Concrete Structure Design Using Mixed-Integer Nonlinear Programming with Complementarity Constraints

Andres Guerra, Alexandra M. Newman, and Sven Leyffer

Contents

Concrete Structure Design Using Mixed-Integer Nonlinear Programming with Complementarity Constraints

Andres Guerra† • Alexandra M. Newman‡ • Sven Leyffer∗

†*Division of Engineering, Colorado School of Mines, Golden, CO 80401*
‡*Division of Economics and Business, Colorado School of Mines, Golden, CO 80401*
∗*Mathematics and Computer Science Division, Argonne National Laboratory, Argonne, IL 60439*
aguerra@mines.edu • newman@mines.edu • leyffer@mcs.anl.gov

November 24, 2009

Abstract

We present a mixed-integer nonlinear programming (MINLP) formulation to achieve minimum-cost designs for reinforced concrete (RC) structures that satisfy building code requirements. The objective function includes material and labor costs for concrete, steel reinforcing bars, and formwork according to typical contractor methods. Restrictions enforce correct geometry of the cross-section dimensions for each element and relative sizes of cross-section dimensions of elements within the structure. Other restrictions define a stiffness and displacement correlation among all structural elements via finite element analysis.

The design of minimum cost RC structures introduces a new class of optimization problems, namely, mixed-integer nonlinear programs with complementarity constraints. The complementarity constraints are used to model RC element strength and ACI code-required safety factors. We reformulate the complementarity constraints as nonlinear equations and show that the resulting ill-conditioned MINLPs can be solved by using an off-the-shelf MINLP solver. Our work provides discrete-valued design solutions for an explicit representation of a process most often performed implicitly with iterative calculations.

We demonstrate the capabilities of a mixed-integer nonlinear algorithm, MINLPBB, to find optimal sizing and reinforcing for cast-in-place beam and column elements in multistory RC structures. Problem instances contain up to 678 variables, of which 214 are integer, and 844 constraints, of which 582 are nonlinear. We solve problems to local optimality within a reasonable amount of computational time, and we find an average cost savings over typical-practice design solutions of 13 percent.

1 Introduction and Literature Review

Reinforced concrete (RC) is commonly used to build cost-efficient and durable structures. Important properties of RC are (1) significant compressive strength that increases over time, (2) low maintenance, (3) fire resistance, and (4) constitution of inexpensive local materials such as sand, gravel, and water. RC consists of large portions of sand and gravel and smaller portions of cement and steel reinforcing bars. The cement and water chemically interact to cohere the sand and gravel into a solid mass surrounding the reinforcing bars. Cast-in-place RC construction refers to methods used to fabricate structural elements in the intended design position and location. Wet concrete is placed into a wooden or steel formwork that holds the concrete in place until it develops sufficient self supporting strength. Reinforcement is placed within the formwork before pouring the concrete so that the concrete hardens around the reinforcement. The combination of concrete and reinforcing bars provides elements that can withstand large forces. The design of RC involves

selecting dimensions consisting of discrete-valued element width and depth, reinforcing bar sizes, and the number of bars ensuring structural integrity. The goal of this paper is to provide a sound mathematical model and solution methodology that yield minimum cost designs of large and complex RC buildings.

The analysis procedures for RC that are typically adopted in practice assume a structural system with a fixed initial stiffness. The demand on the RC elements in terms of displacements and forces depends on the applied loads and relative stiffness of elements, where stiffness is a measure of displacement with respect to force. A fixed initial stiffness distribution is necessary to calculate the demand, or internal forces, in a statically indeterminate structure. Engineers then design element dimensions to resist these internal forces, which, unfortunately, are inconsistent with the internal forces associated with the final design dimensions. This inconsistency creates unnecessarily expensive and overengineered solutions.

An explicit formulation of the RC design problem improves the fidelity of the current-practice structural analysis by resolving inconsistencies between the initial design assumptions and the final design dimensions. We develop an explicit formulation for the RC design problem using continuous and continuously differentiable expressions to enforce relationships between decision variables. We represent Boolean expressions with binary variables whose continuous relaxations yield differentiable expressions. An explicit formulation provides a feasible solution directly and completely computed without an iterative procedure, and in our case, locally optimal solutions that can be easily evaluated. An explicit formulation also allows for the use of a robust solver on larger problems that produce better-quality solutions in less computation time.

In our explicit mixed-integer nonlinear optimization model, we use integer variables to represent the width and depth of each element and the number of reinforcing bars. We employ binary variables to select discrete reinforcing bar sizes from a set of potential sizes and to represent discrete decisions for formwork reuse. Continuous decision variables relate to applied forces and resistive capacity of each element. Formulating the problem with integer variables allows for an objective function including material and labor costs for concrete, steel reinforcing bars, and formwork in accordance with typical methods used by contractors. Constraints or restrictions enforce (1) the correct geometry of the cross-section dimensions for each element, (2) the relative sizes of cross-section dimensions of elements within the structure, (3) finite-element analysis equilibrium equations that define the appropriate relationships between forces, displacements, and stiffness, (4) the resistive capacities of each element, (5) bounds on the applied loads relative to the resistive capacities of each element, and (6) upper and lower bounds on the variables.

Our model provides an example of a new class of challenging optimization problems, namely mixed-integer nonlinear optimization problems with complementarity constraints. The complementarity constraints are necessary to model the resistive forces provided by the concrete, elastic-perfectly plastic material response for the steel reinforcement. We use American Concrete Institute (ACI) code requirements to include the appropriate safety factors and minimum axial load requirements for flexural elements. We reformulate the complementarity constraints as nonlinear inequalities. This approach gives rise to a degenerate MINLP that violates the Mangasarian-Fromowitz constraint qualification at any feasible point; see, for example, Scheel and Scholtes (2000). We show that the resulting MINLP can nonetheless be solved reliably using a suitable off-the-shelf MINLP solver. We demonstrate the capability of our explicit formulation to find lower-cost and more efficient solutions than currently found in practice, extending RC design optimization.

The use of optimization in RC design is not new. The first instances of optimization techniques for RC structures were explicit methods to determine inelastic solutions with fixed element dimensions. Inelastic material behavior incorporates material nonlinearities, whereas elastic material behavior contains linear material properties. De Donato and Maier (1972) were among the first to minimize a quadratic function subject only to sign constraints, incorporating inelastic material behavior for RC structural analysis. Prior to the problem posed by De Donato and Maier, an inelastic solution of an RC structure with fixed element

2

dimensions was difficult to obtain, especially for statically indeterminate problems that require discretization into finite-elements in order to accurately capture structural behavior (most real-world problems require discretization). A study by Corradi et al. (1974) formulates and solves a linear complementarity problem by overrelaxation to find inelastic solutions for multistory frames discretized into finite elements with fixed element dimensions. Kaneko (1977), Maier et al. (1982), and Dinno and Mekha (1995) expand the use of complementarity to perform inelastic analysis of RC structures with fixed element dimensions. When the element dimensions are fixed, finite element methods model constitutive behavior using mathematical operations on linear systems of equations. When element dimensions are variables, finite element analysis becomes increasingly difficult because the system of equations contains nonlinear functions of the element dimensions. We further discuss finite element methods for frames in Section 2.3.1.

While the methods previously discussed use mathematical programming techniques to model inelastic behavior for structures with fixed element dimensions, Krishnamoorthy and Mosi (1981) and Dinno and Mekha (1993) include element dimensions as design variables and search for minimum cost solutions in addition to using complementarity constraints to define inelastic behavior of RC frames. However, to find solutions for reasonably sized buildings, Krishnamoorthy and Mosi (1981) include variables to describe the reinforcing bar areas and determine the width and depth of elements a priori. Dinno and Mekha (1993) include variables to describe the width and depth of elements and to determine the reinforcing bar areas a priori. The objective functions in these studies include simplified costs for concrete, formwork, and reinforcement that do not entirely capture construction practices. Both papers enforce structural stability by ensuring that resistive forces are greater than applied forces, which are determined by finite element analysis. Reducing the number of design variables facilitates solving problems with a larger number of story levels but requires implicit, or iterative, methods to find a feasible solution because the number and size of reinforcing bars depend on the width and height of each structural element.

Ferris and Tin-Loi (1999) present an explicit formulation to find minimum-weight solutions for trusslike structures with complementarity constraints that incorporate inelastic behavior. We consider structures with a greater number of degrees of freedom than trusslike structures have. However, this work demonstrates the use of complementarity constraints in explicit structural optimization problems. Horowitz and Moraes (2005) utilize MINLPBB to solve a problem with complementarity constraints to determine failure loads (rather than actual dimensions) of a continuous RC beam with inelastic material properties. Horowitz and Moraes demonstrate the ability to conduct complex inelastic analysis of RC structures using explicit methods.

Much of the remaining work in optimal design of RC structures excludes complementarity constraints and evaluates structural behavior with elastic material properties to find solutions for buildings with a larger number of design variables. Models using elastic (rather than inelastic) material properties generally preclude the need for complementarity constraints. Fadaee and Grierson (1996), Balling and Yao (1997), and Balling (2002) develop multistep approaches in conjunction with NLP techniques to find minimum-cost solutions with elastic material properties. Fadaee and Grierson (1996) develop implicit methods for three-dimensional RC frames with fewer than 25 continuous-valued variables. Balling and Yao (1997) and Balling (2002) use postprocess rounding operations to obtain discrete-valued, constructible design solutions for two-dimensional RC frames. To find solutions for problems with a larger number of story levels, Balling and Yao (1997) reduce the number of variables that require rounding. However, their simplifying assumptions diminish solution quality. More recently, Guerra and Kiousis (2006) use sequential quadratic programming to determine optimal design solutions of two-dimensional RC structures. They implement a rounding heuristic that requires the solution of a secondary optimization problem in which many of the decision variable values are fixed based on the solution of the monolith. Guerra and Kiousis find solutions for problems with a similar number of story levels as those demonstrated by Balling and Yao (1997) without simplifying assumptions

that reduce solution quality. The objective function presented by Guerra and Kiousis advances previous studies by including changing unit costs as a function of the decision variables to more accurately incorporate construction practices. Lee and Ahn (2003) and Camp et al. (2003) implement genetic algorithms that search for discrete-valued solutions of beam and column elements in RC frames. Like many others, the authors implement elastic material properties. The search for discrete-valued solutions using genetic algorithms is difficult because of the nonlinearities in the model, as well as the large number of combinations of possible element dimensions.

The remainder of this paper is organized such that in Section 2 we present the problem description and formulation and in subsequent sections we present numerical results.

2 Problem Description and Formulation

We begin by describing the model in general terms, then present and justify the mathematical formulation, and finally describe the formulation in detail.

2.1 Parameters and Constants

We consider multistory RC buildings with rectangular beam (horizontally oriented) and column (vertically oriented) elements (see Figure 1). Parameters in the formulation describe structural geometry, element geometry, material properties, applied loads, construction costs, available reinforcement sizes, and upper and lower variable bounds. We use parameter values for beam lengths of element e, l_e, story heights, and applied loading, u_e and $v_{\psi_{je}}$, as shown in Figure 1, based on typical values for RC buildings. The discretization shown includes finite elements with end points identified as nodes, which occur only at the ends of each column and beam. We illustrate typical node number labels in circles, element number labels in squares, and degree-of-freedom labels placed next to corresponding arrows. In Figure 2, we show parameters that define (1) the ACI-specified distance from the concrete edge to the centroid of the reinforcing bars, d^p, (2) the strain in the most compressive concrete fiber, ϵ^{cu}, (3) the ACI-specified strength reduction factor for computing resistive capacity of concrete, β_1, and (4) the compressive strength of the concrete, fc'.

2.2 Variables

Figure 2 illustrates the discrete decision variables that describe the cross-section dimensions for the eth element: integer-valued cross-section width and height, b_e and h_e, respectively, and discrete-valued compressive and tensile reinforcement cross-sectional areas, As_{1e} and As_{2e}, respectively. The reinforcement cross-section areas depend on the size of bar, which we represent with a binary variable, y_{em}, and number of bars, which we represent with integer variables, $As_{1e}^{\#}$ and $As_{2e}^{\#}$. Note that the height of each element is parallel to the plane of the structure shown in Figure 1. We also define continuous-valued auxiliary variables.

2.3 Objective and Constraints

Our formulation seeks to find cross-section dimensions for each beam and column element that result in the lowest labor and material cost while meeting building code requirements for safety and serviceability for axial forces, bending forces, and constructibility. The constraints include restrictions on (1) the geometry of the cross-section dimensions for each element, (2) relative sizes of cross-section dimensions of elements within the structure, (3) a finite element analysis based on the relative stiffness of the elements in the structure, (4) the resistive capacity of each element, (5) bounds on the applied axial forces and bending moments
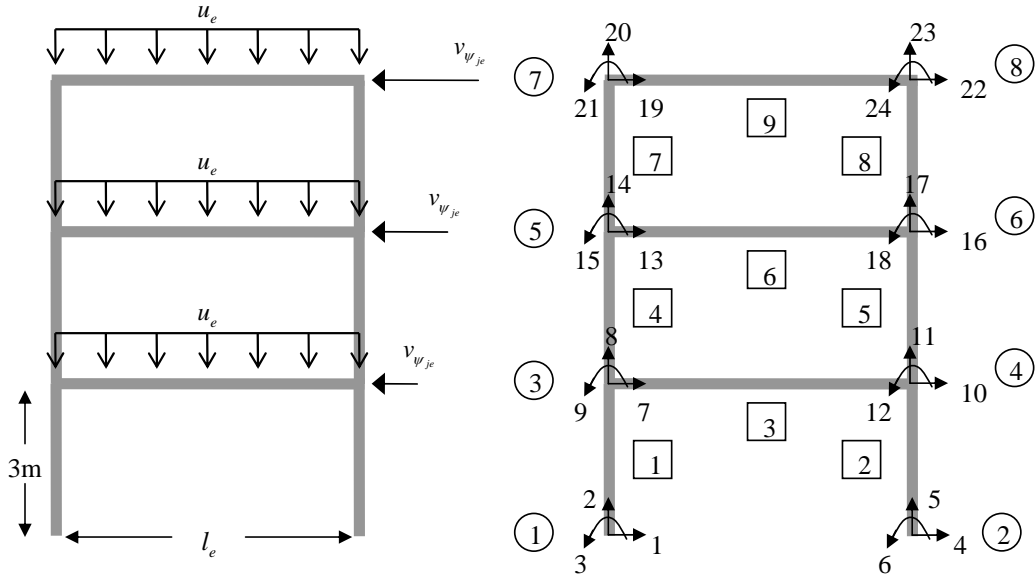
Figure 1: Three-story, one-bay structure with a story height of three meters, beam lengths of five meters and horizontal and vertical loads, including node number labels, element number labels, and degree of freedom number labels. Node numbers are given inside the circles, element numbers are given in the squares, and the degree of freedom numbers are placed next to the arrows.

in relation to the axial and bending moment resistive capacities of each element, and (6) upper and lower bounds on the variables. We enforce the restrictions for the geometry of the cross-section dimensions for each element with linear constraints. The formulation contains nonconvex constraints for the relative sizes of cross-section dimensions of elements within the structure and for the finite element analysis. We model the resistive capacities of each element and bounds on the applied axial forces with complementarity constraints. We restrict the upper and lower bounds with linear constraints.

2.3.1 Finite Element Analysis

The magnitude of axial and bending forces resulting from the applied loads in RC buildings depends on the relative stiffness of elements, which is a function of the width and height of each element. We enforce restrictions for the finite element analysis that ensure equilibrium of forces, stiffnesses, and displacements for the three degrees of freedom at each node: (1) parallel to the beam length, (2) perpendicular to the beam length, and (3) rotational about an axis perpendicular to the plane of the structure. In order to maintain equilibrium at nodes, applied forces that span nodes must be transformed into equivalent nodal forces. For each element in the structure, we define the relative stiffness between the two nodes of each element in matrix form. Based on the degree of freedom labels, we assemble all element stiffness matrices into a global stiffness matrix, which defines the relative stiffnesses between elements with a common node. We assume a rigid, or infinitely stiff, connection between the structure and foundation, which we consider to be fixed degrees of freedom that contain zero displacements. Degrees of freedom one through six in Figure 1 are fixed degrees of freedom. The discretization specifies the structural response with variables that describe axial forces (parallel to the length of an element), shear forces (perpendicular to the length), and bending moments (rotation about an axis perpendicular to the plane of the structure). We design the cross-section dimensions of beam and column elements for a combination of axial and bending forces, but not shear forces

5

because the influence of shear forces is not significant in long, slender elements (Inel and Ozmen, 2006). In the formulation we show all constraints for the finite element analysis but provide only that the relative stiffnesses between elements are functions of the element dimensions. The specific equations for the relative stiffnesses of elements are given in Guerra (2008).

2.3.2 Resistive Capacity of Reinforced Concrete

The capacity to resist applied forces, or resistive capacity, of an RC element is a function of the element width and height as well as the reinforcement areas and strain. Recall that the magnitude of applied forces is also a function of the element width and height. Strain is a unitless variable that describes the change of length relative to the effective length over which the displacements occur. Axial and bending forces in RC buildings result in cross-sections that experience both elongating and shortening strains. We enforce restrictions to develop a linear strain distribution as shown in Figure 2 using variables that describe the location of the neutral axis, c_e, and strain at the location of the compressive and tensile reinforcement, ϵ_e^s and ϵ_e^t, respectively. The neutral axis defines the location of zero strain. Figure 2 illustrates one instance of a linear strain distribution in which compressive strain above the location of the neutral axis is associated with shortening and tensile strain below the neutral axis is associated with elongation. Concrete provides large resistance for compressive strains but cracks when elongated and provides no resistance to tensile strains. Compressive reinforcement resists forces associated with shortening an element, and tensile reinforcement predominately resists forces associated with elongating an element. Tensile reinforcement sometimes provides compressive resistance when the location of the neutral axis is below the location of the tensile reinforcement.

The resistive forces provided by the concrete, compressive reinforcement, and tensile reinforcement are a function of the location of the neutral axis, the strain in the compressive reinforcement, ϵ_e^s, and strain in the tensile reinforcement, ϵ_e^t, respectively. Figure 2 illustrates the resistive forces of the concrete, compressive reinforcement, and tensile reinforcement in relation to the linear strain distribution. The concrete resistive force is a function of the neutral axis reduced by an ACI-specified factor, β_1, 85 percent of the concrete compressive strength, $0.85 \cdot fc'$, and the element width, b_e. For instances in which the location of the neutral axis occurs outside of the concrete cross-section, we use complementarity constraints to enforce the resistive capacity to be a function of the element height rather than the location of the neutral axis. While it is necessary to allow the neutral axis to occur outside the cross-section, appropriate resistive capacity of the concrete includes only cross-section extents.

Resistive forces provided by the compressive and tensile reinforcement are a function of the strain in the reinforcement, the modulus of elasticity of the reinforcement, E_s, and the cross-sectional area of reinforcement. We incorporate elastic-perfectly plastic (i.e., inelastic) reinforcement material behavior using complementarity constraints that enforce a limit on resistive force when strains are greater than the yield strain of the reinforcement.

In RC elements, the locus of combinations of axial and bending forces that results in failure defines the resistive capacity and is termed the interaction diagram in structural engineering. Structural stability is maintained by enforcing that the demand, or applied bending and axial forces, is less than the resistive capacities defined by an interaction diagram. While we enforce structural stability based on ACI code requirements for axial and bending forces, we assume that the cross-section dimensions are not sensitive to connection design between elements or displacements of the structural elements. For structures in Seismic Design Categories A, B, and C, as classified in the ASCE 7 Standard (SEI/ASCE 7-98), these assumptions are acceptable.
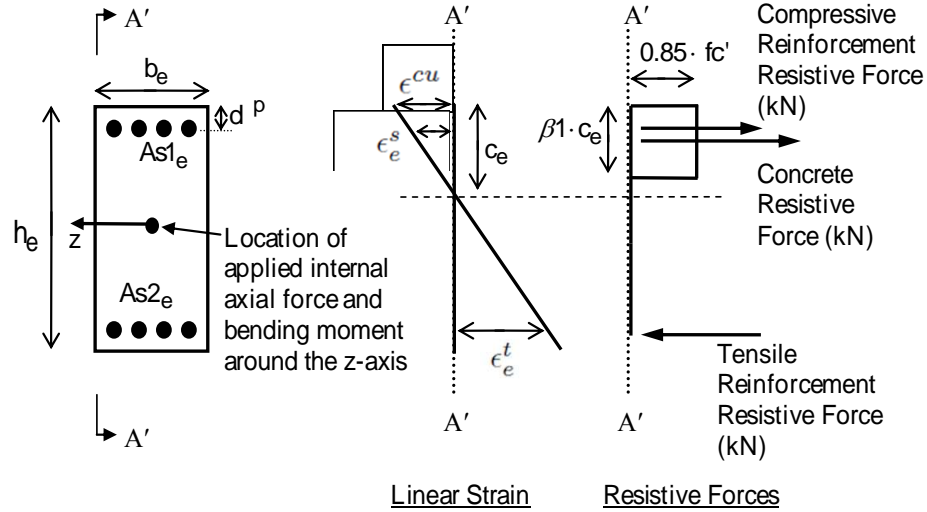
Figure 2: Reinforced concrete cross-section element dimensions, linear strain distribution, and resistive capacities. A' denotes a profile of the cross-section from the left-most figure.

2.3.3 Problem Notation

Our notation largely follows structural engineering notation. We use single, lower-case Roman letters for the index names and single, capital Roman letters with and without superscripts for set names. Parameter names are Greek and Roman letters following structural engineering notation. The notation of the problem and the associated formulation follow.

Sets:

$e \in E$ set of all elements e in the structure

$e \in E^b$ set of all elements e that are beams

$e \in E^c$ set of all elements e that are columns

$e \in E^d$ set of elements e with distributed loads

$e \in E^x$ set of all elements e subject to strength-related constraints

$d \in D_e$ set of global degrees of freedom d for each element e

$d \in D^f$ set of free degrees of freedom d

$d \in D^x$ set of fixed structural degrees of freedom d in each element

$j \in D^{\#}$ set of number of degrees of freedom j in each element (1..6)

$m \in M$ set of reinforcement sizes $= \{\#13, \#16, \#19, \#22\}$

$ACI(b_e, h_e, y_{em}, As_{1e}^{\#}, As_{2e}^{\#})$ set of all elements e that conform to ACI code requirements for reinforcement spacing and percentages

$SYMM(b_e, h_e, As_{1e}, As_{2e})$ set of all elements e that conform to structural geometry symmetry requirements

7

Parameters:

$\underline{b}$ lower bound for the width, b_e, and height, h_e, of all elements [cm]

$\bar{b}$ upper bound for the width, b_e, and height, h_e, of all elements [cm]

d^p distance from the centroid of the compressive and tensile reinforcement to the top and bottom of each element, respectively [cm]

l_e length of structural element e $\forall e \in E$ [m]

l^d development length of reinforcement for all elements [m]

$\underline{x}_e$ near node x-coordinate of element e $\forall e \in E$ [m]

$\underline{As}$ lower bound for the compressive reinforcement, As_{1e}, and tensile reinforcement, As_{2e}, for all elements [cm]

$\overline{As}$ upper bound for the compressive reinforcement, As_{1e}, and tensile reinforcement, As_{2e}, for all elements [cm]

$\underline{As}^{\#}$ lower bound for the number of bars for the compressive reinforcement, $As_{1e}^{\#}$, and tensile reinforcement, $As_{2e}^{\#}$, for all elements

C^C material and placement cost of concrete [$/m^3$]

C^R material and installation cost of reinforcement [\$/kg]

C^F cost to build formwork [\$/square meter contact area]

C^T cost to build and install formwork [\$/square meter contact area]

βa_m area of the mth reinforcing bar size $\forall m \in M = \{1.29,\ 1.99,\ 2.84,\ 3.87\}$ [cm^2]

β_1 concrete strength reduction factor for rectangular stress block [unitless]

fc' compressive strength of concrete [MPa]

fy yield stress of reinforcement [MPa]

Es modulus of elasticity of reinforcement [MPa]

ϵ^{cu} crushing strain of concrete [unitless]

ρs density of reinforcement [kg/m^3]

v_{je} equivalent horizontal earthquake load applied to the jth degree of freedom for element e $\forall e \in E^d$ [kN and kN-m]

Decision Variables:

Integer Variables

$As_{1e}^{\#}$ number of compressive reinforcing bars in element e $\forall e \in E$ [unitless]

$As_{2e}^{\#}$ number of tensile reinforcing bars in element e $\forall e \in E$ [unitless]

<u>Discrete Variables</u>

b_e width of element e $\forall e \in E$ [cm]

h_e height of element e $\forall e \in E$ [cm]

As_{1e} total compressive area of reinforcement in element e $\forall e \in E$ [cm^2]

As_{2e} total tensile area of reinforcement in element e $\forall e \in E$ [cm^2]

<u>Binary Variables</u>

y_{em} $= 1$ if the mth rebar size is used for element e $\forall e \in E^x$, $\forall m \in M$, 0 otherwise

w_e $= 1$ if the eth element formwork is already built $\forall e \in E^x$, 0 otherwise

$z_{ee'}^c$ $= 1$ if column element e is the same size as column element e' $\forall e, e' \in E^x \cap E^c \ni e \leq e'$, 0 otherwise

$z_{ee'}^b$ $= 1$ if beam element e is the same size as beam element e' $\forall e, e' \in E^x \cap E^b \ni e \leq e'$, 0 otherwise

<u>Auxiliary Variables</u>

c_e distance from the most compressive concrete fiber to the neutral axis for element e $\forall e \in E^x$ [cm]

$\hat{c}_e$ reduced distance from the most compressive concrete fiber to incorporate appropriate concrete resistive strength of element e $\forall e \in E^x$ [cm]

f_{ed} applied axial force, shear force, and bending moments at the dth degree of freedom for element e $\forall e \in E, \forall d \in D_e$ [kN and kN/m]

$k_{edd'}$ stiffness between degrees of freedom d and d' for element e $\forall e \in E, \forall d, d' \in D_e$ [kN and kN/m]

$K_{dd'}$ stiffness between the free degrees of freedom d and d' $\forall d, d' \in D^f$ [kN and kN/m]

q_{ed} equivalent nodal load for the dth degree of freedom for element e with distributed loads $\forall e \in E^d, \forall d \in D_e$ [kN and kN-m]

Q_d nodal load for the dth free degree of freedom $\forall d \in D^f$ [kN and kN-m]

s_e^+ positive slack variable for determining strain in the tensile reinforcement in element e $\forall e \in E$ [unitless]

s_e^- negative slack variable for determining strain in the tensile reinforcement in element e $\forall e \in E$ [unitless]

T_e slack variable for minimum applied axial force requirements in element e $\forall e \in E$ [MPa]

Δ_d nodal displacement for the dth free global degree of freedom $\forall d \in D^f$ [m]

δ_{ed} nodal displacement at the dth global degree of freedom for element e $\forall e \in E$ $\forall d \in D_e$ [m]

ϕ_e ACI strength reduction factor for element e $\forall e \in E^x$ [unitless]

ϵ_e^s strain in the compressive reinforcement for element e $\forall e \in E^x$ [unitless]

ϵ_e^t strain in the tensile reinforcement for element e $\forall e \in E^x$ [unitless]

$\hat{\epsilon}_e^s$ strain in compressive reinforcement of element e that remains within yield strain limits $\forall e \in E^x$ [unitless]

$\hat{\epsilon}_e^t$ strain in tensile reinforcement of element e that remains within yield strain limits $\forall e \in E^x$ [unitless]

$\hat{\epsilon}_e^\phi$ strain in tensile reinforcement of element e to formulate the strength reduction factor, ϕ_e [unitless]

u_e^d factored distributed load including the self weight of element e $\forall e \in E^d$ [kN/m]

$\breve{x}_e$ location of the plastic centroid from the most compressive concrete fiber $\forall e \in E$ [cm]

We now state the full model before commenting on some of its aspects in detail.

Problem Statement:

$$(C): \quad \min \quad \sum_{e \in E^b} l_e (C^T - C^F w_e)(\frac{b_e + 2h_e}{100}) + \sum_{e \in E^c} l_e (C^T - C^F w_e)(\frac{2b_e + 2h_e}{100}) + \sum_{e \in E} l_e \cdot C^C (\frac{b_e h_e}{100^2}) +$$

$$\sum_{e \in E^b} \rho s \cdot C^R (\frac{As_{1e}}{100^2}(l_e - l^d) + \frac{As_{2e}}{100^2}(l_e + l^d)) + \sum_{e \in E^c} \rho s \cdot C^R (\frac{As_{1e} + As_{2e}}{100^2})l_e$$

subject to
(see §2.4.2)

$$\underline{b} \leq b_e \leq \bar{b} \qquad \forall e \in E \tag{1a}$$

$$\underline{b} \leq h_e \leq \bar{b} \qquad \forall e \in E \tag{1b}$$

$$\underline{As} \leq As_{1e} \leq \overline{As} \qquad \forall e \in E \tag{1c}$$

$$\underline{As} \leq As_{2e} \leq \overline{As} \qquad \forall e \in E \tag{1d}$$

$$\underline{As}^{\#} \leq As_{1e}^{\#} \quad \forall e \in E^x \tag{1e}$$

$$\underline{As}^{\#} \leq As_{2e}^{\#} \quad \forall e \in E^x \tag{1f}$$

(see §2.4.3)

$$b_e \quad \leq \quad h_e \qquad\qquad \forall e \in E^x \tag{2a}$$

$$h_e \quad \leq \quad 5 \cdot b_e \qquad\qquad \forall e \in E^x \tag{2b}$$

$$As_{1e}^{\#} \quad \leq \quad As_{2e}^{\#} \qquad\qquad \forall e \in E^x \tag{2c}$$

$$(b_e, h_e, As_{1e}, As_{2e}) \quad \in \quad SYMM \qquad\qquad \forall e \in E \tag{2d}$$

$$\sum_{m \in M} y_{em} \quad = \quad 1 \qquad\qquad \forall e \in E^x \tag{2e}$$

$$As_{1e} \quad = \quad As_{1e}^{\#} \sum_{m \in M} \beta a_m \cdot y_{em} \quad \forall e \in E^x \tag{2f}$$

$$As_{2e} \quad = \quad As_{2e}^{\#} \sum_{m \in M} \beta a_m \cdot y_{em} \quad \forall e \in E^x \tag{2g}$$

$$(b_e, h_e, y_{em}, As_{1e}^{\#}, As_{2e}^{\#}) \quad \in \quad ACI \qquad\qquad \forall e \in E \tag{2h}$$

(see §2.4.4)

$$(b_e - b_{e'}) \cdot z_{ee'}^c \quad = \quad 0 \qquad \forall e, e' \in E^x \cap E^c \ni e' > e, \underline{x}_e = \underline{x}_{e'} \tag{3a}$$

$$(h_e - h_{e'}) \cdot z_{ee'}^c \quad = \quad 0 \qquad \forall e, e' \in E^x \cap E^c \ni e' > e, \underline{x}_e = \underline{x}_{e'} \tag{3b}$$

$$(b_e - b_{e'}) \cdot z_{ee'}^b \quad = \quad 0 \qquad \forall e, e' \in E^x \cap E^b \ni e' > e, \underline{x}_e = \underline{x}_{e'} \tag{3c}$$

$$(h_e - h_{e'}) \cdot z_{ee'}^b \quad = \quad 0 \qquad \forall e, e' \in E^x \cap E^b \ni e' > e, \underline{x}_e = \underline{x}_{e'} \tag{3d}$$

$$\sum_{e \in E^x \cap E^c} z_{ee'}^c \quad \geq \quad w_{e'} \quad \forall e' \in E^x \cap E^c \ni e' > e, \underline{x}_e = \underline{x}_{e'} \tag{3e}$$

$$\sum_{e \in E^x \cap E^b} z_{ee'}^b \quad \geq \quad w_{e'} \quad \forall e' \in E^x \cap E^b \ni e' > e, \underline{x}_e = \underline{x}_{e'} \tag{3f}$$

(see §2.4.5)

$$Q_d \quad = \quad \sum_{e \in E^d \ni d \in D_e} q_{ed} \qquad \forall d \in D^f \tag{4a}$$

$$k_{e\psi_{je}\psi_{je}} \quad = \quad f(b_e, h_e) \qquad \forall e \in E, \; j \in D^{\#} \tag{4b}$$

$$K_{dd'} \quad = \quad \sum_{e \in E \ni d, d' \in D_e} k_{edd'} \qquad \forall d, d' \in D^f \tag{4c}$$

$$\sum_{d' \in D^f} K_{dd'} \Delta_{d'} \quad = \quad Q_d \qquad\qquad \forall d \in D^f \tag{4d}$$

$$\delta_{ed} \quad = \quad \Delta_d \qquad\qquad \forall e \in E, \; d \in D^f \ni d \in D_e \tag{4e}$$

$$\delta_{ed} \quad = \quad 0 \qquad\qquad \forall e \in E, \; d \in D^x \ni d \in D_e \tag{4f}$$

$$\sum_{d' \in D_e} k_{edd'} \delta_{ed'} - q_{ed} \quad = \quad f_{ed} \qquad\qquad \forall e \in E, \; d \in D_e \tag{4g}$$

(see §2.4.6)

$$\epsilon^{cu}(c_e - d^p) = \hat{\epsilon}_e^s c_e \qquad \forall e \in E \qquad (5a)$$

$$\epsilon^{cu}(h_e - d^p - c_e) = \hat{\epsilon}_e^t c_e \qquad \forall e \in E \qquad (5b)$$

$$\hat{c}_e \leq \beta_1 \cdot c_e \quad \perp \quad \hat{c}_e \leq h_e \qquad \forall e \in E \qquad (5c)$$

$$c_e \geq d^p \qquad \forall e \in E \qquad (5d)$$

$$\hat{c}_e \geq \beta_1 \cdot d^p \qquad \forall e \in E \qquad (5e)$$

$$\hat{\epsilon}_e^s \leq fy/Es \quad \perp \quad \hat{\epsilon}_e^s \leq \epsilon_e^s \qquad \forall e \in E \qquad (5f)$$

$$\hat{\epsilon}_e^s \geq 0 \qquad \forall e \in E \qquad (5g)$$

$$s_e^+ - s_e^- = \hat{\epsilon}_e^t - \epsilon_e^t \qquad \forall e \in E \qquad (5h)$$

$$-fy/Es \leq \hat{\epsilon}^t \quad \perp \quad s_e^+ \geq 0 \qquad \forall e \in E \qquad (5i)$$

$$\hat{\epsilon}^t \leq fy/Es \quad \perp \quad s_e^- \geq 0 \qquad \forall e \in E \qquad (5j)$$

$$\hat{\epsilon}_e^\phi \geq \epsilon_e^t \quad \perp \quad \hat{\epsilon}_e^\phi \geq 0.002 \qquad \forall e \in E \qquad (5k)$$

$$\phi_e \leq 0.90 \quad \perp \quad \phi_e \leq 0.65 + (250/3)(\hat{\epsilon}_e^\phi - 0.002) \qquad \forall e \in E \qquad (5l)$$

(see §2.4.7)

$$f_{ed} - 0.1 \cdot fc' \cdot b_e \cdot h_e/100^2 \geq -T_e \quad \forall e \in E^b \ni \mathrm{ord}(d) = 1 \qquad (6a)$$

$$f_{ed} - 0.1 \cdot fc' \cdot b_e \cdot h_e/100^2 \geq -T_e \quad \forall e \in E^c \ni \mathrm{ord}(d) = 2 \qquad (6b)$$

$$T_e \geq 0 \qquad \forall e \in E \qquad (6c)$$

$$(0.004 - \epsilon_e^t)T_e \leq 0 \qquad \forall e \in E \qquad (6d)$$

(see §2.4.7)

$$f_{ed} = (0.85 \cdot fc' \cdot \hat{c}_e \cdot \tfrac{b_e}{100^2} + \hat{\epsilon}_e^s \cdot Es \cdot \tfrac{As_{1e}}{100^2} - \hat{\epsilon}_e^t \cdot Es \cdot \tfrac{As_{2e}}{100^2}) \cdot \phi_e \quad \forall e \in E^x \cap E^b \ni \mathrm{ord}(d) = 1 \qquad (7a)$$

$$f_{ed} = (0.85 \cdot fc' \cdot \hat{c}_e \cdot \tfrac{b_e}{100^2} + \hat{\epsilon}_e^s \cdot Es \cdot \tfrac{As_{1e}}{100^2} - \hat{\epsilon}_e^t \cdot Es \cdot \tfrac{As_{2e}}{100^2}) \cdot \phi_e \quad \forall e \in E^x \cap E^c \ni \mathrm{ord}(d) = 2 \qquad (7b)$$

$$\breve{x}_e = h_e/2 \qquad \forall e \in E \qquad (7c)$$

$$\max(-f_{ed}, f_{ed'}) \leq ((0.85 \cdot fc' \cdot \hat{c}_e \cdot \tfrac{b_e}{100^2}) \cdot \tfrac{(\breve{x}_e - \hat{c}_e)}{100} +$$
$$(\hat{\epsilon}_e^s \cdot Es \cdot \tfrac{As_{1e}}{100^2}) \cdot \tfrac{(\breve{x}_e - d^p)}{100} - (\hat{\epsilon}_e^t \cdot Es \cdot \tfrac{As_{2e}}{100^2}) \cdot \tfrac{(h_e - d^p - \breve{x}_e)}{100}) \cdot \phi_e \quad \forall e \in E^x \cap E^b \ni \mathrm{ord}(d,d') = 6, 3 \qquad (7d)$$

$$-((u_e^d \cdot l_e^2)/8 - (f_{ed} - f_{ed'})/2 + \tfrac{(f_{ed} + f_{ed'})^2}{2 \cdot u_e^d \cdot l_e^2})$$
$$\leq ((0.85 \cdot fc' \cdot \hat{c}_e \cdot \tfrac{b_e}{100^2}) \cdot \tfrac{(\breve{x}_e - \hat{c}_e)}{100} + (\hat{\epsilon}_e^s \cdot Es \cdot \tfrac{As_{1e}}{100^2}) \cdot \tfrac{(\breve{x}_e - d^p)}{100}$$
$$-(\hat{\epsilon}_e^t \cdot Es \cdot \tfrac{As_{2e}}{100^2}) \cdot \tfrac{(h_e - d^p - \breve{x}_e)}{100}) \cdot \phi_e \qquad \forall e \in E^x \cap E^b \ni \mathrm{ord}(d,d') = 6, 3 \qquad (7e)$$

$$\pm f_{ed} \leq ((0.85 \cdot fc' \cdot \hat{c}_e \cdot \tfrac{b_e}{100^2}) \cdot \tfrac{(\breve{x}_e - \hat{c}_e)}{100} +$$
$$(\hat{\epsilon}_e^s \cdot Es \cdot \tfrac{As_{1e}}{100^2}) \cdot \tfrac{(\breve{x}_e - d^p)}{100} - (\hat{\epsilon}_e^t \cdot Es \cdot \tfrac{As_{2e}}{100^2}) \cdot \tfrac{(h_e - d^p - \breve{x}_e)}{100}) \cdot \phi_e \quad \forall e \in E^x \cap E^c, \forall d = 3, 6 \qquad (7f)$$

$$f_{ed} \leq 0.80 \cdot \phi_e \cdot 0.85 \cdot fc' \cdot \tfrac{(b_e \cdot h_e - As_e^1 - As_e^2)}{100^2} + fy \cdot \tfrac{(As_e^1 + As_e^2)}{100^2} \quad \forall e \in E^x \cap E^c \ni \mathrm{ord}(d) = 2. \qquad (7g)$$

2.4 Problem Discussion

Next, we describe the objective function and constraints in detail.

2.4.1 Objective Function

The objective function includes, in order of appearance, formwork cost for all elements, concrete cost for all elements, and reinforcement cost for all elements. We separate formwork costs into terms for beams and columns because columns require formwork on all four sides whereas beams require formwork on only three sides. Similarly, we develop reinforcement costs with terms for beams and columns because the length of reinforcement relative to the element length is different for beams and columns. We utilize binary variables, w_e, in the objective function, as explained in Section 2.4.4, to develop a cost model for formwork reuse that follows the estimating methods of RC construction companies.

2.4.2 Upper and Lower Variable Bounds

Constraints (1a) through (1d) define lower and upper bounds on the variables that describe the cross-section dimensions of each element e. Constraints (1e) and (1f) incorporate lower bounds for the number of bars for the compressive and tensile reinforcement, $As_{1e}^{\#}$ and $As_{2e}^{\#}$.

2.4.3 Geometry Restrictions

While constraint (2a) ensures that the width is less than the height, constraint (2b) prevents the creation of tall, slender elements and maintains dimensions corresponding to typical behavior for beams and columns. Constraint (2c) ensures that the area of tensile reinforcement is always greater than the area of compressive reinforcement to maintain flexible elements. Constraint (2d) restricts all symmetric elements with respect to horizontal location to contain the same width, height, and reinforcement area. Constraints (2e) through (2g) define a special ordered set for selecting a discrete reinforcement bar size from the set of $|M|$ sizes, where the parameter βa_m is the cross-sectional area for the mth bar size. Constraint (2h) enforces reinforcing bar spacing and percentage requirements based on ACI code specifications.

2.4.4 Formwork Reuse

Nonlinear equality constraints (3a) and (3b) restrict the formwork reuse binary variables for columns, $z_{ee'}^c$, to equal 1 if and only if the column element e has the same width and height as a column element, e', on a higher story level, indicated by a larger element number, with the same horizontal ordinate ($\underline{x}_e$). The same convention follows for beam elements in constraints (3c) and (3d). When columns or beams, respectively, on a higher story level contain the same width and height as columns or beams on a lower story level, formwork costs are substantially lower because formwork is reused. While constraints (3a) through (3d) compare the dimensions of elements on each story level, constraints (3e) and (3f) define the binary variable, w_e, in terms of the values of the binary variables $z_{ee'}^c$ and $z_{ee'}^b$. Essentially, constraints (3e) and (3f) allow w_e to equal 1 if a beam or column element, respectively, on a higher story level contains the same dimensions as a beam or column on any lower story level. Note that we force columns that are symmetrically located about the horizontal midpoint of the structure to have equal cross-section width and height.

2.4.5 Finite Element Analysis

Constraints (4a) through (4g) define the relations between stiffness, applied forces, and displacements for finite element analysis with beam and column elements. Constraint (4a) defines the equivalent nodal loads for all elements in the structure based on equivalent nodal loads for each element. Equation (4b) defines the element stiffnesses as a function of element widths and heights. Element stiffness is a highly nonlinear function of the element widths and depths. Constraint (4c) assembles the global stiffnesses at each degree of freedom

d and d' from element stiffnesses, $k_{edd'}$. Constraint (4d) defines equilibrium of forces, displacements, and stiffnesses for the finite element analysis. Constraint (4e) defines the nodal displacements for each element by extracting the appropriate values from the displacement vector, Δ_d. Constraint (4f) sets the displacements at each fixed degree of freedom to zero. Constraint (4g) defines internal forces, f_{ed}, at the dth degree of freedom for element e. The internal forces include the shear, axial, and bending moments at the two nodes in each beam and column element. Further details about the finite element analysis can be found in Guerra (2008).

2.4.6 Resistive Forces: Complementarity Constraints

While the finite element analysis provides demand in terms of bending and axial forces that must be resisted in each element, constraints (5a) through (5l) express the capacity in terms of bending and axial forces that each element can resist. In order to meet structural stability, which is discussed in Section 2.4.7, the resistive capacity must be greater than or equal to the demand in each element. Constraints (5a) and (5b) define a linear strain distribution across the height of each element in order to determine the resistive capacity of the reinforced cross-section. Complementarity constraints (5c) through (5e) ensure appropriate concrete compressive resistance when the entire cross-section contains compressive strains and when only a portion of the cross-section contains compressive strains. (See the left hand side of Figure 3.) Complementarity constraints (5f) and (5g) define elastic-perfectly plastic material response for the resistive capacity of the compressive reinforcement when the strain in the compressive reinforcement is less than and greater than the yield strain. Elastic-perfectly plastic material response ensures that the maximum resistive capacity corresponds to the yield strain. (See the first quadrant of the right hand side of Figure 3.) Complementarity constraints (5h) through (5j) define elastic-perfectly plastic material response for the resistive capacity of the tensile reinforcement using positive and negative slack variables, s_e^+ and s_e^-, respectively. (See Figure 3 in its entirety.) Complementarity constraints (5k) and (5l) define the appropriate value for the strength reduction factor according to ACI code requirements as a function of the strain in the tensile reinforcement. The strength reduction factor incorporates factors of safety into the design. (See the left hand side of Figure 4.)

2.4.7 Restrictions for Structural Stability

In general, in order to maintain structural stability, the capacity of each element must be greater than the demand. Constraints (6a) through (6d) enforce ACI code requirements for minimum eccentricity of applied forces using conditional complementarity depending on the value of a slack variable, T_e. For situations in which the strain in the tensile reinforcement is greater than or equal to 0.004 (an ACI-specified value), there is sufficient bending in the element, the value of the slack variable must only be greater than or equal to zero, conditional complementarity is not invoked, and requirements for minimum eccentricity do not apply. For situations in which the strain in the tensile reinforcement is less than 0.004, the value of T_e must equal zero, and the applied axial force must be greater than 10 percent of the maximum axial concrete resistance. (See the right hand side of Figure 4.)

Constraints (7a) and (7b) restrict the resistive axial force of the cross-section to be equal to the applied axial force in each beam and column element, respectively. Because the bending resistive force can increase with increasing applied axial forces, equality of the resistive and applied forces ensures that the magnitude of the resistive bending force corresponds to the appropriate applied axial force. Constraint (7c) defines the location of the plastic centroid from the most compressive concrete fiber, which represents the point about which the bending resistance is computed. Constraints (7d) and (7e) enforce that the corresponding
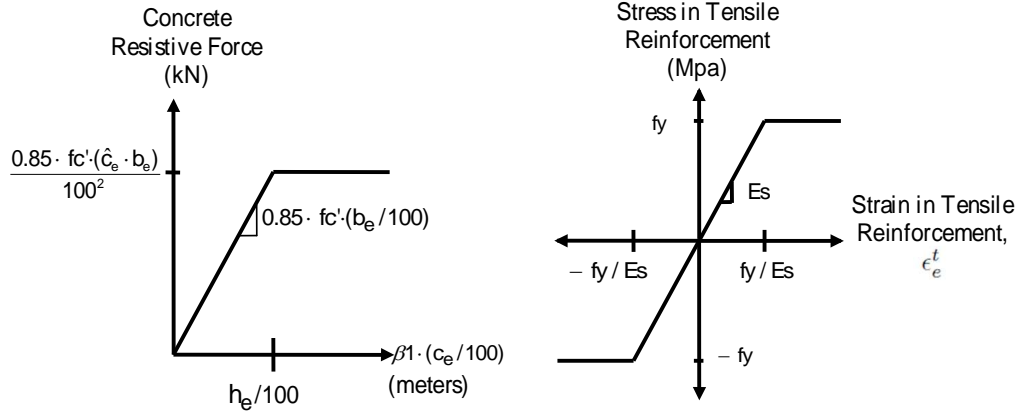
Figure 3: The left figure illustrates the relationship between the concrete resistive force (y-axis) and the location of the neutral axis reduced by the strength reduction factor (x-axis), which defines the amount of concrete that contains compressive strains; see constraints (5c–5e). When the entire cross-section contains compressive strains, the amount of concrete in compression is equal to $h_e/100$; and when only a portion of the cross-section contains compressive strains, the amount of concrete in compression is equal to $\beta 1 \cdot (c_e/100)$. The right figure illustrates the relationship between the stress in the tensile reinforcement(y-axis) and the strain in the tensile reinforcement(x-axis), (5f) and (5g), when the stresses and strains lie in the first quadrant, and illustrates constraints (5h) through (5j) when the stresses and strains lie in the first and third quadrants, respectively.
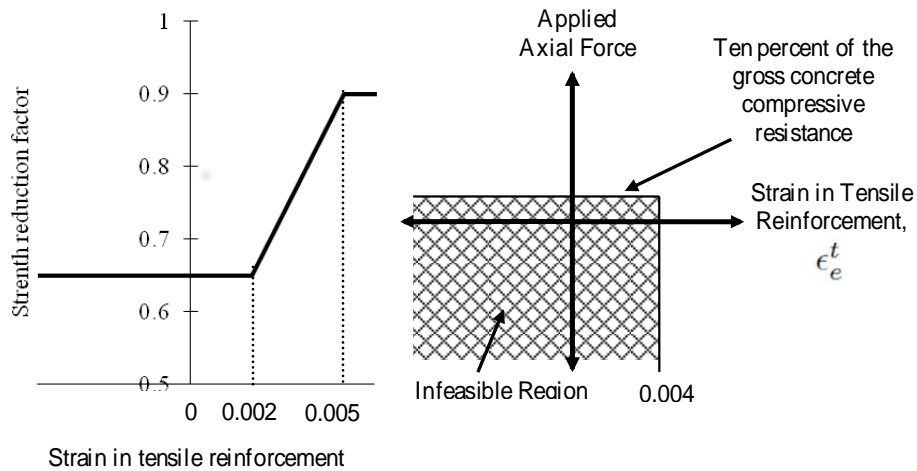

Figure 4: The left figure illustrates the relationship between strain in the tensile reinforcement and the strength reduction factor, see constraints (5k) and (5l). Note the code-specified transition points of 0.002 and 0.005 in the strain of the tensile reinforcement. The right figure illustrates the relationship between the strain in the tensile reinforcement and the applied axial force, see constraints (6a–6d).

applied bending forces at the ends and in the span of the beams, respectively, are less than or equal to the resistive bending force. Constraint (7f) enforces that the applied bending forces at both ends of columns are less than or equal to the resistive bending force. Constraint (7g) ensures that the applied axial force in column elements is less than a ACI-prescribed maximum axial load, which is a function of the strength of the cross-section. Equality of resistive and applied axial forces, (7a) and (7b), and inequality of resistive

and applied bending forces, (7d–7f), ensure that the applied axial and bending forces fall within the locus of failure for any feasible combination of axial and bending forces. Constraint (7g) enforces an upper bound on the applied axial force.

2.4.8 Fixed Case of Formwork Reuse Constraints

We can reduce the combinatorial complexity of the problem by adding logical constraints that force a particular formwork reuse. We do so by fixing the binary variables $z_{ee'}^c$ and $z_{ee'}^b$ to a value of 0 or 1 to force a particular formwork reuse distribution. We include formwork distributions such that all columns contain the same dimensions and all beams contain the same dimensions by forcing: $z_{ee'}^c = 1 \ \forall e, e' \in E^x \cap E^c \ni e' > e, \underline{x}_e = \underline{x}_{e'}$ and $z_{ee'}^b = 1 \ \forall e, e' \in E^x \cap E^b \ni e' > e, \underline{x}_e = \underline{x}_{e'}$. We also include formwork distributions in which subsets of beams and columns contain the same dimensions, by forcing a subset of the binary variable values to 1 and the remaining values to 0. For example, for eight stories, we force two to be one size and six to be another, three to be one size and five to be another; for three form sizes we force, for example, two stories to be one size, two to be another, and four to be yet another. Forcing values for these binary variables eliminates the associated nonlinear equality and inequality constraints (3a–3f) in the formulation. In Section 4, we compare the algorithm performance for the fixed case of the formulation, which contains additional restrictions, and the free case of the formulation, which does not contain additional restrictions on the binary variables.

2.4.9 Potential Convex Reformulations

Rather than using nonlinear constraints as shown in (3a–3f), we also examine the effect on model tractability of a convex hull formulation for formwork reuse. Unfortunately, this reformulation does not result in faster solve times; we give this reformulation in the appendix.

In addition, we study a convex reformulation of the nonlinear equality constraints (2f) and (2g) by replacing them with the following linear constraints:

$$-2 \cdot \overline{As} \cdot (1 - y_{em}) \leq As_{1e} - \beta a_m \cdot As_{1e}^{\#} \leq 2 \cdot \overline{As} \cdot (1 - y_{em}) \quad \forall e \in E^x, \forall m \in M \tag{8a}$$

$$-2 \cdot \overline{As} \cdot (1 - y_{em}) \leq As_{2e} - \beta a_m \cdot As_{2e}^{\#} \leq 2 \cdot \overline{As} \cdot (1 - y_{em}) \quad \forall e \in E^x, \forall m \in M \tag{8b}$$

which is a big-M formulation.

3 Model Instances

We include two-dimensional design examples for one-bay structures with one- to eight-story levels. We use beam lengths of five meters, typical for RC structures, and a load case with both horizontal and vertical loads to demonstrate optimal solutions for combinations of applied axial force and bending moment magnitudes that cover a wide range of values typically observed in RC elements.

3.1 Parameter Values

The parameters in this study contain common values used in the design of RC structures. Table 1 presents the material properties used for all examples. All structures are loaded by their self weight, w^G, an additional gravity dead load, $w^D = 30kN/m$, and a gravity live load, $w^L = 30kN/m$, which are typical values for office buildings. We calculate the parameters that describe horizontal seismic forces, v_{je}, for the jth degree of

freedom for element e using the ASCE 7 (SEI/ASCE 7-98) equivalent lateral load procedure for a structure in Denver, Colorado, the failure of which would result in a substantial public hazard. Buildings subjected to gravity and seismic loads contain factored loads of $1.2w^G+1.2w^D+1.0w^L+1.0v_{je} = 1.2w^G+66kN/m+1.0v_{je}$. Table 2 summarizes the values of the seismic horizontal forces for the multistory design examples with beam lengths of five meters. For all examples, the horizontal loads increase for each story level so that the top story level contains the largest horizontal load.

Table 1: Material properties for concrete and reinforcement

Description	Parameter Name	Value	Units
Lower bound for the width and depth of elements	$\underline{b}$	20	cm
Upper bound for the width and depth of elements	$\bar{\bar{b}}$	200	cm
Concrete cover	d^p	7	cm
Length of structural element e	l_e	3-10	m
Development length of reinforcement	l^d	1.11	m
Near node x-coordinate of element e	$\underline{x}_e$	0-20	m
Lower bound for reinforcement	$\underline{As}$	2.58	cm
Upper bound for reinforcement	$\overline{As}$	2.58	cm
Lower bound for the number of reinforcing bars	$\underline{As}^{\#}$	2	unitless
Material and placement unit price of concrete	C^C	192.80	$\$/m^3$
Material and installation unit price of reinforcement	C^R	1.55	$\$/kg$
Unit price to build formwork	C^F	32.60	$\$/SMCA$
Unit price to build and install formwork	C^T	38.00	$\$/SMCA$
ACI Section 10.2.7.3 reduction factor	β_1	0.85	unitless
Concrete compressive strength	fc'	28	MPa
Reinforcement yield stress	fy	420	MPa
Reinforcement modulus of elasticity	Es	200,000	MPa
Crushing strain of concrete	ϵ^{cu}	0.003	unitless
Density of reinforcement	ρs	7870	kg/m^3

Table 2: Horizontal loads for spans of five meters

Total Number of Story Levels	Horizontal Load on Story Level:							
	One (kN)	Two (kN)	Three (kN)	Four (kN)	Five (kN)	Six (kN)	Seven (kN)	Eight (kN)
1	15.6	–	–	–	–	–	–	–
2	10.4	20.8	–	–	–	–	–	–
3	7.8	15.6	23.4	–	–	–	–	–
4	6.3	12.5	18.8	25.0	–	–	–	–
5	4.8	9.9	14.9	20.0	25.1	–	–	–
6	3.3	6.9	10.7	14.5	18.4	22.3	–	–
7	2.3	5.0	7.9	10.9	13.2	17.0	20.2	–
8	1.9	3.8	6.2	8.6	11.1	13.6	16.3	19.0

3.2 Initialization of Variables

Initial variable values provide a starting point for the algorithm and strongly influence the performance. We set an initial value for all variables in the optimization formulation based on a typical design practice that

meets all the constraints in the problem formulation. This "typical solution" provides a feasible starting point and can be compared with solutions from MINLPBB to determine cost savings potential for the structures we consider. We develop the typical solution with a method often used in practice: (1) approximate an initial width and depth of beam and column elements using applied loads and beams lengths, (2) determine the applied forces and moments on each element using finite element analysis, (3) determine the number and size of reinforcing bars to resist applied forces and moments, and (4) determine whether the number and size of reinforcing bars fits within the cross-section with required spacing and concrete cover. If the reinforcement does not fit, then element widths and/or depths are increased in five-centimeter increments and we repeat steps 2 through 4. While we could use various initial variable values and compare the corresponding objective function values to find the best locally optimal solution, we use only the initial variable values from the typical solution.

3.3   Problem Size and Structure

The mathematical structure of the RC design problem is highly nonlinear and nonconvex. Discrete-valued variables describe constructible design solutions and continuous-valued variables model material response. Nonlinear equality and complementarity constraints make the problem nonconvex. We present information about the size of the problems in terms of the number of story levels, number of variables (including integer variables), number of constraints (including nonlinear constraints) and number of complementarity constraint pairs for the fixed and free cases of our model runs (see Section 2.4.8). We solve problems with up to 678 variables, of which 214 are integer, 844 constraints, of which 582 are nonlinear, and with 120 complementarity constraint pairs (see Table 3).

Table 3: Problem size for the one- through eight-story, one-bay structures with beam lengths of five meters and vertical and horizontal loads when the binary variables for formwork reuse are free and fixed

Number of Story Levels	Number of Variables (Integer)		Number of Constraints (Nonlinear)		Number of Complementarity Constraint Pairs
	Fixed	Free	Fixed	Free	Fixed and Free
1	76 (18)	–	95 (64)	–	15
2	152 (36)	156 (40)	188 (122)	190 (126)	30
3	228 (54)	238 (64)	285 (180)	289 (192)	45
4	304 (72)	322 (90)	386 (238)	392 (262)	60
5	380 (90)	408 (118)	491 (296)	499 (336)	75
6	456 (108)	496 (148)	600 (354)	610 (414)	90
7	532 (126)	586 (180)	713 (412)	725 (496)	105
8	608 (144)	678 (214)	830 (470)	844 (582)	120

3.4   Choice of MINLP Algorithm

A range of different MINLP solution techniques exists, and we briefly comment on their suitability for our problem instances. Outer approximation (Duran and Grossman, 1986) and generalized Benders decomposition (Geoffrion, 1972) solve a sequence of MILP master problem and NLP subproblems. The extended cutting plane method (Westerlund and Pettersson, 1995) solves only a sequence of MILP problems. We considered BARON (Tawarmalani and Sahinidis, 2002), which can solve nonconvex MINLPs to global optimality using global underestimators and branch-and-reduce techniques. However, the size and complexity of

the nonlinear expressions in the finite element analysis make it unlikely that BARON can succeed in solving our problem (Sahinidis, 2006). All four methods rely heavily on the convexity of the problem functions or the ready availability of underestimators. Unfortunately, our models do not easily admit underestimators, and we cannot apply these techniques. We also considered using NOMADm (Abramson, 2005, 2006), a pattern search technique that does not require derivative information (Torczon, 1997).

An alternative to these techniques is nonlinear branch-and-bound. This method solves a sequence of nonlinear problems at every node of a tree. One advantage of this approach is that nonlinear solvers often find good solutions even to nonconvex problems. The MINLPBB solver is attractive because the underlying SQP method has been shown to be an efficient and robust solver for optimization problems with complementarity constraints (Fletcher and Leyffer, 2004).

We solve the RC design problem with a software package, MINLPBB (Leyffer 1999), on Linux-based servers with dual-core AMD Opteron processors with a CPU speed of 2.41 GHz and random access memory of approximately one gigabyte. MINLPBB uses a branch-and-bound framework with a sequential quadratic programming technique to solve the continuous relaxations of the problem (Fletcher and Leyffer, 1998; Leyffer, 1999; Leyffer, 2001). The branch and bound method in MINLPBB uses a depth-first search and solves the NLP relaxations using filterSQP (Fletcher and Leyffer, 1998). The user can influence branching by supplying priorities for integer variables, as well as by choosing node selection strategies and branching rules (Leyffer, 1999). We use a depth-first search strategy, branching on the variable with the highest user-defined priority. MINLPBB guarantees global optimality only for convex problems. However, it also provides a more robust solution technique for nonconvex MINLP problems in comparison to outer approximation and Benders decomposition (Leyffer, 1999). Outer approximation and Benders decomposition often reduce the size of the feasible region by using cuts, which can eliminate optimal solutions in nonconvex problems like our RC design problem. Fletcher and Leyffer (2004) have demonstrated the abilities of the NLP solver in MINLPBB, filterSQP, to solve over 150 optimization problems with complementarity constraints.

4 Numerical Results

We present design examples to illustrate three key aspects: (1) MINLPBB algorithm performance for the RC design problem, (2) cost savings over typical practice, and (3) characteristics of the optimal solution, namely, optimal formwork reuse and stiffness distributions. We detail these aspects and the corresponding results in the following three subsections.

4.1 Algorithm Performance

Recall that we solve two cases of the optimization formulation: the original formulation that allows solutions with any formwork reuse distribution, termed the free case, and the formulation that contains restrictions that enforce a particular formwork reuse distribution, the fixed case. While we find solutions for up to an eight-story, one-bay structure for the original problem formulation, we can find solutions in less CPU time when a particular formwork reuse distribution is enforced. Figure 5 illustrates on a logarithmic scale the CPU time as a function of the number of integer variables for both the fixed and free cases.

Table 4 summarizes the number of nonlinear programs solved and the algorithm performance (in terms of solution time) for the the fixed and free cases for one-bay structures with up to eight story levels. In each case, we run MINLPBB until termination or until we reach a memory limit. The fixed cases require significantly less CPU time, on average, than do the free cases. As expected, the CPU time increases exponentially as the problem size increases in both cases. In the fixed case, however, the amount of CPU time is less than
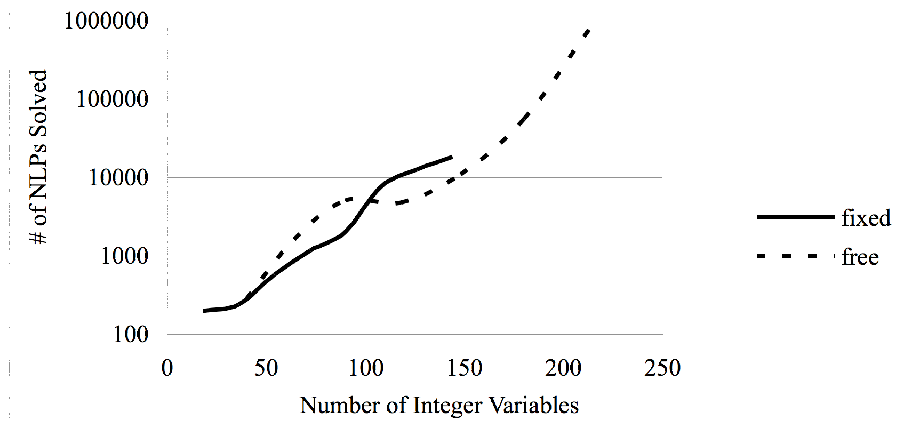
Figure 5: Number of nonlinear programs solved to reach the optimal solution vs. number of integer variables for the one-bay structures for the fixed case of the optimization formulation

two hours for the eight-story scenario, whereas for the free case, the largest scenario requires days of CPU time.

We study a convex formulation of constraints (2f) and (2g), which is a special ordered set formulation, using a Big-M construct; see constraints (8a) and (8b). As expected, the objective function value is the same as that of the original formulation. However, the CPU time is about 100 times longer than that resulting from the special ordered set formulation for the three-story one-bay structure with vertical loads only. We also study the convex formulation of constraints (3a) through (3f), which are nonlinear equality constraints, using a convex hull formulation (see the appendix). The objective function value for the NLP relaxation of the convex hull is slightly lower than the NLP relaxation of the original formulation, indicating a looser lower bound. The objective function value for the convex hull formulation is the same as that of the original formulation. However, the CPU time for the three-story case is about 375 times longer than that of the original formulation with nonlinear equality constraints.

4.2   Objective Function Values

To justify the use of integer variables, we compare rounded solutions from the root node found with MINLPBB and discrete-valued solutions from MINLPBB for the one- through eight-story, one-bay structures with beam lengths of five meters and vertical and horizontal loads. In both rounding and MINLP methods, the initial variable values equal those of the typical solution. Additionally, rounding and branching operations both begin from a root node in which the binary variables for formwork reuse are fixed such that all beams contain the same dimensions and all columns contain the same dimensions. We also enforce this formwork distribution in rounding operations. We allow the solver to determine discrete reinforcing bar sizes before rounding.

We obtain an integer feasible solution by rounding as follows: (1) if element width and depth, b_e and h_e, are continuous-valued, then round b_e and h_e up to the nearest five-centimeter increment; (2) if the number of compressive and tensile reinforcing bars, $As^{\#}_{1e}$ and $As^{\#}_{2e}$, are continuous-valued, then round up $As^{\#}_{1e}$ and $As^{\#}_{2e}$ to the nearest integer value; (3) if the number and size of reinforcing bars does not fit with appropriate cover and spacing, then round up b_e to the next five-centimeter increment; (4) if minimum reinforcement ratios are violated, then round up the bar size to the next largest size or round up the number of bars if

already at largest bar size; (5) if maximum reinforcement ratios are violated, then round up b_e one increment and, if needed after rounding b_e one increment, round up h_e one increment; (6) if the location of the neutral axis is less than the lower bound of 7 centimeters, then round up the reinforcing bar size one increment, then round up number of bars one increment, if needed, and then round up b_e one increment, if needed; (7) if the bending force resistive capacity contains a smaller magnitude than the applied bending force, then round up the bar size one increment and then h_e one increment, if needed; and (8) repeat steps 1 through 7 until a feasible discrete solution is obtained.

To evaluate the relative quality of our solutions given various methods of increasing difficulty in obtaining them, we make the following comparisons relative to the objective function value from a typical solution: (1) the objective function value at the root node of the fixed case of the nonlinear programming relaxation; (2) the objective function value from a rounded, feasible root node solution to the fixed problem; (3) the objective function value found using MINLPBB with fixed formwork; and (4) the objective function value found using MINLPBB allowing formwork reuse to vary. The last four columns of Table 4 show these results.

On average, the objective function value at the root node is 19 percent better (lower) than that provided by the typical solution. We expect this marked contrast because the root node solution is not necessarily integer feasible. On average, the solutions rounded from the fixed case show no improvement from the typical solutions; some cases show a slight degradation, whereas others show a slight improvement. These data demonstrate that rounding is not an effective means of improving on typical practice and underscores the necessity of solving an *integer* nonlinear program to achieve savings over typical practice. The integer feasible solution given by the fixed formwork case is 13 percent better than the typical solution, with no clear trend in solution quality improvement as the number of story levels increases. The fixed solution generally gives the same quality objective as the free solution, though the latter requires considerably more CPU time to obtain. (In the largest two cases, the objective degrades slightly, indicating that an inferior local optimum is found.) These results indicate that it is generally sufficient for these examples (cost structures) to consider just the fixed case, though, in general, this would not be true.

Table 4: Number of nonlinear programs and the CPU time (seconds) for the fixed and free cases. Also reported are the percent savings over the typical solution (obtained heuristically) as given by the root node solution and the rounded solution (both heuristic solutions) and the local optimal solutions for the fixed and free cases obtained with the MINLP solver.

Number of	Number of		CPU Time		Percent Savings with Respect to Typical Solution			
Story Levels	Nonlinear Programs		(seconds)		Heuristic		MINLP	
					Root	Rounded	Fixed	Free
	Fixed	Free	Fixed	Free	Solution	Solution	Solution	Solution
1	198	-	1	–	13.1	2.2	9.4	–
2	239	284	4	5	19.1	-0.1	13.7	13.7
3	576	1637	20	56	22.0	0.8	16.0	16.0
4	1166	5133	84	357	21.4	2.6	14.2	14.9
5	2014	4776	215	562	20.6	4.9	13.9	13.9
6	7707	11038	1218	1527	18.9	-4.3	11.6	11.6
7	12611	55622	2651	14486	19.4	-2.4	12.2	11.6
8	18318	842945	5244	196606	18.0	0.7	10.6	10.2

In Figure 6, we show the distribution of reinforcement, concrete, and forming costs for the one-bay structures with beam lengths of five meters and vertical and horizontal loads for the fixed case of the
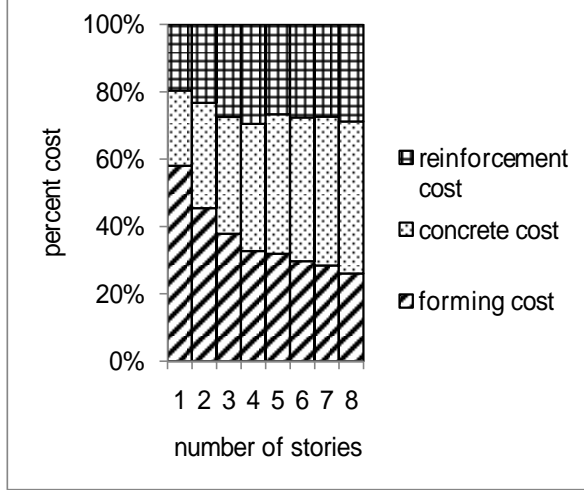
Figure 6: Reinforcement, concrete, and forming cost percentage for one-bay structures with beam lengths of five meters and vertical and horizontal loads

formulation. In the figure, the large initial percentage of cost attributed to forming decreases to nearly 25 percent for the eight-story structure. Figure 6 illustrates an asymptotic tendency for the percentage of cost attributed to forming, indicating that there is a limit at which the optimal solution contains more than one form size for columns, for beams, or for both.

4.3 Solution Characteristics

We observe that, for all our examples, the lowest cost solution contains exactly one formwork size for columns and one formwork size for beams, if we use the typical formwork cost parameter values, as defined in Table 1. We find the solution with the lowest cost contains one formwork size for columns and one formwork size for beams. To ensure the validity of this result, we conduct an exhaustive search with both the fixed and free cases of the formulation to find solutions with more than one formwork size for beams and/or columns. For the free case, we use a wide range of initial variable values such that the starting point is a structure with two or more form sizes, including the collective set of potential distributions of two formwork sizes. For the fixed case, we force formwork reuse distributions with two or more form sizes for the collective set of potential distributions of two formwork sizes. These collectively exhaustive searches for solutions with two or more form sizes show that one formwork size always results in a lower objective function value for the typical formwork cost parameter values. As prices can vary greatly depending on the year and location of construction, we demonstrate optimal solutions with more than one form size with a parametric study of the forming cost parameters for a six-story one-bay structure with vertical and horizontal loads.

To illustrate how changing the cost parameters affects the choice of formwork reuse, we conduct the following experiment. Recall that the parameters C^T and C^F are the cost of building and installing formwork and the cost of building formwork, respectively. Hence, the difference between the two is the cost of installing formwork. We change the formwork cost parameters, C^T and C^F, such that the installation cost remains constant but the benefit for reusing formwork decreases. We change the values from \$38.00/square meter of contact area (SMCA) and \$32.60/SMCA, respectively, to values of \$10.80/SMCA and \$5.40/SMCA, respectively, in order for the optimal solution for the six-story, one-bay structure to contain two form sizes for columns and two form sizes for beams. The original formwork costs result in a ratio of installation to building and installation of approximately seven, while the new values result in a ratio of two. We illustrate
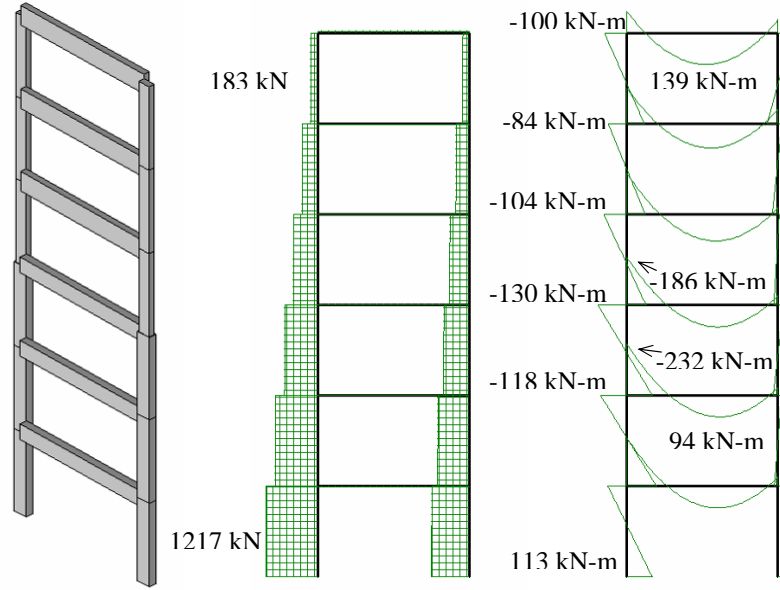
22

Figure 7: Optimal design dimensions, axial force distribution of the optimal solution, and bending moment distribution of the optimal solution for the six-story, one-bay structure with beam lengths of five meters and horizontal and vertical loads when C^T equals \$10.80/SMCA and C^F equals \$5.40/SMCA

the optimal design dimensions, axial force distribution, and bending moment distribution in Figure 7.

Consider the relative stiffness distribution of the six-story structure with two formwork sizes for columns and two for beams. Recall that the demand on the RC elements in terms of displacements and forces depends on the applied loads and relative stiffness of elements, where stiffness increases for increasing cross-sectional element dimensions. For the six-story example presented in Figure 7, the optimal column width and height equal 20 and 45 centimeters, respectively, for the bottom three story levels and equal 20 and 35 centimeters, respectively, for the top three story levels. The optimal beam width and height equal 20 and 85 centimeters, respectively, for the bottom three story levels and equal 20 and 75 centimeters, respectively, for the top three story levels. The column and beam dimensions decrease for the top three story levels compared to the bottom three story levels. Decreases in element dimensions result in decreases in stiffness. The distribution of forces depends on the relative stiffnesses of elements such that elements with smaller stiffness contain relatively smaller forces.

In the six-story structure shown in Figure 7, the axial force distribution of the optimal solution shows the largest forces applied to columns on the bottom story level (1231 kN) and a linear decrease for higher story levels. Horizontal loads that push the structure from right to left cause larger axial forces in the left hand side columns compared to the right hand side columns. The bending moment distribution of the optimal solution illustrates moments with the largest magnitude occur at the connection between the beams and columns on all but the top story level in which the largest moment occurs in the span of the beam. The beam stiffness must be significantly larger than the column stiffness in order for the largest magnitude moment to occur in the span of the beam. On the first five story levels the stiffnesses provided by two columns at the end of each beam (one below each beam and one above each beam) cause the largest moment to occur at the ends of

the beams. On the top story level in which the beam is attached only to one column at the end points, the relative stiffness is such that the largest moment occurs in the span of the beam. The moment distribution remains the same for the first five story levels because the relative stiffness of the beams and columns is about the same for these story levels. This six-story example with two formwork sizes demonstrates the range of stiffness distributions that we find for all other examples.

5   Conclusions

We present, for the first time, an explicit MINLP formulation of the RC design problem. We find optimal solutions for RC structures that result in an average of 13 percent cost savings over the typical solution, and we demonstrate the ability of MINLPBB to solve a new class of hard optimization problems: mixed integer nonlinear problems with complementarity constraints. For the size of the problems considered in this study, we find locally optimal solutions that contain one form size for columns and one form size for beams. We find various distributions of relative element widths and depths, or stiffness, depending on the structure size, geometry, and applied loads. In general, the most efficient concrete cross-section is one that resists only compressive axial forces. The loads applied to multistory RC structures cause bending of the elements, which creates elements of which only a portion of the concrete provides resistance.

Efficient designs for RC structures contain a large number of same-size elements to substantially reduce formwork cost. For typical formwork unit prices we find optimal solutions with one form size and must drastically decrease the unit price to find solutions with two form sizes. Further exploration of the transition between one and two form sizes would provide a significant contribution to this research topic. Solving systems with a larger number of story levels is necessary to explore such a transition. We find that the performance of MINLPBB is affected by the number of variables and constraints and by the number of different size cross-sections that must be determined. We find a decrease in the number of NLPs solved and CPU time for the fixed case of the optimization formulation in comparison to the free case. We also find that the free and fixed cases result in nearly the same quality of solutions. While the fixed case contains a smaller number of integer variables, number of NLPs solved, and CPU time than does the free case, we find that the fixed case quickly reaches a very large number of NLPs and CPU time. We show the limits of attainable results when using MINLPBB for the RC design problem and find that we can solve one-bay structures with up to eight story levels. To find solutions for structures with a larger number of story levels, we suggest reducing the number of integer and binary variables and improving the branching scheme. One method to reduce the number of integer and binary variables in the RC design problem without losing model fidelity would be to describe the cross-section width and depth of a group of elements with only two variables so that all elements in the group contain the same dimensions. Although the method would require a dynamic group of elements in order to find the optimal number of elements that should contain the same dimensions, the large number of same-size elements indicates that there is always some formwork reuse. Improvements to the branching scheme could also increase tractability for larger problems. Any improvements to the branching scheme in reference to the behavior of the RC design problem could substantially increase the tractability of the RC design problem and allow the solution of instances with a larger number of degrees of freedom. A sensitivity study of the variables in the problem could be used to determine the best types of improvements for the branching scheme specific to the RC design problem. A better understanding of the algorithm performance for the RC design problem could also help improve the branching scheme.

Acknowledgments

A   Convex Hull Reformulation

We replace the nonlinear equality constraints given in (3a–3f) with the convex hull formulation for formwork reuse, as shown below.

$$k \in K \text{ set of sizes for element width and depth, } b_e \text{ and } h_e, \text{ respectively}$$

Binary Variables

$$z^b_{ek} = \begin{cases} 1 & \text{if the } k^{th} \text{ size is used for the width of element } e \quad \forall e \in E^x, \forall k \in K \\ 0 & \text{otherwise} \end{cases}$$

$$z^h_{ek} = \begin{cases} 1 & \text{if the } k^{th} \text{ size is used for the depth of element } e \quad \forall e \in E^x, \forall k \in K \\ 0 & \text{otherwise} \end{cases}$$

Continuous Variables

b_e width of element e $\forall e \in E$ (cm)

h_e height of element e $\forall e \in E$ (cm)

$$w_e = \begin{cases} 1 & \text{if the } e^{th} \text{ element formwork is already built} \quad \forall e \in E^x \\ 0 & \text{otherwise} \end{cases}$$

$$z^{cb}_{ee'k} = \begin{cases} 1 & \text{if width of column element } e \text{ is the same } k^{th} \text{ size as the width of column element } e' \\ & \forall e, e' \in E^x \cap E^c, \forall k \in K \ni e \leq e' \\ 0 & \text{otherwise} \end{cases}$$

$$z^{ch}_{ee'k} = \begin{cases} 1 & \text{if the depth of column element } e \text{ is the same } k^{th} \text{ size as the depth of column element } e' \\ & \forall e, e' \in E^x \cap E^c, \forall k \in K \ni e \leq e' \\ 0 & \text{otherwise} \end{cases}$$

$$z^{bb}_{ee'k} = \begin{cases} 1 & \text{if the width of beam element } e \text{ is the same } k^{th} \text{ size as the width of beam element } e' \\ & \forall e, e' \in E^x \cap E^b, \forall k \in K \ni e \leq e' \\ 0 & \text{otherwise} \end{cases}$$

$$z^{bh}_{ee'k} = \begin{cases} 1 & \text{if the depth of beam element } e \text{ is the same } k^{th} \text{ size as the depth of beam element } e' \\ & \forall e, e' \in E^x \cap E^b, \forall k \in K \ni e \leq e' \\ 0 & \text{otherwise} \end{cases}$$

$$y^c_{ee'} = \begin{cases} 1 & \text{if column element } e \text{ contains the same width and depth as column element } e' \\ & \forall e, e' \in E^x \cap E^c \ni e \leq e' \\ 0 & \text{otherwise} \end{cases}$$

$$y^b_{ee'} = \begin{cases} 1 & \text{if beam element } e \text{ contains the same width and depth as beam element } e' \\ & \forall e, e' \in E^x \cap E^b \ni e \le e' \\ 0 & \text{otherwise} \end{cases}$$

$$b_e = \sum_{k \in K} k \cdot z^b_{ek} \ \ \forall e \in E^x \tag{9a}$$

$$h_e = \sum_{k \in K} k \cdot z^h_{ek} \ \ \ \forall e \in E^x \tag{9b}$$

$$\sum_{k \in K} z^b_{ek} = 1 \ \ \forall e \in E^x \tag{9c}$$

$$\sum_{k \in K} z^h_{ek} = 1 \ \ \forall e \in E^x \tag{9d}$$

$$z^{cb}_{ee'k} \ge z^b_{ek} + z^b_{e'k} - 1 \ \ \forall e, e' \in E^x \cap E^c, \forall k \in K \ni e \le e' \tag{9e}$$

$$z^{cb}_{ee'k} \le z^b_{ek} \ \ \forall e, e' \in E^x \cap E^c, \forall k \in K \ni e \le e' \tag{9f}$$

$$z^{cb}_{ee'k} \le z^b_{e'k} \ \ \forall e, e' \in E^x \cap E^c, \forall k \in K \ni e \le e' \tag{9g}$$

$$z^{ch}_{ee'k} \ge z^h_{ek} + z^h_{e'k} - 1 \ \ \forall e, e' \in E^x \cap E^c, \forall k \in K \ni e \le e' \tag{9h}$$

$$z^{ch}_{ee'k} \le z^h_{ek} \ \ \forall e, e' \in E^x \cap E^c, \forall k \in K \ni e \le e' \tag{9i}$$

$$z^{ch}_{ee'k} \le z^h_{e'k} \ \ \forall e, e' \in E^x \cap E^c, \forall k \in K \ni e \le e' \tag{9j}$$

$$y^c_{ee'} \ge \sum_{k \in K} (z^{cb}_{ee'k} + z^{ch}_{ee'k}) - 1 \ \ \forall e, e' \in E^x \cap E^c, \ni e \le e' \tag{9k}$$

$$y^c_{ee'} \le \sum_{k \in K} z^{cb}_{ee'k} \ \ \forall e, e' \in E^x \cap E^c, \ni e \le e' \tag{9l}$$

$$y^c_{ee'} \le \sum_{k \in K} z^{ch}_{ee'k} \ \ \forall e, e' \in E^x \cap E^c, \ni e \le e' \tag{9m}$$

$$z^{bb}_{ee'k} \ge z^b_{ek} + z^b_{e'k} - 1 \ \ \forall e, e' \in E^x \cap E^b, \forall k \in K \ni e \le e' \tag{9n}$$

$$z^{bb}_{ee'k} \le z^b_{ek} \ \ \forall e, e' \in E^x \cap E^b, \forall k \in K \ni e \le e' \tag{9o}$$

$$z^{bb}_{ee'k} \le z^b_{e'k} \ \ \forall e, e' \in E^x \cap E^b, \forall k \in K \ni e \le e' \tag{9p}$$

$$z^{bh}_{ee'k} \ge z^h_{ek} + z^h_{e'k} - 1 \ \ \forall e, e' \in E^x \cap E^b, \forall k \in K \ni e \le e' \tag{9q}$$

$$z^{bh}_{ee'k} \le z^h_{ek} \ \ \forall e, e' \in E^x \cap E^b, \forall k \in K \ni e \le e' \tag{9r}$$

$$z^{bh}_{ee'k} \le z^h_{e'k} \ \ \forall e, e' \in E^x \cap E^b, \forall k \in K \ni e \le e' \tag{9s}$$

$$y^b_{ee'} \ge \sum_{k \in K} (z^{bb}_{ee'k} + z^{bh}_{ee'k}) - 1 \ \ \forall e, e' \in E^x \cap E^b, \ni e \le e' \tag{9t}$$

$$y^b_{ee'} \le \sum_{k \in K} z^{bb}_{ee'k} \ \ \forall e, e' \in E^x \cap E^b, \ni e \le e' \tag{9u}$$

$$y^b_{ee'} \le \sum_{k \in K} z^{bh}_{ee'k} \ \ \forall e, e' \in E^x \cap E^b, \ni e \le e' \tag{9v}$$

$$\sum_{e \in E^x \cap E^c, \ni e \le e'} y^c_{ee'} \ge w_{e'} \ \ \forall e' \in E^x \cap E^c \tag{9w}$$

$$\sum_{e \in E^x \cap E^b, \ni e \le e'} y^b_{ee'} \ge w_{e'} \ \ \forall e' \in E^x \cap E^b \tag{9x}$$

We note that the reformulated MINLP is nonconvex because of the presence of nonlinear equations describing the finite element analysis.

References

Abramson M.A. (2006, May). *Air Force Institute of Technology*, Retrieved from

http://www.afit.edu/en/enc/Faculty/MAbramson/nomadm.html

Abramson, M.A. (2005). "Second Order Behavior of Pattern Search," *SIAM Journal on Optimization.* 16(2): 315-330.

American Concrete Institute (ACI). (2005). *Committee 318 Building Code Requirements for Structural Concrete* (ACI 318-05) *and Commentary* (ACI 318R-05), Detroit.

AMPL Optimization LLC. (2001). AMPL Version 10.6.16. www.ampl.com.

Balling, R.J. (2002). "Decomposition Methods in Structural Optimization," in *ASCE Recent Advances in Optimal Structural Design*, Burns, S.A., editor. 41-53. New York.

Balling, R.J., and Yao, X. (1997). "Optimization of Reinforced Concrete Frames," *J. of Structural Engineering*, ASCE. 123(2): 193-202.

Bussieck, M.R., and Pruessner, A. (2003). "Mixed-Integer Nonlinear Programming," GAMS Development Corporation.

Camp, C.V., Pezeshk, S., and Hansson, H. (2003). "Flexural Design of Reinforced Concrete Frames using a Genetic Algorithm," *Journal of Structural Engineering.* 129(1): 105.

Corradi, L., De Donato, O., and Maier, G. (1974). "Inelastic Analysis of Reinforced Concrete Frames, Journal of the Structural Division," *J. of the Structural Division*, ASCE. 100(9): 1925-1942.

De Donato, O., and Maier, G. (1972). "Mathematical Programming Methods for the Inelastic Analysis of Reinforced Concrete Frames Allowing for Limited Rotation Capacity," *Int. J. Num. Meth. Engng.* 4: 307-329.

Dinno K.S., and Mekha, B.B. (1993). "Optimal Design of Reinforced Concrete Frames Based on Inelastic Analysis," *Computers and Structures.* 47(2): 245-252.

Dinno K.S., Mekha, B.B. (1995). "A Developed Algorithm for the Inelastic Analysis of Reinforced Concrete Frames," *Computers and Structures.* 57(2): 227-232.

Duran M., and Grossman, I.E. (1986). "An Outer-Approximation Algorithm for a Class of Mixed Integer Nonlinear Programs," *Mathematical Programming* 36: 307.

Fadaee, M.J., and Grierson, D.E. (1996). "Design Optimization of 3D Reinforced Concrete Structures," *Structural Optimization*, Springer-Verlag. 12(1-2): 127-134.

Ferris, M.C., and Tin-Loi, F. (1999). "On the Solution of a Minimum Weight Elastoplastic Problem Involving Displacement and Complementarity Constraints," *Computer Methods in Applied Mechanics and Engineering.* 174: 107-120.

Fletcher, R., and Leyffer, S. (2004). "Solving Mathematical Program with Complementarity Constraints as Nonlinear Programs," *Optimization Methods and Software*, 19(1): 15-40.

Fletcher, R., and Leyffer, S. (1998). User Manual for filterSQP. Numerical Analysis Report NA/181, Dundee University, Dundee, Scotland.

Fourer, R., Gay D., and Kernighan, B. (2003). <u>AMPL A Modeling Language for Mathematical Programming</u>, Thompson Learning, Pacific Grove, CA.

Geoffrion, A.M. (1972). "Generalized Benders Decomposition," *Journal of Optimization Theory and Applications* 10: 237-260.

Guerra, A., and Kiousis P.D. (2006). "Design Optimization of Reinforced Concrete Structures," *Computers and Concrete.* 3(5): 313.

Guerra, A. (2008). "Design of Reinforced Concrete Using Mixed Integer Nonlinear Programming," *Ph.D. Dissertation.* Colorado School of Mines, Golden CO.

Horowitz, B., and Moraes, G.R. (2005). "Mathematical Programming with Complementarity Constraints Applied to Limited Ductility Analysis of Concrete Beams," *6th World Congresses of Structural and Multidisciplinary Optimization.* Rio De Janeiro, Brazil.

Inel, M., and Ozmen, H.B. (2006). "Effects of Plastic Hinge Properties in Nonlinear Analysis of Reinforced Concrete Buildings," *Engineering Structures.* 28: 1494-1502.

Kaneko, I. (1977). "A Mathematical Programming Method for the Inleastic Analysis of Reinforced Concrete Frames," *International J. for Numerical Methods in Engineering*, Wiley and Sons. 11: 1137.

Krishnamoorthy, C.S., and Mosi, D.R. (1981). "Optimal Design of Reinforced Concrete Frames Based on Inelastic Analysis," *Engineering Optimization.* 5: 151-167.

Lee, C., and Ahn, J. (2003). "Flexural Design of Reinforced Concrete Frames by Genetic Algorithm," *Journal of Structural Engineering.* 129(6): 762.

Leyffer, S. (1999). User Manual for MINLPBB, University of Dundee Numerical Analysis Report NA/XXX. Dundee University, Dundee, Scotland.

Leyffer, S. (2001). "Integrating SQP and Branch-and-Bound for Mixed Integer Nonlinear Programming," *Computational Optimization and Applications.* 18: 295-309.

Maier, G., Giannessi, F., and Nappi, A. (1982). "Indirect Identification of Yield Limits by Mathematical Programming," *Engineering Structures.* 4: 86-98.

Scheel, H., and Scholtes, S. (2000). "Mathematical Programs with Complementarity Constraints: Stationarity, Optimality, and Sensitivity," *Mathematics of Operations Research.* 25(1): 1-22.

Sahinidis, N.V. (2006). Private communication.

SEI/ASCE 7-98. (2000). *Minimum Design Loads for Buildings and Other Structures*, American Society of Civil Engineers, Reston, VA.

Tawarmalani, M., and Sahinidis, N.V. (2002). "Convexification and Global Optimization in Continuous and Mixed-Integer Nonlinear Programming: Theory, Algorithms, Software, and Applications, Nonconvex Optimization and its Applications series," Kluwer Academic Publishers. Boston MA. 65.

Torczon, V. (1997). "On the Convergence of Pattern Search Algorithms," *SIAM Journal on Optimization.* 7(1): 1-25.