathcal{A}^T_{111}$-module $\underline{A}^*$ if and only if $T(\alpha)$ has maximal rank.*

Proof. Take any $\alpha \in A^*$ and $r = (X, Y, Z) \in \mathcal{A}^T_{111}$. Using (5.1)-(5.2) we have

$$T(r\alpha) = T(X^{\mathbf{t}}(\alpha)) = (X \circ_A T)(\alpha) = (Z \circ_C T)(\alpha) = Z \cdot T(\alpha). \tag{5.3}$$

Suppose first that T is 1_A-generic with $T(\alpha)$ of full rank. If $r \neq 0$, then $Z \neq 0$ by the description in Lemma 3.1, so $Z \cdot T(\alpha)$ is nonzero. This shows that the homomorphism $\mathcal{A}^T_{111} \to \underline{A}^*$ of $\mathcal{A}^T_{111}$-modules given by $r \mapsto r\alpha$ is injective. Since $\dim \mathcal{A}^T_{111} \geq m = \dim A^*$, this homomorphism is an isomorphism and so $\underline{A}^* \simeq \mathcal{A}^T_{111}$ as $\mathcal{A}^T_{111}$-modules.

Now suppose that $\underline{A}^*$ is generated by an element $\alpha \in A^*$. This means that for every $\alpha' \in A^*$ there is an $r = (X, Y, Z) \in \mathcal{A}^T_{111}$ such that $r\alpha = \alpha'$. From (5.3) it follows that $\ker T(\alpha) \subseteq \ker T(\alpha')$. This holds for every α', hence $\ker T(\alpha)$ is in the joint kernel of $T(A^*)$. By conciseness this joint kernel is zero, hence $\ker T(\alpha) = 0$ and $T(\alpha)$ has maximal rank. ☐

Theorem 5.4. *Let $T \in A \otimes B \otimes C$ and let $\mathcal{A}$ be a tri-presented algebra. Then $\mathcal{A} \subseteq \mathcal{A}^T_{111}$ if and only if the map $T^{\mathbf{t}}_C : A^* \otimes B^* \to C$ factors through $\pi : A^* \otimes B^* \to \underline{A}^* \otimes_{\mathcal{A}} \underline{B}^*$ and induces an $\mathcal{A}$-module homomorphism $\varphi \colon \underline{A}^* \otimes_{\mathcal{A}} \underline{B}^* \to \underline{C}$. If this holds, then $T = T_\varphi$.*

Proof. By the universal property of the tensor product over $\mathcal{A}$, the map $T^{\mathbf{t}}_C : A^* \otimes B^* \to C$ factors through π if and only if the bilinear map $A^* \times B^* \to C$ given by $(\alpha, \beta) \mapsto T(\alpha, \beta)$ is $\mathcal{A}$-bilinear. That is, for every $r = (X, Y, Z) \in \mathcal{A}$, $\alpha \in A^*$, and $\beta \in B^*$ one has $T(r\alpha, \beta) = T(\alpha, r\beta)$. By (5.2), $T(r\alpha, \beta) = (X \circ_A T)(\alpha, \beta)$ and $T(\alpha, r\beta) = (Y \circ_B T)(\alpha, \beta)$. It follows that the factorization exists if and only if for every $r = (X, Y, Z) \in \mathcal{A}$ we have $X \circ_A T = Y \circ_B T$. Suppose that this holds and consider the obtained map $\varphi \colon \underline{A}^* \otimes_{\mathcal{A}} \underline{B}^* \to \underline{C}$. Thus for $\alpha \in A^*$ and $\beta \in B^*$ we have $\varphi(\alpha \otimes_{\mathcal{A}} \beta) = T(\alpha, \beta)$. The map φ is a homomorphism of $\mathcal{A}$-modules if and only if for every $r = (X, Y, Z) \in \mathcal{A}$ we have $\varphi(r\alpha \otimes_{\mathcal{A}} \beta) = r\varphi(\alpha \otimes_{\mathcal{A}} \beta)$. By (5.1), $r\varphi(\alpha \otimes_{\mathcal{A}} \beta) = (Z \circ_C T)(\alpha, \beta)$ and by (5.2), $\varphi(r\alpha \otimes_{\mathcal{A}} \beta) = (X \circ_A T)(\alpha, \beta)$. These are equal for all α, β if and only if $X \circ_A T = Z \circ_C T$. The equality $T = T_\varphi$ follows directly from definition of T_φ. ☐

Theorem 5.5 (characterization of concise 111-abundant tensors)**.** *A concise tensor that is 111-abundant is isomorphic to a tensor T_φ associated to a surjective homomorphism of $\mathcal{A}$-modules*

$$\varphi \colon N_1 \otimes_{\mathcal{A}} N_2 \to N_3, \tag{5.4}$$

where $\mathcal{A}$ is a commutative associative unital algebra, N_1, N_2, N_3 are $\mathcal{A}$-modules and $\dim N_1 = \dim N_2 = \dim N_3 = m \leq \dim \mathcal{A}$, *and moreover for every $n_1 \in N_1, n_2 \in N_2$ the maps $\varphi(n_1 \otimes_{\mathcal{A}} -) \colon N_2 \to N_3$ and $\varphi(- \otimes_{\mathcal{A}} n_2) \colon N_1 \to N_3$ are nonzero. Conversely, any such T_φ is 111-abundant and concise.*

The conditions $\varphi(n_1 \otimes_{\mathcal{A}} -) \neq 0$, $\varphi(- \otimes_{\mathcal{A}} n_2) \neq 0$ for any nonzero n_1, n_2 have appeared in the literature. Bergman [7] calls φ *nondegenerate* if they are satisfied.

Proof. By Theorem 5.4 a concise tensor T that is 111-abundant is isomorphic to T_φ where $\mathcal{A} = \mathcal{A}^T_{111}$, $N_1 = \underline{A}^*$, $N_2 = \underline{B}^*$, $N_3 = \underline{C}$. Since T is concise, the homomorphism φ is onto and the restrictions $\varphi(\alpha \otimes_{\mathcal{A}} -)$, $\varphi(- \otimes_{\mathcal{A}} \beta)$ are nonzero for any nonzero $\alpha \in A^*$, $\beta \in B^*$. Conversely, if we take (5.4) and set $A := N_1^*$, $B := N_2^*$, $C := N_3$, then T_φ is concise by the conditions on φ and by Theorem 5.4, $\mathcal{A} \subseteq \mathcal{A}^{T_\varphi}_{111}$ hence T_φ is 111-abundant. ☐

Example 5.6. By Proposition 5.3 we see that for a concise 1_A-generic tensor T the tensor product $\underline{A}^*\otimes_{\mathcal{A}}\underline{B}^*$ simplifies to $\mathcal{A}\otimes_{\mathcal{A}}\underline{B}^* \simeq \underline{B}^*$. The homomorphism $\varphi\colon \underline{B}^* \to \underline{C}$ is surjective, hence an isomorphism of $\underline{B}^*$ and $\underline{C}$, so the tensor T_φ becomes the multiplication tensor $\mathcal{A}\otimes_{\mathbb{C}}\underline{C} \to \underline{C}$ of the $\mathcal{A}$-module $\underline{C}$. One can then choose a surjection $S \to \mathcal{A}$ from a polynomial ring such that $S_{\le 1}$ maps isomorphically onto $\mathcal{A}$. This shows how the results of this section generalize §2.2.

In the setting of Theorem 5.5, since T is concise it follows from Lemma 3.1 that the projections of $\mathcal{A}_{111}^T$ to $\mathrm{End}(A)$, $\mathrm{End}(B)$, $\mathrm{End}(C)$ are one to one. This translates into the fact that no nonzero element of $\mathcal{A}_{111}^T$ annihilates A, B or C. The same is then true for A^*, B^*, C^*.

5.2. **Two new obstructions to minimal border rank.**

Lemma 5.7. *Let $T \in \mathbb{C}^m\otimes\mathbb{C}^m\otimes\mathbb{C}^m$ be concise, 111-sharp and of minimal border rank. Then $\mathcal{A}_{111}^T$ is smoothable.*

Proof. By 111-sharpness, the degeneration $T_\epsilon \to T$ from a minimal rank tensor induces a family of triple intersection spaces, hence by semicontinuity it is enough to check for T_ϵ of *rank* m. By Example 4.5 each T_ϵ has 111-algebra $\prod_{i=1}^m \mathbb{C}$. Thus the 111-algebra of T is the limit of algebras isomorphic to $\prod_{i=1}^m \mathbb{C}$, hence smoothable. □

Recall from §2 that for $m \le 7$ every algebra is smoothable.

As in section §2.2 view $\mathcal{A}_{111}^T$ as a quotient of a fixed polynomial ring S. Then the $\mathcal{A}_{111}^T$-modules $\underline{A}$, $\underline{B}$, $\underline{C}$ become S-modules.

Lemma 5.8. *Let $T \in \mathbb{C}^m\otimes\mathbb{C}^m\otimes\mathbb{C}^m$ be concise, 111-sharp and of minimal border rank. Then the S-modules $\underline{A}$, $\underline{B}$, $\underline{C}$ lie in the principal component of the Quot scheme.*

Proof. As in the proof above, the degeneration $T_\epsilon \to T$ from a minimal rank tensor induces a family of $\mathcal{A}_{111}^{T_\epsilon}$ and hence a family of S-modules $\underline{A}_\epsilon$, $\underline{B}_\epsilon$, $\underline{C}_\epsilon$. These modules are semisimple when T_ϵ has minimal border rank by Example 2.1. □

Already for $m = 4$ there are S-modules outside the principal component [34, §6.1], [29].

Example 5.9. In [37, Example 5.3] the authors exhibit a 1_A-generic, End-closed, commuting tuple of seven 7×7-matrices that corresponds to a tensor T of border rank higher than minimal. By Proposition 3.2 this tensor is 111-sharp. However, the associated module $\underline{C}$ is *not* in the principal component, in fact it is a smooth point of another (elementary) component. This can be verified using Białynicki-Birula decomposition, as in [34, Proposition 5.5]. The proof of non-minimality of border rank in [37, Example 5.3] used different methods. We note that the tensor associated to this tuple does *not* satisfy all $p = 1$ Koszul flattenings.

6. Conditions where tensors of bounded rank fail to be concise

Proposition 6.1. *Let $T \in \mathbb{C}^5\otimes\mathbb{C}^5\otimes\mathbb{C}^5$ be such that the matrices in $T(A^*)$ have the shape*

$$\begin{pmatrix} 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \\ * & * & * & * & * \end{pmatrix}.$$

If T is concise, then $T(C^*)$ contains a matrix of rank at least 4.

Proof. Write the elements of $T(A^*)$ as matrices

$$K_i = \begin{pmatrix} 0 & \star \\ u_i & \star \end{pmatrix} \in \operatorname{Hom}(B^*, C) \quad \text{for } i = 1, 2, \ldots, 5$$

where $u_i \in \mathbb{C}^3$. Suppose T is concise. Then the joint kernel of $\langle K_1, \ldots, K_5 \rangle$ is zero, so $u_1, \ldots, u_5$ span $\mathbb{C}^3$. After a change of coordinates we may assume u_1, u_2, u_3 are linearly independent while $u_4 = 0$, $u_5 = 0$. Since $K_4 \neq 0$, choose a vector $\gamma \in C^*$ such that $\gamma \cdot K_4 \neq 0$. Choose $\xi \in \mathbb{C}$ such that $(\gamma_5 + \xi\gamma) \cdot K_4 \neq 0$. Note that $T(\gamma_5) : B^* \to A$ has matrix whose rows are the last rows of $K_1, \ldots, K_5$. We claim that the matrix $T(\gamma_5 + \xi\gamma) \colon B^* \to A$ has rank at least four. Indeed, this matrix can be written as

$$\begin{pmatrix} u_1 & \star & \star \\ u_2 & \star & \star \\ u_3 & \star & \star \\ 0 & \multicolumn{2}{c}{(\gamma_5 + \xi\gamma) \cdot K_4} \\ 0 & \star & \star \end{pmatrix}.$$

This concludes the proof. ∎

Proposition 6.2. *Let $T \in A \otimes B \otimes C$ with $m = 5$ be a concise tensor. Then one of its associated spaces of matrices contains a full rank or corank one matrix.*

Proof. Suppose that $T(A^*)$ is of bounded rank three. We use [4, Theorem A] and its notation, in particular $r = 3$. By this theorem and conciseness, the matrices in the space $T(A^*)$ have the shape

$$\begin{pmatrix} \star & \star & \star \\ \star & \mathcal{Y} & 0 \\ \star & 0 & 0 \end{pmatrix}$$

where the starred part consists of p rows and q columns, for some $p, q \geq 0$, and $\mathcal{Y}$ forms a primitive space of bounded rank at most $3 - p - q$. Furthermore, since $r + 1 < m$ and $r < 2 + 2$, by [4, Theorem A, "Moreover" part] we see that $T(A^*)$ is not primitive itself, hence at least one of p, q is positive. If just one is positive, say p, then by conciseness $\mathcal{Y}$ spans $5 - p$ rows and bounded rank $3 - p$, which again contradicts [4, Theorem A, "Moreover"]. If both are positive, we have $p = q = 1$ and $\mathcal{Y}$ is of bounded rank one, so by [5, Lemma 2], up to coordinate change, after transposing $T(A^*)$ has the shape as in Proposition 6.2. ∎

Proposition 6.3. *In the setting of Proposition 3.3, write $T' = a_1 \otimes \mathbf{x}_1 + \cdots + a_{m-1} \otimes \mathbf{x}_{m-1} \in \mathbb{C}^{m-1} \otimes \mathbb{C}^{m-1} \otimes \mathbb{C}^{m-1} =: A' \otimes C'^* \otimes C'$, where $\mathbf{x}_1 = \mathrm{Id}_{C'}$. If T is 1-degenerate, then T' is $1_{C'^*}$ and $1_{C'}$-degenerate.*

Proof. Say T' is $1_{C'^*}$-generic with $T'(c')$ of rank $m-1$. Then $T(c' + \lambda u^*)$ has rank m for almost all $\lambda \in \mathbb{C}$, contradicting 1-degeneracy. The $1_{C'}$-generic case is similar. ∎

Corollary 6.4. *In the setting of Proposition 6.3, the module $\underline{C'}$ associated to $T'(A'^*)$ via the ADHM correspondence as in §2.2 cannot be generated by a single element. Similarly, the module $\underline{C'^*}$ associated to $(T'(A'^*))^{\mathbf{t}}$ cannot be generated by a single element.*

Proof. By Proposition 2.3 the module $\underline{C'}$ is generated by a single element if and only if T' is $1_{C'^*}$-generic. The claim follows from Proposition 6.3. The second assertion follows similarly since T' is not $1_{C'}$-generic. □

7. Proof of Theorem 1.6 in the 1-degenerate case and Theorem 1.7

Throughout this section $T \in \mathbb{C}^5 \otimes \mathbb{C}^5 \otimes \mathbb{C}^5$ is a concise 1-degenerate 111-abundant tensor.

We use the notation of Proposition 3.3 throughout this section.

We begin, in §7.1 with a few preliminary results. We then, in §7.2 prove a variant of the $m = 5$ classification result under a more restricted notion of isomorphism and only require 111-abundance. Then the $m = 5$ classification of corank one 111-abundant tensors follows easily in §7.3 as does the orbit closure containment in §7.4. Finally we give two proofs that these tensors are of minimal border rank in §7.5.

7.1. Preliminary results.

We first classify admissible three dimensional spaces of 4×4 matrices $\langle \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\rangle \subseteq \mathrm{End}(\mathbb{C}^4)$. One could proceed by using the classification [50, §3] of abelian subspaces of $\mathrm{End}(\mathbb{C}^4)$ and then impose the additional conditions of Proposition 3.3. We instead utilize ideas from the ADHM correspondence to obtain a short, self-contained proof.

Proposition 7.1. *Let $\langle \mathbf{x}_1 = \mathrm{Id}_4, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\rangle \subset \mathrm{End}(\mathbb{C}^4)$ be a 4-dimensional subspace spanned by pairwise commuting matrices. Suppose there exist nonzero subspaces $V, W \subseteq \mathbb{C}^4$ with $V \oplus W = \mathbb{C}^4$ which are preserved by $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$. Then either these exists a vector $v \in \mathbb{C}^4$ with $\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\rangle \cdot v = \mathbb{C}^4$ or there exists a vector $v^* \in \mathbb{C}^{4*}$ with $\langle \mathbf{x}_1^{\mathbf{t}}, \mathbf{x}_2^{\mathbf{t}}, \mathbf{x}_3^{\mathbf{t}}, \mathbf{x}_4^{\mathbf{t}}\rangle v^* = \mathbb{C}^{4*}$.*

Proof. For $h = 1,2,3,4$ the matrix $\mathbf{x}_h$ is block diagonal with blocks $\mathbf{x}'_h \in \mathrm{End}(V)$ and $\mathbf{x}''_h \in \mathrm{End}(W)$.

Suppose first that $\dim V = 2 = \dim W$. In this case we will prove that v exists. The matrices $\mathbf{x}'_h$ commute and commutative subalgebras of $\mathrm{End}(\mathbb{C}^2)$ are at most 2-dimensional and are, up to a change of basis, spanned by $\mathrm{Id}_{\mathbb{C}^2}$ and either $\begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix}$ or $\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix}$. In each of of the two cases, applying the matrices to the vector $(1,1)^{\mathbf{t}}$ yields the space $\mathbb{C}^2$. Since the space $\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\rangle$ is 4-dimensional, it is, after a change of basis, a direct sum of two maximal subalgebras as above. Thus applying $\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\rangle$ to the vector $v = (1,1,1,1)^{\mathbf{t}}$ yields the whole space.

Suppose now that $\dim V = 3$. If some $\mathbf{x}'_h$ has at least two distinct eigenvalues, then consider the generalized eigenspaces V_1, V_2 associated to them and suppose $\dim V_1 = 1$. By commutativity, the subspaces V_1, V_2 are preserved by the action of every $\mathbf{x}'_h$, so the matrices $\mathbf{x}_h$ also preserve the subspaces $W \oplus V_1$ and V_2. This reduces us to the previous case. Hence, every $\mathbf{x}'_h$ has a single eigenvalue. Subtracting multiples of $\mathbf{x}_1$ from $\mathbf{x}_s$ for $s = 2,3,4$, the $\mathbf{x}'_s$ become nilpotent, hence up to a change of basis in V, they have the form

$$\mathbf{x}'_s = \begin{pmatrix} 0 & (\mathbf{x}'_s)_{12} & (\mathbf{x}'_s)_{13}\\ 0 & 0 & (\mathbf{x}'_s)_{23}\\ 0 & 0 & 0\end{pmatrix}.$$

The space $\langle \mathbf{x}'_2, \mathbf{x}'_3, \mathbf{x}'_4\rangle$ cannot be 3-dimensional, as it would fill the space of 3×3 upper triangular matrices, which is non-commutative. So $\langle \mathbf{x}'_2, \mathbf{x}'_3, \mathbf{x}'_4\rangle$ is 2-dimensional and so some linear combination of the matrices $\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ is the identity on W and zero on V.

We subdivide into four cases. First, if $(\mathbf{x}'_s)_{12} \neq 0$ for some s and $(\mathbf{x}'_t)_{23} \neq 0$ for some $t \neq s$, then change bases so $(\mathbf{x}'_s)_{23} = 0$ and take $v = (0,p,1,1)^{\mathbf{t}}$ such that $p(\mathbf{x}'_s)_{12} + (\mathbf{x}'_s)_{13} \neq 0$. Second, if the above fails and $(\mathbf{x}'_s)_{12} \neq 0$ and $(\mathbf{x}'_s)_{23} \neq 0$ for some s, then there must be a t such that $(\mathbf{x}'_t)_{13} \neq 0$ and all other entries are zero, so we may take $v = (0,0,1,1)^{\mathbf{t}}$. Third, if $(\mathbf{x}'_s)_{12} = 0$ for all $s = 2,3,4$, then for dimensional reasons we have

$$\langle \mathbf{x}'_2, \mathbf{x}'_3, \mathbf{x}'_4 \rangle = \begin{pmatrix} 0 & 0 & \star \\ 0 & 0 & \star \\ 0 & 0 & 0 \end{pmatrix}$$

and again $v = (0,0,1,1)^{\mathbf{t}}$ is the required vector. Finally, if $(\mathbf{x}'_s)_{23} = 0$ for all $s = 2,3,4$, then arguing as above $v^* = (1,0,0,1)$ is the required vector. □

We now prove a series of reductions that will lead to the proof of Theorem 1.7.

Proposition 7.2. *Let $m = 5$ and $T \in A \otimes B \otimes C$ be a concise, 1-degenerate, 111-abundant tensor with $T(A^*)$ of corank one. Then up to $\mathrm{GL}(A) \times \mathrm{GL}(B) \times \mathrm{GL}(C)$ action it has the form as in Proposition 3.3 with*

$$\mathbf{x}_s = \begin{pmatrix} 0 & \chi_s \\ 0 & 0 \end{pmatrix}, \quad 2 \leq s \leq 4, \tag{7.1}$$

where the blocking is $(2,2) \times (2,2)$.

Proof. We apply Proposition 3.3. It remains to prove the form (7.1).

By Proposition 3.3(4) zero is an eigenvalue of every $\mathbf{x}_s$. Suppose some $\mathbf{x}_s$ is not nilpotent, so has at least two different eigenvalues. By commutativity, its generalized eigenspaces are preserved by the action of $\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$, hence yield V and W as in Proposition 7.1 and a contradiction to Corollary 6.4. We conclude that every $\mathbf{x}_s$ is nilpotent.

We now prove that the codimension of $\sum_{s=2}^4 \operatorname{Im} \mathbf{x}_s \subseteq C'$ is at least two. Suppose the codimension is at most one and choose $c \in C'$ such that $\sum_{s=2}^4 \operatorname{Im} \mathbf{x}_s + \mathbb{C}c = C'$. Let $\mathcal{A} \subset \operatorname{End}(C')$ be the unital subalgebra generated by $\mathbf{x}_2$, $\mathbf{x}_3$, $\mathbf{x}_4$ and let $W = \mathcal{A} \cdot c$. The above equality can be rewritten as $\langle \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \rangle C' + \mathbb{C}c = C'$, hence $\langle \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \rangle C' + W = C'$. We repeatedly substitute the last equality into itself, obtaining

$$C' = \langle \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \rangle C' + W = (\langle \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \rangle)^2 C' + W = \ldots = (\langle \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \rangle)^{10} C' + W = W,$$

since $\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ commute and satisfy $\mathbf{x}_s^4 = 0$. This proves that $C' = \mathcal{A} \cdot c$, again yielding a contradiction with Corollary 6.4.

Applying the above argument to $\mathbf{x}_2^{\mathbf{t}}, \mathbf{x}_3^{\mathbf{t}}, \mathbf{x}_4^{\mathbf{t}}$ proves that joint kernel of $\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ is at least two-dimensional.

We now claim that $\bigcap_{s=2}^4 \ker(\mathbf{x}_s) \subseteq \sum_{s=2}^4 \operatorname{Im} \mathbf{x}_s$. Suppose not and choose $v \in C'$ that lies in the joint kernel, but not in the image. Let $W \subseteq C'$ be a subspace containing the image and such that $W \oplus \mathbb{C}v = C'$. Then $\langle \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \rangle W \subseteq \langle \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \rangle C' \subseteq W$, hence $V = \mathbb{C}v$ and W yield a decomposition as in Proposition 7.1 and a contradiction. The containment $\bigcap_{s=2}^4 \ker(\mathbf{x}_s) \subseteq \sum_{s=2}^4 \operatorname{Im} \mathbf{x}_s$ together with the dimension estimates yield the equality $\bigcap_{s=2}^4 \ker(\mathbf{x}_s) = \sum_{s=2}^4 \operatorname{Im} \mathbf{x}_s$. To obtain the form (7.1) it remains to choose a basis of C' so that the first two basis vectors span $\bigcap_{s=2}^4 \ker(\mathbf{x}_s)$. □

7.2. **Classification of 111-abundant tensors under restricted isomorphism.** Refining Proposition 7.2, we now prove the following classification.

Theorem 7.3. *Let $m = 5$. Up to $\mathrm{GL}(A)\times\mathrm{GL}(B)\times\mathrm{GL}(C)$ action and swapping the B and C factors, there are exactly seven concise 1-degenerate, 111-abundant tensors in $A\otimes B\otimes C$ with $T(A^*)$ of corank one. To describe them explicitly, let*

$$T_{\mathrm{M1}} = a_1\otimes(b_1\otimes c_1+b_2\otimes c_2+b_3\otimes c_3+b_4\otimes c_4)+a_2\otimes b_3\otimes c_1+a_3\otimes b_4\otimes c_1+a_4\otimes b_4\otimes c_2+a_5\otimes(b_5\otimes c_1+b_4\otimes c_5)$$

and

$$T_{\mathrm{M2}} = a_1\otimes(b_1\otimes c_1+b_2\otimes c_2+b_3\otimes c_3+b_4\otimes c_4)+a_2\otimes(b_3\otimes c_1-b_4\otimes c_2)+a_3\otimes b_4\otimes c_1+a_4\otimes b_3\otimes c_2+a_5\otimes(b_5\otimes c_1+b_4\otimes c_5).$$

Then the tensors are

$(T_{\mathcal{O}_{58}})$	$T_{\mathrm{M2}}+a_5\otimes(b_1\otimes c_2-b_3\otimes c_4)$
$(T_{\mathcal{O}_{57}})$	T_{M2}
$(\widetilde{T}_{\mathcal{O}_{57}})$	$T_{\mathrm{M1}}+a_5\otimes(b_5\otimes c_2-b_1\otimes c_2+b_3\otimes c_3)$
$(\widetilde{T}_{\mathcal{O}_{56}})$	$T_{\mathrm{M1}}+a_5\otimes b_5\otimes c_2$
$(T_{\mathcal{O}_{56}})$	$T_{\mathrm{M1}}+a_5\otimes b_2\otimes c_2$
$(T_{\mathcal{O}_{55}})$	$T_{\mathrm{M1}}+a_5\otimes b_3\otimes c_2$
$(T_{\mathcal{O}_{54}})$	T_{M1}

These tensors are pairwise non-isomorphic, as we explain below. For a tensor $T\in A\otimes B\otimes C$ its annihilator in $\mathfrak{gl}(A)\times\mathfrak{gl}(B)\times\mathfrak{gl}(C)$ is called its *symmetry Lie algebra*. The symmetry Lie algebra intersected with $\mathfrak{gl}(A)\times\mathfrak{gl}(B)$ is called the *AB-part* etc. We list the dimensions of these Lie algebras below.

A linear algebra computation (see, e.g., [19]) shows that the dimensions of the symmetry Lie algebras are

case	$(T_{\mathcal{O}_{58}})$	$(T_{\mathcal{O}_{57}})$	$(\widetilde{T}_{\mathcal{O}_{57}})$	$(\widetilde{T}_{\mathcal{O}_{56}})$	$(T_{\mathcal{O}_{56}})$	$(T_{\mathcal{O}_{55}})$	$(T_{\mathcal{O}_{54}})$
full	16	17	17	18	18	19	20
AB-part	5	5	5	5	6	6	6
BC-part	5	6	5	6	5	6	6
CA-part	5	5	6	6	6	6	6

Proof of Theorem 7.3. We utilize Proposition 7.2 and its notation. By conciseness, the matrices $\mathbf{x}_2$, $\mathbf{x}_3$, $\mathbf{x}_4$ are linearly independent, hence form a codimension one subspace of $\mathrm{End}(\mathbb{C}^2)$. We utilize the perfect pairing on $\mathrm{End}(\mathbb{C}^2)$ given by $(A,B)\mapsto \mathrm{Tr}(AB)$, so that $\langle\chi_2,\chi_3,\chi_4\rangle^\perp\subseteq\mathrm{End}(\mathbb{C}^2)$ is one-dimensional, spanned by a matrix P. Conjugation with an invertible 4×4 block diagonal matrix with 2×2 blocks M, N maps χ_s to $M\chi_s N^{-1}$ and P to NPM^{-1}. Under such conjugation the orbits are matrices of fixed rank, so after changing bases in $\langle a_2,a_3,a_4\rangle$, we reduce to the cases

$$\text{(M1)}\qquad P=\begin{pmatrix}0&1\\0&0\end{pmatrix}\qquad \chi_2=\begin{pmatrix}1&0\\0&0\end{pmatrix},\quad \chi_3=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad \chi_4=\begin{pmatrix}0&0\\0&1\end{pmatrix},\quad \textit{and}$$

$$\text{(M2)}\qquad P=\begin{pmatrix}1&0\\0&1\end{pmatrix}\qquad \chi_2=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad \chi_3=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad \chi_4=\begin{pmatrix}0&0\\1&0\end{pmatrix}.$$

In both cases the joint right kernel of our matrices is $(*,*,0,0)^{\mathbf{t}}$ while the joint left kernel is $(0,0,*,*)$, so $w_5=(w_{5,1},w_{5,2},0,0)^{\mathbf{t}}$ and $u_5=(0,0,u_{5,3},u_{5,4})$.

7.2.1. *Case* (M2). In this case there is an involution, namely conjugation with

$$\begin{pmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&0&1&0\\0&0&1&0&0\\0&0&0&0&1\end{pmatrix}\in \mathrm{GL}_5$$

that preserves P, and hence $\langle \mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4\rangle$, while it swaps $w_{5,1}$ with $w_{5,2}$ and $u_{5,1}$ with $u_{5,2}$. Using this involution and rescaling c_5, we assume $w_{5,1}=1$. The matrix

$$\begin{pmatrix}u_{5,3}&u_{5,4}\\u_{5,3}w_{5,2}&u_{5,4}w_{5,2}\end{pmatrix}$$

belongs to $\langle\chi_2,\chi_3,\chi_4\rangle$ by Proposition 3.3(3), so it is traceless. This forces $u_{5,4}\neq 0$. Rescaling b_5 we assume $u_{5,4}=1$. The trace is now $u_{5,3}+w_{5,2}$, so $u_{5,3}=-w_{5,2}$. The condition (3.2) applied for $s=2,3,4$ gives linear conditions on the possible matrices $\mathbf{x}_5$ and jointly they imply that

$$\mathbf{x}_5=\begin{pmatrix}p_1&p_2&*&*\\p_3&p_4&*&*\\0&0&p_4-w_{5,2}(p_1+p_5)&p_5\\0&0&-p_3-w_{5,2}(p_6-p_1)&p_6\end{pmatrix}\tag{7.2}$$

for arbitrary $p_i\in\mathbb{C}$ and arbitrary starred entries. Using (3.4) with $u^*=(1,0,0,0)^{\mathbf{t}}$ and $w^*=(0,0,0,1)$, we may change coordinates to assume that the first row and last column of $\mathbf{x}_5$ are zero, and subtracting a multiple of $\mathbf{x}_4$ from $\mathbf{x}_5$ we obtain further that the $(3,2)$ entry of $\mathbf{x}_5$ is zero, so

$$\mathbf{x}_5=\begin{pmatrix}0&0&0&0\\p_3&p_4&0&0\\0&0&p_4&0\\0&0&-p_3&0\end{pmatrix}$$

Subtracting p_4X_1 from X_5 and then adding p_4 times the last row (column) to the fourth row (column) we arrive at

$$\mathbf{x}_5=\begin{pmatrix}0&0&0&0\\p_3&0&0&0\\0&0&0&0\\0&0&-p_3&0\end{pmatrix}\tag{7.3}$$

for possibly different values of the parameter p_3. Conjugating with the 5×5 block diagonal matrix

$$\begin{pmatrix}1&0&0&0&0\\w_{5,2}&1&0&0&0\\0&0&1&0&0\\0&0&w_{5,2}&1&0\\0&0&0&0&1\end{pmatrix}$$

does not change P, hence $\langle \mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4\rangle$, and it does not change $\mathbf{x}_5$ as well, but it makes $w_{5,2}=0$. Thus we arrive at the case when $w_5=(1,0,0,0)^{\mathbf{t}}$, $u_5=(0,0,0,1)$ and $\mathbf{x}_5$ is as in (7.3). There are two subcases: either $p_3=0$ or $p_3\neq 0$. In the latter case, conjugation with the diagonal matrix with diagonal $(1,p_3,1,p_3,1)$ does not change $\langle \mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4\rangle$ and it maps $\mathbf{x}_5$ to the same matrix but with $p_3=1$. In summary, in this case we obtain the types ($T_{\mathcal{O}_{57}}$) and ($T_{\mathcal{O}_{58}}$).

7.2.2. *Case* (M1). For every $t\in\mathbb{C}$ conjugation with

$$\begin{pmatrix}1&t&0&0&0\\0&1&0&0&0\\0&0&1&t&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}$$

preserves $\langle \mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4\rangle$ and maps u_5 to $(0,0,u_{5,3},u_{5,4}-tu_{5,3})$ and w_5 to $(w_{5,1}+tw_{5,2},w_{5,2},0,0)^{\mathbf{t}}$. Taking t general, we obtain $w_{5,1},u_{5,4}\neq 0$ and rescaling b_5,c_5 we obtain $u_{5,4}=1=w_{5,1}$. Since $w_5u_5\in\langle \mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4\rangle$, this forces $u_{5,3}=0$ or $w_{5,2}=0$. Using (3.2) again, we obtain that

$$\mathbf{x}_5=\begin{pmatrix}q_1&*&*&*\\w_{5,2}(q_1-q_3)&q_2&*&*\\0&0&q_3&*\\0&0&u_{5,3}(q_4-q_2)&q_4\end{pmatrix}\tag{7.4}$$

for arbitrary $q_1,q_2,q_3,q_4\in\mathbb{C}$ and arbitrary starred entries. We normalize further. Transposing (this is the unique point of the proof where we swap the B and C coordinates) and swapping 1 with 4 and 2 with 3 rows and columns (which is done by conjugation with appropriate permutation matrix) does not change the space $\langle \mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4\rangle$ or $\mathbf{x}_1$ and it maps u_5, w_5 to $(0,0,w_{5,2},w_{5,1})$, $(u_{5,4},u_{5,3},0,0)^{\mathbf{t}}$. Using this operation if necessary, we may assume $u_{5,3}=0$. By subtracting multiples of u_5, w_5 and $\mathbf{x}_2$, $\mathbf{x}_3$, $\mathbf{x}_4$ we obtain

$$\mathbf{x}_5=\begin{pmatrix}0&0&0&0\\-q_3w_{5,2}&q_2&q_4&0\\0&0&q_3&0\\0&0&0&0\end{pmatrix}\tag{7.5}$$

Rescaling the second row and column we reduce to two cases:

$$w_{5,2}=1\tag{M1a}$$

$$w_{5,2}=0\tag{M1b}$$

Case (M1a). In this case we have $w_5=(1,1,0,0)^{\mathbf{t}}$ and $u_5=(0,0,0,1)$. We first add $q_4\mathbf{x}_2$ to $\mathbf{x}_5$ and subtract q_4w_5 from the fourth column. This sets $q_4=0$ in (7.5). Next, we subtract $-q_2X_1$ from X_5 and then add q_2u_5 to the first column and q_2w_5 to the fourth row. This makes $q_2=0$ (and changes q_3). Finally, if q_3 is nonzero, we can rescale $\mathbf{x}_5$ by q_3^{-1} and rescale the fifth row and column. This yields $q_3=1$. In summary, we have two cases: $(q_2,q_3,q_4)=(0,1,0)$ and $(q_2,q_3,q_4)=(0,0,0)$. These are the types ($\widetilde{T}_{\mathcal{O}_{56}}$) and ($\widetilde{T}_{\mathcal{O}_{57}}$).

Case (M1b). In this case we have $w_5=(1,0,0,0)^{\mathbf{t}}$ and $u_5=(0,0,0,1)$.

Subtract $-q_3\mathbf{x}_1$ from $\mathbf{x}_5$ and then add q_3u_5 to the first column and q_3w_5 to the fourth row. This makes $q_3=0$ (and changes q_2).

Subcase $q_2=0$: Then either $q_4=0$ and we obtain type ($T_{\mathcal{O}_{54}}$) or we rescale X_5 and the fifth row and column to obtain $q_4=1$. Here $(q_2,q_3,q_4)=(0,0,1)$. This is type ($T_{\mathcal{O}_{55}}$).

Subcase $q_2\neq 0$: Then we rescale X_5 and the fifth row and column to obtain $q_2=1$. Subtract q_4 times the second column from the third and add q_4 times the third row to the second. This does not change $\mathbf{x}_1, \ldots, \mathbf{x}_4$ and it changes $\mathbf{x}_5$ by making $q_4 = 0$. Here $(q_2, q_3, q_4) = (1, 0, 0)$, this is type ($T_{\mathcal{O}_{56}}$).

We have shown that there are at most seven isomorphism types up to $\mathrm{GL}(A) \times \mathrm{GL}(B) \times \mathrm{GL}(C)$ action, while the dimensions of the Lie algebras and restricted Lie algebras show that they are pairwise non-isomorphic. This concludes the proof of Theorem 7.3. □

7.3. **Proof of Theorem 1.7.**

Proof. We first prove that there are exactly five isomorphism types of concise 1-degenerate 111-abundant up to action of $\mathrm{GL}_5(\mathbb{C})^{\times 3} \rtimes \mathfrak{S}_3$. By Proposition 6.2, after possibly permuting A, B, C, the space $T(A^*)$ has corank one. It is enough to prove that in the setup of Theorem 7.3 the two pairs of tensors with the symmetry Lie algebras of the same dimension of are isomorphic. Swapping the A and C coordinates of the tensor in case ($T_{\mathcal{O}_{56}}$) and rearranging rows, columns, and matrices gives case ($\widetilde{T}_{\mathcal{O}_{56}}$). Swapping the A and B coordinates of the tensor in case ($\widetilde{T}_{\mathcal{O}_{57}}$) and rearranging rows and columns, we obtain the tensor

$$a_1(b_1c_1 + b_2c_2 + b_3c_3 + b_4c_4) + a_2b_3c_2 + a_3(b_4c_1 + b_4c_2) + a_4(b_3c_1 - b_4c_2) + a_5(b_3c_5 + b_5c_1 + b_4c_5)$$

The space of 2×2 matrices associated to this tensor is perpendicular to $\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$ which has full rank, hence this tensor is isomorphic to one of the (M2) cases. The dimension of the symmetry Lie algebra shows that it is isomorphic to ($T_{\mathcal{O}_{57}}$). This concludes the proof that there are exactly five isomorphism types.

7.4. **Proof of the degenerations.** Write $T \trianglerighteq T'$ if T degenerates to T' and $T \simeq T'$ if T and T' lie in the same orbit of $\mathrm{GL}_5(\mathbb{C})^{\times 3} \rtimes \mathfrak{S}_3$. The above yields $(T_{\mathcal{O}_{56}}) \simeq (\widetilde{T}_{\mathcal{O}_{56}})$ and $(\widetilde{T}_{\mathcal{O}_{57}}) \simeq (T_{\mathcal{O}_{57}})$. Varying the parameters in §7.2.1, §7.2.2, §7.2.2 we obtain degenerations which give

$$(T_{\mathcal{O}_{58}}) \trianglerighteq (T_{\mathcal{O}_{57}}) \simeq (\widetilde{T}_{\mathcal{O}_{57}}) \trianglerighteq (\widetilde{T}_{\mathcal{O}_{56}}) \simeq (T_{\mathcal{O}_{56}}) \trianglerighteq (T_{\mathcal{O}_{55}}) \trianglerighteq (T_{\mathcal{O}_{54}}),$$

which proves the required nesting. For example, in §7.2.2 we have a two-parameter family of tensors parameterized by $(q_2, q_4) \in \mathbb{C}^2$. As explained in that subsection, their isomorphism types are

$q_2 \neq 0$	$q_2 = 0,\ q_4 \neq 0$	$q_2 = q_4 = 0$
($T_{\mathcal{O}_{56}}$)	($T_{\mathcal{O}_{55}}$)	($T_{\mathcal{O}_{54}}$)

This exhibits the last two degenerations; the others are similar.

To complete the proof, we need to show that these tensors have minimal border rank. By degenerations above, it is enough to show this for ($T_{\mathcal{O}_{58}}$). We give two proofs.

7.5. **Two proofs that the tensors have minimal border rank.**

7.5.1. *Proof one: the tensor ($T_{\mathcal{O}_{58}}$) lies in the closure of minimal border rank 1_A-generic tensors.* Our first approach is to prove that ($T_{\mathcal{O}_{58}}$) lies in the closure of the locus of 1_A-generic concise minimal border rank tensors. We do this a bit more generally, for all tensors in the case (M2). By the discussion above every such tensor is isomorphic to one where $\mathbf{x}_5$ has the form (7.3) and we will assume that our tensor T has this form for some $p_3 \in \mathbb{C}$.

Recall the notation from Proposition 3.3. Take $u_2 = 0$, $w_2 = 0$, $u_3 := (0, 0, -p_3, 0)$, $w_3^{\mathbf{t}} = (0, p_3, 0, 0)$, $u_4 = 0$, $w_4 = 0$. We see that $u_s\mathbf{x}_m = 0$, $\mathbf{x}_m w_s = 0$, and $w_s u_t = w_t u_s$ for all s, t, so for every $\epsilon \in \mathbb{C}^*$

we have a commuting quintuple

$$\mathrm{Id}_5, \quad \begin{pmatrix} \mathbf{x}_s & w_s \\ u_s\epsilon & 0 \end{pmatrix} \quad s = 2,3,4, \quad \text{and} \quad \begin{pmatrix} \mathbf{x}_5 & w_5\epsilon^{-1} \\ u_5 & 0 \end{pmatrix}$$

We check directly that the tuple is End-closed, hence by Theorem 1.4 it corresponds to a tensor of minimal border rank. (Here we only use the $m = 5$ case of the theorem, which is significantly easier than the $m = 6$ case.) Multiplying the matrices of this tuple from the right by the diagonal matrix with entries $1, 1, 1, 1, t$ and then taking the limit with $t \to 0$ yields the tuple of matrices corresponding to our initial tensor T.

While we have shown all (M2) cases are of minimal border rank, it can be useful for applications to have an explicit border rank decomposition. What follows is one such:

7.5.2. *Proof two: explicit proof of minimal border rank for* ($T_{\mathcal{O}_{58}}$). For $t \in \mathbb{C}^*$, consider the matrices

$$B_1 = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & t & 1 & 0 & 0 \\ 0 & t^2 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, B_4 = \begin{pmatrix} -t & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ t^2 & 0 & 0 & -t & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_5 = (1, -t, 0, -t, t^2)^{\mathbf{t}} \cdot (-t, 0, t, 1, t^2) = \begin{pmatrix} -t & 0 & t & 1 & t^2 \\ t^2 & 0 & -t^2 & -t & -t^3 \\ 0 & 0 & 0 & 0 & 0 \\ t^2 & 0 & -t^2 & -t & -t^3 \\ -t^3 & 0 & t^3 & t^2 & t^4 \end{pmatrix}$$

The limit at $t \to 0$ of this space of matrices is the required tuple. This concludes the proof of Theorem 1.7. □

8. Proof (1)=(4) in Theorem 1.4

8.1. **Preliminary remarks.** Let $T \in A \otimes B \otimes C = \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ be 1_A-generic and satisfy the A-Strassen equations. Let $E \subseteq \mathfrak{sl}(C)$ be the associated $m-1$-dimensional space of commuting traceless matrices as in §2.2. Let $\underline{C}$ be the associated module and S the associated polynomial ring, as in §2.2. By §2.2 the tensor T has minimal border rank if and only if the space E is a limit of spaces of simultaneously diagonalizable matrices if and only if $\underline{C}$ is a limit of semisimple modules.

The *principal component* of the Quot (resp. Hilbert) scheme is the closure of the set of semisimple modules (resp. algebras). Similarly, the *principal component* of the space of commuting matrices is the closure of the space of simultaneously diagonalizable matrices. A tensor T has minimal border rank if and only if E lies in the principal component of the space of commuting matrices if and only if $\underline{C}$ lies in the principal component of the Quot scheme.

Write $\mathrm{Ann}(\underline{C}) = \{s \in S \mid s(\underline{C}) = 0\}$. Let α_i be a basis of A^* with $T(\alpha_1)$ of full rank and $X_i = T(\alpha_i)T(\alpha_1)^{-1} \in \mathrm{End}(C)$, for $1 \le i \le m$. The algebra of matrices generated by $\mathrm{Id}, X_2, \dots, X_m$ is isomorphic to $S/\mathrm{Ann}(\underline{C})$. The End-closed condition in the language of modules becomes the requirement that the algebra of matrices has dimension (at most) m. The tensor T is assumed to be A-concise, i.e., $\dim\langle \mathrm{Id}, X_2, \dots, X_m \rangle = m$, so the algebra is equal to this linear span: $X_iX_j \in \langle \mathrm{Id} = X_1, X_2, \dots, X_m \rangle$.

Our argument proceeds by examining the possible structures of $\underline{C}$ and $S/\mathrm{Ann}\,(\underline{C})$ and, in each case, proving that $\underline{C}$ lies in the principal component. Let r be the minimal number of generators of $\underline{C}$.

In this section we introduce the additional index range

$$2 \le y, z, q \le m.$$

When $S/\mathrm{Ann}\,(\underline{C})$ is *local*, i.e., there is a unique maximal ideal $\mathfrak{m}$, we consider the Hilbert function $H_{\underline{C}}(k) := \dim(\mathfrak{m}^k \underline{C}/\mathfrak{m}^{k+1}\underline{C})$ and by Nakayama's Lemma $H_{\underline{C}}(0) = r$. Similarly, we consider the Hilbert function $H_{S/\mathrm{Ann}\,(\underline{C})}(k) := \dim(\mathfrak{m}^k/\mathfrak{m}^{k+1})$. Since the algebra is local, $H_{S/\mathrm{Ann}\,(\underline{C})}(0) = 1$. Observe that if $X_y X_z X_w = 0$ for all y, z, w, then $\mathrm{Ann}\,(\underline{C})$ contains $S_{\ge 3}$, which implies $S/\mathrm{Ann}\,(\underline{C})$ is local. When $H_{S/\mathrm{Ann}\,(\underline{C})}(1) = k < m-1$, we may work with a polynomial ring in k variables, $\widetilde{S} = \mathbb{C}[y_1, \dots, y_k]$.

We will use the following results, which significantly restrict the possible structure of $\underline{C}$ and $S/\mathrm{Ann}\,(\underline{C})$.

(i) For a finite algebra $\mathcal{A} = \Pi \mathcal{A}_t$, with the $\mathcal{A}_t$ local, the algebra $\mathcal{A}$ can be generated by q elements if and only if $H_{\mathcal{A}_t}(1) \le q$ for all t. From the geometric perspective, the number of generators needed is the smallest dimension of an affine space the associated scheme can be realized inside, and one just chooses the support of each $\mathcal{A}_t$ to be a different point of $\mathbb{A}^q$.

(ii) When the module $\underline{C}$ is generated by a single element (so we are in the Hilbert scheme), and $m \le 7$, all such modules lie in the principal component [17].

(iii) By [34, Cor. 4.3], when $m \le 10$ and the algebra of matrices generated by $\mathrm{Id}, X_2, \dots, X_m$ is generated by at most three generators, then the module lies in the principal component. When $S/\mathrm{Ann}\,(\underline{C})$ is local, this happens when $H_{S/\mathrm{Ann}\,(\underline{C})}(1) \le 3$.

(iv) When $m - 1 \le 6$, if $X_y X_z = 0$ for all y, z, then the module lies in the principal component by [34, Thm. 6.14]. This holds when $S/\mathrm{Ann}\,(\underline{C})$ is local with $H_{S/\mathrm{Ann}\,(\underline{C})}(2) = 0$.

(v) If $X_y X_z X_w = 0$ for all y, z, w (i.e., $H_{S/\mathrm{Ann}\,(\underline{C})}(3) = 0$), $\dim \sum \mathrm{Im}(X_y X_z) = 1$ (i.e., $H_{S/\mathrm{Ann}\,(\underline{C})}(2) = 1$), and $\dim \cap_{y,z} \ker(X_y X_z) = m - 1$, then $(X_2, \dots, X_m)$ deforms to a tuple with a matrix having at least two eigenvalues. Explicitly, there is a normal form so that

$$X_y = \begin{pmatrix} 0 & 0 & H_y & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & G_y \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where $X_2^2 \neq 0$ and all other products are zero. Then

$$Y := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & G_2 H_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

commutes with all the X_i, and the deformation (to a not necessarily traceless tuple) is $(X_2 + \lambda Y, X_3, \dots, X_m)$ by [34, Lem. 6.13].

We now show that all End-closed subspaces $\widetilde{E} = \langle \mathrm{Id}, E\rangle$ lie in the principal component when $m = 5, 6$ by, in each possible case, assuming the space is not in the principal component and obtaining a contradiction.

8.2. **Case** $m = 5$**.**

8.2.1. *Case: E contains an element with more than one eigenvalue, i.e., E is not nilpotent.* By [34, Lem. 3.12] this is equivalent to saying the algebra $S/\mathrm{Ann}\,(\underline{C})$ is a nontrivial product of algebras $\Pi_t \mathcal{A}_t$. Since $\dim(S/\mathrm{Ann}\,(\underline{C})) = 5$, we have for each t that $\dim(\mathcal{A}_t) \leq 4$ and thus $H_{\mathcal{A}_t}(1) \leq 3$.Using (i), we see $S/\mathrm{Ann}\,(\underline{C})$ is generated by at most three elements, a contradiction by (iii).

8.2.2. *Case: all elements of E are nilpotent.* In this case $\mathrm{Ann}\,(\underline{C})$ contains $S_{\geq (m-1)m}$ because any nilpotent $m \times m$ matrix raised to the m-th power is zero and we have $m-1$ commuting matrices that we could multiply together. Thus $S/\mathrm{Ann}\,(\underline{C})$ is local and we can speak about Hilbert functions. By (iii) we assume $H_{S/\mathrm{Ann}\,(C)}(1) \geq 4$, so $H_{S/\mathrm{Ann}\,(\underline{C})}(2) = 0$. Thus for all z, w, $y_z y_w \in \mathrm{Ann}\,(\underline{C})$ and we conclude by (iv).

8.3. **Case** $m = 6$**.** For non-local $S/\mathrm{Ann}\,(\underline{C})$, arguing as in §8.2.1 the only case is $S/\mathrm{Ann}\,(\underline{C}) \simeq \mathcal{A}_1 \times \mathcal{A}_2$ with $\dim \mathcal{A}_1 = 1$ and $H_{\mathcal{A}_2}(1) = 4$, $H_{\mathcal{A}_2}(2) = 0$. Correspondingly the module $\underline{C}$ is a direct sum of modules $\underline{C}_1 \oplus \underline{C}_2$, where $\mathcal{A}_2 \simeq S/\mathrm{Ann}\,(\underline{C}_2)$. By (iii) and (iv) the module $\underline{C}_2$ lies in the principal component and trivially so does $\underline{C}_1$. Hence $\underline{C}$ lies in the principal component.

We are reduced to the case $S/\mathrm{Ann}\,(\underline{C})$ is local. By (iii) we assume $H_{S/\mathrm{Ann}\,(\underline{C})}(1) > 3$. Moreover, if $H_{S/\mathrm{Ann}\,(\underline{C})}(1) = 5$, we have $H_{S/\mathrm{Ann}\,(\underline{C})}(2) = 0$ and we conclude by (iv). Thus the unique Hilbert function $H_{S/\mathrm{Ann}\,(\underline{C})}$ left to consider is $(1, 4, 1)$.

8.3.1. *Case* $\dim \sum_{y,z} \mathrm{Im}(X_y X_z) = 1$*, i.e.,* $H_{S/Ann(\underline{C})}(2) = 1$. Since for all y, z, $X_y X_z$ lies in the m dimensional space $\langle \mathrm{Id}, X_2, \dots, X_m\rangle$, we must have $\dim(\cap_{y,z} \ker(X_y X_z)) = m-1$ and thus (v) applies. Let $\underline{C}(\lambda)$ denote $\underline{C}$ with this deformed module structure. The assumption that $X_1 X_y = X_y X_1 = 0$ for $2 \leq y \leq m$ implies $H_1 K_y = 0$ and $H_y K_1 = 0$ which implies that $\underline{C}(\lambda)$ also satisfies the End-closed condition. Since $\underline{C}(\lambda)$ is not supported at a point, it cannot have Hilbert function $(1, 4, 1)$ so it is in the principal component, and thus so is $\underline{C} = \underline{C}(0)$.

8.3.2. *Case* $\dim \sum_{y,z} \mathrm{Im}(X_y X_z) > 1$. This hypothesis says $H_{\underline{C}}(2) \geq 2$. Since $H_{S/\mathrm{Ann}\,(\underline{C})}(3) = 0$ also $H_{\underline{C}}(3) = 0$. We have $H_{\underline{C}}(0) + H_{\underline{C}}(1) + H_{\underline{C}}(2) = 6$. If $H_{\underline{C}}(0) = 1$ then (ii) applies, so assume $H_{\underline{C}}(0) \geq 2$. If $H_{\underline{C}}(1) = 1$, then a near trivial case of Macaulay's growth bound for modules [8, Cor. 3.5], says $H_{\underline{C}}(2) < 2$, so the Hilbert function $H_{\underline{C}}$ is $(2, 2, 2)$, and the minimal number of generators of $\underline{C}$ is $H_{\underline{C}}(0) = 2$. Let $F = Se_1 \oplus Se_2$ be a free S-module of rank two. Fix an isomorphism $\underline{C} \simeq F/\mathcal{R}$, where $\mathcal{R}$ is the subspace generated by the relations.

We briefly recall the apolarity theory for modules from [34, §4.1]. Let $\widetilde{S} = \mathbb{C}[y_1, \dots, y_4]$ which we may use instead of S because $H_{S/\mathrm{Ann}\,(\underline{C})}(1) = 4$. Let $\widetilde{S}^* = \bigoplus_j \mathrm{Hom}(\widetilde{S}_j, \mathbb{C}) =: \mathbb{C}[z_1, \dots, z_4]$ be the dual polynomial ring. Let $F^* := \bigoplus_j \mathrm{Hom}(F_j, \mathbb{C}) = \widetilde{S}^* e_1^* \oplus \widetilde{S}^* e_2^* = \mathbb{C}[z_1, \dots, z_4] e_1^* \oplus \mathbb{C}[z_1, \dots, z_4] e_2^*$. The action of $\widetilde{S}$ on F^* is the usual contraction action. In coordinates it is the "coefficientless" differentiation: $y_i^d(z_j^u) = \delta_{ij} z_j^{u-d}$ when $u \geq d$ and is zero otherwise. The subspace $\mathcal{R}^\perp \subseteq F^*$ is an $\widetilde{S}$-submodule.

Consider a minimal set of generators of $\mathcal{R}^\perp \subseteq F^*$. The assumption $H_{\underline{C}}(2) = 2$ implies there are two generators in degree two, write their leading terms as $\sigma_{11} e_1^* + \sigma_{12} e_2^*$ and $\sigma_{21} e_1^* + \sigma_{22} e_2^*$, with $\sigma_{uv} \in \widetilde{S}_2$. Then $\mathrm{Ann}\,(\underline{C}) \cap \widetilde{S}_{\geq 2} = \langle \sigma_{11}, \dots, \sigma_{22}\rangle^\perp \cap \widetilde{S}_{\geq 2}$. But $H_{\widetilde{S}/\mathrm{Ann}\,(\underline{C})}(2) = 1$, so all the σ_{uv} must be a multiple of some σ and after changing bases we write the leading terms as σe_1^*, σe_2^*. We see $\langle y_i\sigma e_1^* + \dots, y_i\sigma e_2^* + \dots, 1 \leq i \leq 4\rangle \subseteq \mathcal{R}^\perp$, where y_i acts on $\widetilde{S}^*$ by contraction and the "..." are lower order terms. Now $H_{\underline{C}}(1) = 2$ says this is a 2-dimensional space, i.e., that σ is a square. Change coordinates so $\sigma = z_1^2$. Thus the generators of $\mathcal{R}^\perp$ include $Q_1 \coloneqq z_1^2 e_1^* + \ell_{11}e_1^* + \ell_{12}e_2^*, Q_2 \coloneqq z_1^2 e_2^* + \ell_{21}e_1^* + \ell_{22}e_2^*$ for some linear forms ℓ_{uv}. These two generators plus their contractions (by y_1, y_1^2) span a six dimensional space, so these must be all the generators. Our module is thus a degeneration of the module where the z_1, ℓ_{uv} are all independent linear forms. Take a basis of the module $\mathcal{R}^\perp \subseteq F^*$ as $Q_1, Q_2, y_1Q_1, y_1Q_2, y_1^2Q_1, y_1^2Q_2$. Then the matrix associated to the action of y_1 is

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and if we deform our module to a space where the linear forms z_1, ℓ_{uv} are all independent and change bases such that $\ell_{11} = y_2^*$, $\ell_{12} = y_3^*$, $\ell_{21} = y_4^*$, $\ell_{22} = y_5^*$, we may write our space of matrices as

$$\begin{pmatrix} 0 & 0 & z_1 & 0 & z_2 & z_3\\ 0 & 0 & 0 & z_1 & z_4 & z_5\\ 0 & 0 & 0 & 0 & z_1 & 0\\ 0 & 0 & 0 & 0 & 0 & z_1\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Using *Macaulay2 VersalDeformations* [33] we find that this tuple is a member of the following family of tuples of commuting matrices parametrized by $\lambda \in \mathbb{C}$. Their commutativity is straightforward if tedious to verify by hand

$$\begin{pmatrix} 0 & \lambda^2 z_4 & z_1 & -\lambda z_5 & z_2 & z_3\\ -\lambda z_1 & 0 & -\lambda z_4 & z_1 & z_4 & z_5\\ -\lambda^3 z_4 & \lambda^2 z_1 & 0 & \lambda^2 z_4 & z_1 & -\lambda z_5\\ 0 & 0 & 0 & -\lambda^2 z_5 & \lambda(z_2 - z_4) & \lambda z_3 + z_1\\ 0 & 0 & 0 & 0 & -\lambda^2 z_5 & 0\\ 0 & 0 & 0 & 0 & 0 & -\lambda^2 z_5 \end{pmatrix}.$$

Here there are two eigenvalues, each with multiplicity three, so the deformed module is a direct sum of two three dimensional modules, each of which thus has an associated algebra with at most three generators and we conclude by (iii). $\square$

9. Minimal cactus and smoothable rank

For a degree m zero-dimensional subscheme $\mathrm{Spec}(R)$ with an embedding $\mathrm{Spec}(R) \subseteq Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C) \subseteq \mathbb{P}(A \otimes B \otimes C)$, its *span* $\langle \mathrm{Spec}(R)\rangle$ is the zero set of $I_1(\mathrm{Spec}(R)) \subseteq A^* \otimes B^* \otimes C^*$, where $I_1(\mathrm{Spec}(R))$ is the degree one component of the homogeneous ideal I of the embedded $\mathrm{Spec}(R)$. We say that the embedding $\mathrm{Spec}(R) \subseteq Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$ is *nondegenerate* if its span projects surjectively to $\mathbb{P}A$, $\mathbb{P}B$, and $\mathbb{P}C$. For a nondegenerate embedding, the maps $\mathrm{Spec}(R) \to \mathbb{P}A$, $\mathrm{Spec}(R) \to \mathbb{P}B$, $\mathrm{Spec}(R) \to \mathbb{P}C$, induced by projections, are embeddings as well. If $\langle \mathrm{Spec}(R)\rangle$ contains a concise tensor, then the embedding of $\mathrm{Spec}(R)$ is automatically nondegenerate.

The *cactus rank* [12] of $T \in A \otimes B \otimes C$ is the smallest r such that there exists a degree r zero-dimensional subscheme $\mathrm{Spec}(R) \subseteq Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C) \subseteq \mathbb{P}(A \otimes B \otimes C)$ with $[T] \in \langle Spec(R) \rangle$. (Recall that the smoothable rank has the same definition except that one additionally requires R to be smoothable.)

Given a degree ρ zero-dimensional scheme R, for each $\varphi \in R^*$, one gets a tensor $T^\varphi \in R^* \otimes R^* \otimes R^* \simeq \mathbb{C}^\rho \otimes \mathbb{C}^\rho \otimes \mathbb{C}^\rho$ defined by $T^\varphi(r_1, r_2, r_3) \coloneqq \varphi(r_1 r_2 r_3)$. Given any non-degenerate embedding $\mathrm{Spec}(R) \subseteq Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C) \subseteq \mathbb{P}(A \otimes B \otimes C)$, the space of tensors T^φ is isomorphic to the space of tensors $\langle \mathrm{Spec}(R) \rangle$ as will be shown in the proof of Proposition 9.1 below.

In this section we show that the scheme (resp. smoothable scheme) $\mathrm{Spec}(R)$ which witnesses that a tensor $T \in A \otimes B \otimes C$ has minimal cactus (resp. smoothable) rank is unique, in fact, the algebra R is isomorphic to $\mathcal{A}^T_{111}$.

Proposition 9.1. *Let* $\mathrm{Spec}(R)$ *be a degree* m *zero-dimensional subscheme and let* $T \in A \otimes B \otimes C$. *The following are equivalent:*

(1) *There exists a nondegenerate embedding* $\mathrm{Spec}(R) \subseteq Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$ *with* $T \in \langle \mathrm{Spec}(R) \rangle$*, so in particular* T *has cactus rank at most* m.

(2) *there exists* $\varphi \in R^*$ *such that* T *is isomorphic to the tensor in* $R^* \otimes R^* \otimes R^*$ *given by the trilinear map* $(r_1, r_2, r_3) \mapsto \varphi(r_1 r_2 r_3)$.

If T *is concise and satisfies the above, then it is* 1*-generic and has cactus rank* m.

Proof. We first show (1) implies (2). An embedding $\mathrm{Spec}(R) \subseteq \mathbb{P}A$ with $\langle \mathrm{Spec}(R) \rangle = \mathbb{P}A$ is induced from an embedding $\mathrm{Spec}(R) \subseteq A$ with $\langle \mathrm{Spec}(R) \rangle = A$, which in turn induces a vector space isomorphism $\tau_a \colon A^* \to R \cong \mathrm{Sym}(A^*)/I_{R,A}$ as follows: let $I_{R,A}$ denote the ideal of $\mathrm{Spec}(R) \subseteq A$, then $\tau_a(\alpha) \coloneqq \alpha \bmod I_{R,A}$. Hence, a nondegenerate embedding of $\mathrm{Spec}(R)$ induces a triple of vector space isomorphisms $\tau_a \colon A^* \to R$, $\tau_b \colon B^* \to R$, $\tau_c \colon C^* \to R$.

More generally, for each (s,t,u), with $s,t,u \geq 1$, the map

$$\tau_{s,t,u} : S^s A^* \otimes S^t B^* \otimes S^u C^* \to S^s A^* \otimes S^t B^* \otimes S^u C^* / (I_{R, A \otimes B \otimes C})_{s,t,u}$$

is a surjection onto $R \cong S^s A^* \otimes S^t B^* \otimes S^u C^* / (I_{R, A \otimes B \otimes C})_{s,t,u}$, and these maps are all compatible with multiplication, in particular $\tau_{1,1,1}(\alpha \otimes \beta \otimes \gamma) = \tau_a(\alpha) \tau_b(\beta) \tau_c(\gamma)$. Then

$$\langle \mathrm{Spec}(R) \rangle = (\ker \tau_{1,1,1})^\perp \subseteq (A^* \otimes B^* \otimes C^*)^* = A \otimes B \otimes C.$$

By duality, the space $(\ker \tau_{1,1,1})^\perp$ is the image of the map $R^* \to A \otimes B \otimes C$ defined by requiring that $\varphi \in R^*$ maps to the trilinear form $(\alpha, \beta, \gamma) \mapsto \varphi(\tau_a(\alpha) \tau_b(\beta) \tau_c(\gamma))$.

If T is the image of φ, then it is isomorphic to the trilinear map $(r_1, r_2, r_3) \mapsto \varphi(r_1 r_2 r_3)$ via $\tau_a^{\mathbf{t}} \otimes \tau_b^{\mathbf{t}} \otimes \tau_c^{\mathbf{t}}$, proving (1) implies (2).

Assuming (2), choose vector space isomorphisms τ_a, τ_b, τ_c and define a map $A^* \otimes B^* \otimes C^* \to R$, by $\alpha \otimes \beta \otimes \gamma \mapsto \tau_a(\alpha) \tau_b(\beta) \tau_c(\gamma)$. (For readers familiar with border apolarity, the kernel of this map is I_{111}.) Then extend it to $S^s A^* \otimes S^t B^* \otimes S^u C^*$ by $\tau_a(\alpha_1 \cdots \alpha_s) = \tau_a(\alpha_1) \cdots \tau_a(\alpha_i)$ and similarly. This yields the required nondegenerate embedding of $\mathrm{Spec}(R)$. The tensor T' corresponding to $(\alpha, \beta, \gamma) \mapsto \varphi(\tau_a(\alpha) \tau_b(\beta) \tau_c(\gamma))$ is isomorphic to T and lies in $\langle \mathrm{Spec}(R) \rangle$. This proves (1).

Finally, if T satisfies the above, then it is isomorphic to $(r_1, r_2, r_3) \mapsto \varphi(r_1 r_2 r_3)$ for some φ. If T is additionally concise, then for every $r \in R$ there exists an $r' \in R$ such that $\varphi(r r') \neq 0$. Hence the map $(r_1, r_2) \mapsto \varphi(r_1, r_2)$ has full rank. But this map is $\varphi(1_R)$. This shows that T is 1-generic. It has cactus rank at least m by conciseness and at most m by assumption. □

In particular, a concise tensor $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ has *minimal smoothable rank* if there exists a smoothable degree m algebra R satisfying the conditions of Proposition 9.1.

Theorem 9.2. *Let $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ be a concise tensor. The following are equivalent*

(1) T has minimal smoothable rank,

(2) T is 1-generic, 111-sharp and its 111-algebra is smoothable and Gorenstein.

(3) T is 1-generic, 111-abundant and its 111-algebra is smoothable.

We emphasize that in Theorem 9.2 one does not need to find the smoothable scheme to show the tensor has minimal smoothable rank, which makes the theorem effective by reducing the question of determining minimal smoothable rank to proving smoothability of a given algebra.

Proof of Theorem 9.2. Suppose (1) holds and so there exists a smoothable algebra R and an embedding of it into $Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$ with $T \in \langle \mathrm{Spec}(R) \rangle$. By Proposition 9.1 T is 1-generic and isomorphic to the tensor in the vector space $R^* \otimes R^* \otimes R^*$ given by the trilinear map $(r_1, r_2, r_3) \mapsto \varphi(r_1 r_2 r_3)$ for some functional $\varphi \in R^*$, in particular $T \in \mathrm{Hom}(R \otimes R \otimes R, \mathbb{C})$. Suppose that there exists a nonzero $r \in R$ such that $\varphi(Rr) = 0$. Then for all $r_1, r_2 \in R$, $(r_1, r_2, r) \mapsto 0$ so T is not concise. Hence no such r exists and so φ is nondegenerate. This shows that R is Gorenstein.

For an element $r \in R$, the multiplication by r on the first position gives a map

$$\mu_r^{(1)} : \mathrm{Hom}(R \otimes R \otimes R, \mathbb{C}) \to \mathrm{Hom}(R \otimes R \otimes R, \mathbb{C})$$

and similarly we obtain $\mu_r^{(2)}$ and $\mu_r^{(3)}$. Observe that for $i = 1, 2, 3$ and every $r \in R$ the map corresponding to the tensor $\mu_r^{(i)}(T)$ is the composition of the multiplication $R \otimes R \otimes R \to R$, the multiplication by r map $R \to R$ and $\varphi : R \to \mathbb{C}$. Therefore $\mu_r^{(1)}(T) = \mu_r^{(2)}(T) = \mu_r^{(3)}(T)$. Moreover, for any nonzero r we have $\mu_r^{(i)}(T) \neq 0$ since φ is nondegenerate. This shows that $\langle \mu_r^{(i)}(T) \mid r \in R \rangle$ is an m-dimensional subspace of $\mathcal{A}_{111}^T \cdot T \subseteq A \otimes B \otimes C$.

Since T has minimal smoothable rank, it has minimal border rank so it is 111-abundant and by Proposition 3.2 is it 111-sharp, so its 111-algebra is $\langle \mu_r^{(i)}(T) \mid r \in R \rangle$, which is isomorphic to R. This proves (1) implies (2). That (2) implies (3) is vacuous.

Suppose (3) holds and take $R = \mathcal{A}_{111}^T$. Then T is 111-sharp by Proposition 3.2, which also implies the tensor T is isomorphic to the multiplication tensor of R. The algebra R is Gorenstein as T is 1-generic (see §2.5). Since R is Gorenstein, the R-module R^* is isomorphic to R. Take one such isomorphism $\Phi : R \to R^*$ and let $\varphi = \Phi(1_R)$. Then the composition $R \otimes R \otimes R \to R \to \mathbb{C}$ can be rewritten as $R \otimes R \to R \to R^*$, where the first map is the multiplication and the second one sends r to $r\varphi$; this second map is equal to Φ. Composing further with Φ^{-1} we obtain a map $R \otimes R \to R \to R^* \to R$ which is simply the multiplication. All this shows that the tensor in $R^* \otimes R^* \otimes R^*$ associated to (R, φ) is isomorphic to the multiplication tensor of R, hence to T. By Proposition 9.1 and smoothability of R such a tensor has minimal smoothable rank. □

Remark 9.3. There is a version of Theorem 9.2 without smoothability assumptions: a concise tensor has minimal cactus rank if and only if it is 1-generic and 111-abundant with Gorenstein 111-algebra.

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