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CONDITIONAL BOUNDS FOR THE LEAST QUADRATIC NON-RESIDUE AND RELATED PROBLEMS

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Abstract. This paper studies explicit and theoretical bounds for several interesting quantities in number theory, conditionally on the Generalized Riemann Hypothesis. Specifically, we improve the existing explicit bounds for the least quadratic non-residue and the least prime in an arithmetic progression. We also refine the classical conditional bounds of Littlewood for L-functions at $s = 1$. In particular, we derive explicit upper and lower bounds for $L(1, \chi)$ and $\zeta(1 + it)$, and deduce explicit bounds for the class number of imaginary quadratic fields. Finally, we improve the best known theoretical bounds for the least quadratic non-residue, and more generally, the least k-th power non-residue.

1. Introduction

Let $q > 1$ be a natural number and let $G = (\mathbb{Z}/q\mathbb{Z})^*$ denote the group of reduced residues (mod q). Given a proper subgroup H of G, two natural and interesting questions in number theory are:

(1) Determine or estimate the least prime p not dividing q and lying outside the subgroup H, and
(2) Given a coset of H in G, determine or estimate the least prime p lying in that coset.

These problems have a long history, and have attracted the attention of many mathematicians. There are two particular cases that are especially well known. If q is prime and H is the group of quadratic residues the first problem amounts to the famous question of Vinogradov on the least quadratic non-residue. For any q, if $H = \{1\}$ is the trivial subgroup, then the second problem is that of estimating the least prime in an arithmetic progression. In this paper we assume the truth of the Generalized Riemann Hypothesis and obtain explicit as well as asymptotic bounds on these problems. Our work improves upon previous results in this area, notably the works of Bach [2], and Bach and Sorenson [4].

1.1. **The least prime outside a subgroup.** Assuming GRH, Ankeny [1] established that the least prime lying outside a proper subgroup H of $(\mathbb{Z}/q\mathbb{Z})^*$ is $O((\log q)^2)$. Ankeny's work was refined by Bach [2], who gave explicit as well as asymptotic bounds for the least such prime. Thus Bach showed, on GRH, that the least prime p that does not lie in H is less than $2(\log q)^2$, and if the additional

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natural requirement that $p \nmid q$ is imposed then p is less than $3(\log q)^2$. He further established that the least prime $p \nmid q$ with p not lying in H satisfies the asymptotic bound $p \le (1+o(1))(\log q)^2$. In the special case when H consists of the square residue classes (mod q), Wedeniwski [16] has recently given the improved explicit bound $\frac{3}{2}(\log q)^2 - \frac{44}{5}\log q + 13$. We begin with an explicit result on this question. We shall assume that $q \ge 3000$, since questions for smaller values of q may easily be settled by direct calculation. As usual we denote by $\omega(q)$ the number of distinct prime factors of q.

Theorem 1.1. *Assume GRH. Let $q \ge 3000$ be an integer, and let H be a proper subgroup of $G = (\mathbb{Z}/q\mathbb{Z})^*$.*

(1) Let $A(q) = \max(0, 2\log\log q - 8/5 - \sum_{p|q}(\log p)/(p-1))$, and put

$$B(q) = \max(0, 2\log\log q + 3 + 2\omega(q)(\log\log q)^2/\log q - 2A(q)).$$

The least prime ℓ with $\ell \nmid q$ and ℓ not lying in the subgroup H satisfies the bound $\ell \le (\log q + B(q))^2$.

(2) Suppose q is not divisible by any prime below $(\log q)^2$. Then there is a prime $\ell \le (\log q)^2$ with ℓ not lying in the subgroup H.

The quantity $B(q)$ in Theorem 1.1 is zero when q is large and does not have too many small prime factors. Asymptotically $B(q) = o(\log\log q)$, and in any given situation it may easily be directly estimated.

Consider now the special case when q is prime and H is the subgroup of quadratic residues (mod q). Combining Theorem 1.1 with a computer calculation for the primes below 3000 we obtain the following corollary.

Corollary 1.1. *Assume GRH. If $q \ge 5$ is prime, the least quadratic non-residue* (mod q) *lies below* $(\log q)^2$.

Theorem 1.1 establishes an explicit bound of the same strength as the asymptotic bound given in Bach [2]. Interestingly it turns out that the asymptotic bound in this question may also be improved.

Theorem 1.2. *Assume GRH. Let q be a large integer and let H be a subgroup of $G = (\mathbb{Z}/q\mathbb{Z})^*$ with index $h = [G : H] > 1$. Then the least prime p not in H satisfies*

$$p < (\alpha(h) + o(1))(\log q)^2,$$

where $\alpha(2) = 0.42$, $\alpha(3) = 0.49$ and in general $\alpha(h) = 0.51$ for all $h > 3$.

In particular the least quadratic non-residue is bounded by $(0.42 + o(1))(\log q)^2$. Theorem 1.2 gives good bounds when h is small, but when h is very large the following stronger result holds, which appears to be the limit of our method.

Theorem 1.3. *Assume GRH. Let q be a large integer and let H be a subgroup of $G = (\mathbb{Z}/q\mathbb{Z})^*$ with index $h = [G : H] \ge 4$. Then the least prime p not in H satisfies*

$$p < \Big(\frac{1}{4} + o(1)\Big)\Big(1 - \frac{1}{h}\Big)^2 \Big(\frac{\log(2h)}{\log(2h) - 2}\Big)^2 (\log q)^2.$$

1.2. **The least prime in a given coset.** Now we turn to the second of our questions: the problem of determining the least prime lying in a given coset of H.

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Theorem 1.4. *Assume GRH. Let $q \geq 20000$ and let H be a subgroup of $G = (\mathbb{Z}/q\mathbb{Z})^*$ with index $h = [G : H] > 1$. Let p be the smallest prime lying in a given coset aH. Then either $p \leq 10^9$, or*

$$p \leq \Big((h-1)\log q + 3(h+1) + \frac{5}{2}(\log\log q)^2\Big)^2.$$

From the proof one may obtain more precise estimates, but the form above seems easiest to state.

We next single out the case when H is the trivial subgroup comprised of just the identity. Here the problem amounts to getting bounds on the least prime $P(a,q)$ in the arithmetic progression $a \pmod q$ (where we assume that $(a,q) = 1$). A celebrated (unconditional) theorem of Linnik gives that $P(a,q) \ll q^L$ for some absolute constant L. The work of Heath-Brown [10] shows that $L = 5.5$ is an admissible value for Linnik's constant, and this has been recently improved to $L = 5.2$ by Xylouris [17].

Under the assumption of the Generalized Riemann Hypothesis, Bach and Sorenson [4] showed that

$$P(a,q) \leq (1 + o(1))(\phi(q)\log q)^2, \tag{1.1}$$

and they also derived the explicit bound

$$P(a,q) \leq 2(q\log q)^2. \tag{1.2}$$

Our work leading up to Theorem 1.4 and some computation for $q \leq 20000$ permits the following refinement of these bounds.

Corollary 1.2. *Assume GRH. Let $q > 3$, and let $a \pmod q$ be a reduced residue class. The least prime in the arithmetic progression $a \pmod q$ satisfies*

$$P(a,q) \leq (\phi(q)\log q)^2.$$

By modifying the argument for Theorems 1.2 and 1.3, and using the Brun-Titchmarsh theorem, one may derive an asymptotic bound of the form $P(a,q) \leq (1 - \delta + o(1))(\phi(q)\log q)^2$ for some small $\delta > 0$.

1.3. **Bounds for values of L-functions at $s = 1$.** The Generalized Riemann Hypothesis implies the Generalized Lindelöf Hypothesis, thus furnishing good upper bounds for the values of L-functions. Such results go back to Littlewood [13], [14], and the best known asymptotic bounds may be found in [7] (for L-values on the critical line) and [5] (for L-values inside the critical strip). Explicit bounds on GRH for L-values on the critical line may be found in [6]. The methods of this paper allow us to give, assuming GRH, new explicit upper and lower bounds for L-values at the edge of the critical strip.

Theorem 1.5. *Assume GRH. Let q be a positive integer and χ be a primitive character modulo q. For $q \geq 10^{10}$ we have*

$$|L(1,\chi)| \leq 2e^\gamma \left(\log\log q - \log 2 + \frac{1}{2} + \frac{1}{\log\log q}\right)$$

and

$$\frac{1}{|L(1,\chi)|} \leq \frac{12e^\gamma}{\pi^2}\Big(\log\log q - \log 2 + \frac{1}{2} + \frac{1}{\log\log q} + \frac{14\log\log q}{\log q}\Big).$$

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By Dirichlet's class number formula, we can deduce explicit bounds for the class number of an imaginary quadratic field under the assumption of GRH. More specifically, let $-q$ be a fundamental discriminant with $q > 4$ and let $\chi_{-q}(n) = \left(\frac{-q}{n}\right)$ be the Kronecker symbol, which is a primitive Dirichlet character (mod q). Moreover, denote by $h(-q)$ the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-q})$. Then, the Dirichlet class number formula reads

$$h(-q) = \frac{\sqrt{q}}{\pi} L(1, \chi_{-q}). \tag{1.3}$$

For discussions on the computation of these class numbers, we refer to [3], [9], [12], [15]. From Theorem 1.5 we obtain the following corollary.

Corollary 1.3. *Assume GRH. Let $-q$ be a fundamental discriminant, and let $h(-q)$ denote the class number of $\mathbb{Q}(\sqrt{-q})$. If $q \geq 10^{10}$ then*

$$h(-q) \geq \frac{\pi}{12e^\gamma}\sqrt{q}\Big(\log\log q - \log 2 + \frac{1}{2} + \frac{1}{\log\log q} + \frac{14\log\log q}{\log q}\Big)^{-1}$$

and

$$h(-q) \leq \frac{2e^\gamma}{\pi}\sqrt{q}\left(\log\log q - \log 2 + \frac{1}{2} + \frac{1}{\log\log q}\right).$$

The work of [12] computes class numbers and class groups for all imaginary quadratic fields with absolute discriminant below 10^{11}. From our corollary it follows that on GRH $h(-q) \geq 9053$ when $q \geq 10^{11}$.

Our methods would also lead to explicit bounds for more general L-functions. We do not pursue this here, and content ourselves by stating that for real numbers $t \geq 10^{10}$ we have (assuming RH)

$$|\zeta(1+it)| \leq 2e^\gamma \left(\log\log t - \log 2 + \frac{1}{2} + \frac{1}{\log\log t}\right)$$

and

$$\frac{1}{|\zeta(1+it)|} \leq \frac{12e^\gamma}{\pi^2}\Big(\log\log t - \log 2 + \frac{1}{2} + \frac{1}{\log\log t} + \frac{14\log\log t}{\log t}\Big).$$

2. Preliminary lemmas

This section gathers together several preliminary results that will be useful in our subsequent work. We first introduce a convenient notation that will be in place for the rest of the paper. We let θ stand for a complex number of magnitude at most one. In each occurrence θ might stand for a different value, so that we may write $\theta - \theta = 2\theta$, $\theta \times \theta = \theta$ and so on.

Recall from Chapter 12 of [8] the following properties of the Riemann zeta-function and Dirichlet L-functions. Set

$$\xi(s) = s(s-1)\pi^{-\frac{s}{2}}\Gamma\Big(\frac{s}{2}\Big)\zeta(s),$$

which is an entire function of order 1, satisfies the functional equation $\xi(s) = \xi(1-s)$, and for which the Hadamard factorization formula gives

$$\xi(s) = e^{Bs}\prod_\rho \Big(1 - \frac{s}{\rho}\Big)e^{s/\rho}. \tag{2.1}$$

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Above, ρ runs over the non-trivial zeros of $\zeta(s)$, and B is a real number given by

$$B = -\sum_{\rho} \mathrm{Re}\frac{1}{\rho} = \frac{1}{2}\log(4\pi) - 1 - \frac{\gamma}{2} = -0.02309\ldots. \tag{2.2}$$

Let $\chi \pmod q$ denote a primitive Dirichlet character, and let $L(s,\chi)$ denote the associated Dirichlet L-function. Let $\mathfrak{a} = (1-\chi(-1))/2 = 0$ or 1 depending on whether the character χ is even or odd. Let $\xi(s,\chi)$ denote the completed L-function

$$\xi(s,\chi) = \Big(\frac{q}{\pi}\Big)^{\frac{s}{2}}\Gamma\Big(\frac{s+\mathfrak{a}}{2}\Big)L(s,\chi).$$

The completed L-function satisfies the functional equation

$$\xi(s,\chi) = \epsilon_\chi \xi(1-s,\overline{\chi}),$$

where ϵ_χ is a complex number of size 1. The zeros of $\xi(s,\chi)$ are precisely the non-trivial zeros of $L(s,\chi)$, and letting $\rho_\chi = \frac{1}{2} + i\gamma_\chi$ denote such a zero (throughout the paper we assume the truth of the GRH), we have Hadamard's factorization formula:

$$\xi(s,\chi) = \exp(A(\chi) + sB(\chi))\prod_{\rho_\chi}\Big(1 - \frac{s}{\rho_\chi}\Big)e^{s/\rho_\chi}. \tag{2.3}$$

Above, $A(\chi)$ and $B(\chi)$ are constants, with $\mathrm{Re}\, B(\chi)$ being of particular interest for us. Note that

$$\mathrm{Re}(B(\chi)) = \mathrm{Re}(B(\overline{\chi})) = \mathrm{Re}\frac{\xi'}{\xi}(0,\chi) = -\sum_{\rho_\chi}\mathrm{Re}\frac{1}{\rho_\chi}. \tag{2.4}$$

Recall the digamma function $\psi_0(z) = \frac{\Gamma'}{\Gamma}(z)$, and its derivative the trigamma function $\psi_1(z) = \psi_0'(z)$. The following special values will be useful:

$$\psi_0(1) = -\gamma,\quad \psi_1(1) = \zeta(2) = \frac{\pi^2}{6},\quad \psi_0(1/2) = -2\log 2 - \gamma,\ \text{ and } \psi_1(1/2) = \frac{\pi^2}{2}.$$

Lemma 2.1. *Assume RH. For $x > 1$ there exists some $|\theta| \le 1$ such that*

$$\sum_{n\le x}\Lambda(n)\log(x/n) = x - (\log 2\pi)\log x - 1 + \sum_{k=1}^{\infty}\frac{1-x^{-2k}}{4k^2} + 2\theta|B|(\sqrt{x}+1).$$

Lemma 2.2. *Assume GRH. Let $q \ge 3$ and let $\chi \pmod q$ be a primitive Dirichlet character. Define*

$$S(x,\chi) := \sum_{n\le x}\Lambda(n)\chi(n)\log(x/n).$$

For $x > 1$, there exists some $|\theta| \le 1$ such that

$$S(x,\chi) = |\mathrm{Re}(B(\chi))|(2\theta\sqrt{x} + 2\theta) - \frac{\xi'}{\xi}(0,\chi)\log x + \frac{1}{2}\Big(\log\frac{q}{\pi}\Big)\log x + \widetilde{E}_{\mathfrak{a}}(x),$$

where

$$\widetilde{E}_0(x) = \frac{\pi^2}{24} - \frac{\gamma}{2}\log x - \frac{1}{2}(\log x)^2 - \sum_{k=1}^{\infty}\frac{x^{-2k}}{(2k)^2}$$

and

$$\widetilde{E}_1(x) = \frac{\pi^2}{8} - (\log 2 + \gamma/2)\log x - \sum_{k=0}^{\infty}\frac{x^{-2k-1}}{(2k+1)^2}.$$

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In particular,

$$\mathrm{Re}S(x,\chi) = |\mathrm{Re}(B(\chi))|(2\theta\sqrt{x} + 2\theta + \log x) + \frac{1}{2}\Big(\log\frac{q}{\pi}\Big)\log x + \widetilde{E}_{\mathfrak{a}}(x).$$

We shall confine ourselves to proving Lemma 2.2; the proof of Lemma 2.1 is essentially the same.

Proof of Lemma 2.2. We begin with

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} -\frac{\xi'}{\xi}(s,\chi)\frac{x^s}{s^2}ds,$$

and first evaluate the integral by moving the line of integration to the left. There are poles at the non-trivial zeros ρ_χ of $L(s,\chi)$, and a double pole at $s = 0$. Evaluating the residues here we find that our integral equals

$$-\sum_{\rho_\chi}\frac{x^{\rho_\chi}}{\rho_\chi^2} - \frac{\xi'}{\xi}(0,\chi)\log x - \Big(\frac{\xi'}{\xi}\Big)'(0,\chi).$$

Invoking GRH,

$$-\sum_{\rho_\chi}\frac{x^{\rho_\chi}}{\rho_\chi^2} = \theta\sqrt{x}\sum_{\rho_\chi}\frac{1}{|\rho_\chi|^2} = 2\theta\sqrt{x}|\mathrm{Re}(B(\chi))|.$$

Further,

$$\Big(\frac{\xi'}{\xi}\Big)'(0,\chi) = -\sum_{\rho_\chi}\frac{1}{\rho_\chi^2} = 2\theta|\mathrm{Re}(B(\chi))|.$$

Putting these observations together we obtain

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} -\frac{\xi'}{\xi}(s,\chi)\frac{x^s}{s^2}ds = |\mathrm{Re}(B(\chi))|\Big(2\theta\sqrt{x} + 2\theta\Big) - \frac{\xi'}{\xi}(0,\chi)\log x. \tag{2.5}$$

On the other hand, the LHS of (2.5) equals

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Big(-\frac{1}{2}\log\frac{q}{\pi} - \frac{1}{2}\psi_0\Big(\frac{s+\mathfrak{a}}{2}\Big) - \frac{L'}{L}(s,\chi)\Big)\frac{x^s}{s^2}ds. \tag{2.6}$$

The first and third terms above contribute

$$-\frac{1}{2}\Big(\log\frac{q}{\pi}\Big)\log x + \sum_{n\le x}\Lambda(n)\chi(n)\log(x/n).$$

If $\mathfrak{a} = 1$, the middle term in (2.6) equals (here there are simple poles at the negative odd integers, and a double pole at 0)

$$\sum_{k=0}^{\infty}\frac{x^{-2k-1}}{(2k+1)^2} - \frac{1}{2}\psi_0\Big(\frac{1}{2}\Big)\log x - \frac{1}{4}\psi_1\Big(\frac{1}{2}\Big) = -\widetilde{E}_1(x).$$

If $\mathfrak{a} = 0$ the middle term in (2.6) equals (here there are simple poles at the negative even integers, and a triple pole at 0)

$$\sum_{k=1}^{\infty}\frac{x^{-2k}}{(2k)^2} + \frac{1}{2}(\log x)^2 - \frac{1}{2}\psi_0(1)\log x - \frac{1}{4}\psi_1(1) = -\widetilde{E}_0(x).$$

This completes our proof of the first assertion. The second assertion follows by taking the real part, and noting that Re $\frac{\xi'}{\xi}(0,\chi) = \mathrm{Re}(B(\chi))$ from (2.4). □

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Lemma 2.3. *Assume GRH. Let $q \geq 3$ and let $\chi \pmod q$ be a primitive Dirichlet character. For any $x > 1$ we have, for some $|\theta| \leq 1$,*

$$-\frac{\xi'}{\xi}(0,\bar{\chi}) - \frac{1}{x}\frac{\xi'}{\xi}(0,\chi) + \frac{2\theta}{\sqrt{x}}|\mathrm{Re}(B(\chi))| = \frac{1}{2}\Big(1-\frac{1}{x}\Big)\log\frac{q}{\pi} - \sum_{n\leq x}\frac{\Lambda(n)}{n}\chi(n)\Big(1-\frac{n}{x}\Big) + E_{\mathfrak{a}}(x),$$

where

$$E_0(x) = -\log 2 - \frac{\gamma}{2}\Big(1-\frac{1}{x}\Big) + \frac{\log x + 1}{x} - \sum_{k=1}^{\infty}\frac{x^{-2k-1}}{2k(2k+1)}$$

and

$$E_1(x) = -\sum_{k=0}^{\infty}\frac{x^{-2k-2}}{(2k+1)(2k+2)} - \frac{\gamma}{2}\Big(1-\frac{1}{x}\Big) + \frac{\log 2}{x}.$$

In particular, $|\mathrm{Re}(B(\chi))|$ equals

$$\Big(1+\frac{2\theta}{\sqrt{x}}+\frac{1}{x}\Big)^{-1}\Big(\frac{1}{2}\Big(1-\frac{1}{x}\Big)\log\frac{q}{\pi} - \mathrm{Re}\sum_{n\leq x}\frac{\Lambda(n)}{n}\chi(n)\Big(1-\frac{n}{x}\Big) + E_{\mathfrak{a}}(x)\Big).$$

Proof. The proof is similar to that of Lemma 2.2, and so we will be brief. We consider

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{\xi'}{\xi}(s,\chi)\frac{x^{s-1}}{s(s-1)}ds, \tag{2.7}$$

and begin by evaluating the integral by moving the line of integration to the left. There are poles at $s=1$, $s=\rho_\chi$ and $s=0$ and therefore our integral above equals

$$\frac{\xi'}{\xi}(1,\chi) - \frac{1}{x}\frac{\xi'}{\xi}(0,\chi) + \sum_{\rho_\chi}\frac{x^{\rho_\chi-1}}{\rho_\chi(\rho_\chi-1)}.$$

Note that $\frac{\xi'}{\xi}(1,\chi) = -\frac{\xi'}{\xi}(0,\bar{\chi})$. Further, since we are assuming GRH, we have for some $|\theta| \leq 1$,

$$\sum_{\rho_\chi}\frac{x^{\rho_\chi-1}}{\rho_\chi(\rho_\chi-1)} = \frac{\theta}{\sqrt{x}}\sum_{\rho_\chi}\frac{1}{|\rho_\chi|^2} = \frac{2\theta}{\sqrt{x}}|\mathrm{Re}(B(\chi))|.$$

Thus the quantity in (2.7) equals the LHS of the first identity claimed in our lemma.

On the other hand, we may rewrite the integral (2.7) as

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Big(\frac{1}{2}\log\frac{q}{\pi} + \frac{1}{2}\psi_0\Big(\frac{s+\mathfrak{a}}{2}\Big) + \frac{L'}{L}(s,\chi)\Big)\frac{x^{s-1}}{s(s-1)}ds. \tag{2.8}$$

Now the first and the last terms above give

$$\frac{1}{2}\Big(1-\frac{1}{x}\Big)\log\frac{q}{\pi} - \sum_{n\leq x}\frac{\Lambda(n)}{n}\chi(n)\Big(1-\frac{n}{x}\Big).$$

If $\mathfrak{a}=1$ then the middle term in (2.8) equals (there are simple poles at $s=1$, 0, and the negative odd integers)

$$-\sum_{k=0}^{\infty}\frac{x^{-2k-2}}{(2k+1)(2k+2)} + \frac{1}{2}\psi_0(1) - \frac{1}{2x}\psi_0\Big(\frac{1}{2}\Big) = E_1(x).$$

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If $\mathfrak{a} = 0$ then the middle term equals (here there are simple poles at $s = 1$ and at the negative even integers, and a double pole at $s = 0$)

$$\frac{1}{2}\psi_0\Big(\frac{1}{2}\Big) + \frac{\log x + 1 + \gamma/2}{x} - \sum_{k=1}^{\infty} \frac{x^{-2k-1}}{2k(2k+1)} = E_0(x).$$

This proves the first part of our lemma.

Now, by the functional equation and (2.4), we have $\mathrm{Re}\frac{\xi'}{\xi}(1,\chi) = -\mathrm{Re}\frac{\xi'}{\xi}(0,\overline{\chi}) = -\mathrm{Re}(B(\chi))$. Thus taking real parts in the identity just established, we deduce the stated identity for $|\mathrm{Re}(B(\chi))|$. ☐

Lemma 2.4. *Assume RH. For $x > 1$ we have, for some $|\theta| \le 1$,*

$$\sum_{n\le x} \frac{\Lambda(n)}{n}\Big(1 - \frac{n}{x}\Big) = \log x - (1+\gamma) + \frac{\log(2\pi)}{x} - \sum_{n=1}^{\infty} \frac{x^{-2n-1}}{2n(2n+1)} + 2\theta\frac{|B|}{\sqrt{x}}.$$

Proof. We argue as in Lemma 2.3, starting with

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} -\frac{\zeta'}{\zeta}(s)\frac{x^{s-1}}{s(s-1)}ds = \sum_{n\le x}\frac{\Lambda(n)}{n}\Big(1-\frac{n}{x}\Big).$$

Moving the line of integration to the left, and computing residues, we find that the above equals

$$\log x - (1+\gamma) + \frac{\log(2\pi)}{x} - \sum_{\rho}\frac{x^{\rho-1}}{\rho(\rho-1)} - \sum_{n=1}^{\infty}\frac{x^{-2n-1}}{2n(2n+1)}.$$

The lemma follows. ☐

Lemma 2.5. *Let $q \ge 3$ and let $\chi \pmod q$ be a primitive Dirichlet character. Suppose that GRH holds for $L(s,\chi)$. For any $x \ge 2$, there exists a real number $|\theta| \le 1$ such that*

$$\log|L(1,\chi)| = \mathrm{Re}\sum_{n\le x}\frac{\chi(n)\Lambda(n)}{n\log n}\frac{\log\left(\frac{x}{n}\right)}{\log x} + \frac{1}{\log x}\Big(\frac{1}{2}\log\frac{q}{\pi} + \frac{1}{2}\psi_0\Big(\frac{1+\mathfrak{a}}{2}\Big)\Big)$$
$$- \Big(\frac{1}{\log x} + \frac{2\theta}{\sqrt{x}(\log x)^2}\Big)|\mathrm{Re}(B(\chi))| + \frac{2\theta}{x\log^2 x}.$$

Proof. For any $\sigma \ge 1$, we consider

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} -\frac{L'}{L}(\sigma+s,\chi)\frac{x^s}{s^2}ds = \sum_{n\le x}\frac{\chi(n)\Lambda(n)}{n^\sigma}\log(x/n).$$

Shifting the contour to the left, we find that the integral also equals

$$-\frac{L'}{L}(\sigma,\chi)\log x - \left(\frac{L'}{L}\right)'(\sigma,\chi) - \sum_{\rho_\chi}\frac{x^{\rho_\chi-\sigma}}{(\rho_\chi-\sigma)^2} - \sum_{n=0}^{\infty}\frac{x^{-2n-\mathfrak{a}-\sigma}}{(2n+\mathfrak{a}+\sigma)^2}.$$

Therefore,

$$-\frac{L'}{L}(\sigma,\chi) = \sum_{n\le x}\frac{\chi(n)\Lambda(n)}{n^\sigma}\frac{\log(x/n)}{\log x} + \frac{1}{\log x}\left(\frac{L'}{L}\right)'(\sigma,\chi)$$
$$+ \frac{\theta x^{\frac{1}{2}-\sigma}}{\log x}\sum_{\rho_\chi}\frac{1}{|\rho_\chi|^2} + \frac{\theta x^{-\sigma}}{\log x}\sum_{n=0}^{\infty}\frac{x^{-2n}}{(2n+1)^2}.$$

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Integrating both sides over σ from 1 to ∞ and taking real parts we conclude that

$$\log|L(1,\chi)| = \mathrm{Re}\sum_{n\le x}\frac{\chi(n)\Lambda(n)}{n\log n}\frac{\log(x/n)}{\log x} - \frac{1}{\log x}\mathrm{Re}\frac{L'}{L}(1,\chi) + \frac{\theta}{\sqrt{x}(\log x)^2}\sum_{\rho_\chi}\frac{1}{|\rho_\chi|^2} + \frac{2\theta}{x(\log x)^2}.$$

The lemma follows upon noting that

$$\begin{aligned}-\mathrm{Re}\frac{L'}{L}(1,\chi) &= -\mathrm{Re}\frac{\xi'}{\xi}(1,\chi) + \frac{1}{2}\log\frac{q}{\pi} + \frac{1}{2}\psi_0\Big(\frac{1+\mathfrak{a}}{2}\Big)\\ &= -|\mathrm{Re}(B(\chi))| + \frac{1}{2}\log\frac{q}{\pi} + \frac{1}{2}\psi_0\Big(\frac{1+\mathfrak{a}}{2}\Big).\end{aligned}$$

□

Lemma 2.6. *Assume RH. For all $x \ge e$ we have*

$$\sum_{n\le x}\frac{\Lambda(n)}{n\log n}\frac{\log(x/n)}{\log x} = \log\log x + \gamma - 1 + \frac{\gamma}{\log x} + \frac{2B\theta}{\sqrt{x}(\log x)^2} + \frac{\theta}{3x^3(\log x)^2}.$$

Proof. Analogously to the proof of Lemma 2.5 we begin with, for any $\sigma > 1$,

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} -\frac{\zeta'}{\zeta}(s+\sigma)\frac{x^s}{s^2}ds = \sum_{n\le x}\frac{\Lambda(n)}{n^\sigma}\log(x/n),$$

and moving the line of integration to the left, this equals

$$\frac{x^{1-\sigma}}{(\sigma-1)^2} - \frac{\zeta'}{\zeta}(\sigma)\log x - \Big(\frac{\zeta'}{\zeta}\Big)'(\sigma) - \sum_{\rho}\frac{x^{\rho-\sigma}}{(\rho-\sigma)^2} - \sum_{n=1}^{\infty}\frac{x^{-2n-\sigma}}{(2n+\sigma)^2}.$$

Let σ_0 denote a parameter that will tend to 1 from above, and integrate the two expressions above for σ from σ_0 to ∞. Thus

$$\sum_{n\le x}\frac{\Lambda(n)}{n^{\sigma_0}\log n}\log(x/n) = \int_{\sigma_0}^{\infty}\frac{x^{1-\sigma}}{(\sigma-1)^2}d\sigma + (\log x)\log\zeta(\sigma_0) + \frac{\zeta'}{\zeta}(\sigma_0) + \frac{2B\theta}{\sqrt{x}\log x} + \frac{\theta}{3x^3\log x}.$$

Now, with $\alpha = (\sigma_0 - 1)\log x$,

$$\int_{\sigma_0}^{\infty}\frac{x^{1-\sigma}}{(\sigma-1)^2}d\sigma = (\log x)\int_{\alpha}^{\infty}\frac{e^{-y}}{y^2}dy,$$

and upon integrating by parts we have

$$\int_{\alpha}^{\infty}\frac{e^{-y}}{y^2}dy = \frac{e^{-\alpha}}{\alpha} - \int_1^{\infty}\frac{e^{-y}}{y}dy + \int_{\alpha}^{1}\frac{1-e^{-y}}{y}dy + \log\alpha.$$

Now let $\sigma_0 \to 1^+$, so that $\alpha \to 0^+$, and note that $\log\zeta(\sigma_0) = -\log(\sigma_0 - 1) + O(\sigma_0 - 1)$, $-\zeta'/\zeta(\sigma_0) = 1/(\sigma_0 - 1) - \gamma + O(\sigma_0 - 1)$, and that $\int_0^1(1-e^{-y})dy/y - \int_1^{\infty}e^{-y}dy/y = \gamma$. The lemma then follows. □

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3. Proof of Theorem 1.1

Unlike the other results of this paper, the following simple lemma is unconditional.

Lemma 3.1. *Let $m \geq 3$ be an integer and $x \geq 2$ be a real number. Then*

$$\sum_{\substack{n\leq x\\(n,m)>1}} \Lambda(n)\log(x/n) \leq \frac{1}{2}\omega(m)(\log x)^2$$

and

$$\sum_{\substack{n\leq x\\(n,m)>1}} \frac{\Lambda(n)}{n}\left(1-\frac{n}{x}\right) \leq \sum_{\substack{n\\(n,m)>1}} \frac{\Lambda(n)}{n} = \sum_{p|m}\frac{\log p}{p-1}.$$

Proof. If $(n,m) > 1$ and $n \leq x$ is a prime power, then $n = p^\alpha$ where $p|m$ and $\alpha \leq \log x/\log p$. Hence

$$\begin{aligned}\sum_{\substack{n\leq x\\(n,m)>1}} \Lambda(n)\log(x/n) &= \sum_{\substack{p\leq x\\p|m}}\sum_{\alpha\leq \log x/\log p} \log p \log\left(\frac{x}{p^\alpha}\right)\\ &= \sum_{\substack{p\leq x\\p|m}} \log p\log x\left[\frac{\log x}{\log p}\right] - \sum_{\substack{p\leq x\\p|m}}\log^2 p \sum_{\alpha\leq\log x/\log p}\alpha\\ &\leq \frac{1}{2}\sum_{\substack{p\leq x\\p|m}}\log p\log x\left[\frac{\log x}{\log p}\right] \leq \frac{1}{2}\omega(m)(\log x)^2.\end{aligned}$$

The second part of the lemma is evident. □

Let $q \geq 3000$ and let H be a proper subgroup of $G = (\mathbb{Z}/q\mathbb{Z})^*$. Let X be such that all primes $\ell \nmid q$ with $\ell \leq X$ lie in the subgroup H. Let $\tilde{H}$ denote the group of Dirichlet characters $\chi \pmod q$ such that $\chi(n) = 1$ for all $n \in H$. Thus $\tilde{H}$ is a subgroup of the group of Dirichlet characters mod q, and note that $|\tilde{H}|$ equals $[G : H]$. The assumption that all primes $\ell \leq X$ with $\ell \nmid q$ lie in H is equivalent to $\chi(\ell) = 1$ for all $\chi \in \tilde{H}$. Since H is a proper subgroup, there exists a non-principal character $\chi \pmod q$ in $\tilde{H}$. Our strategy is to compare upper and lower bounds for $\mathrm{Re}(S(X,\chi))$ where, as defined earlier, $S(X,\chi) = \sum_{n\leq X}\Lambda(n)\chi(n)\log(X/n)$. Note that the character χ need not be primitive, and we let $\tilde{\chi} \pmod{\tilde{q}}$ denote the primitive character that induces χ.

3.1. Proof of Part 1.

Here we assume that $X = (\log q + B(q))^2 \geq (\log q)^2 \geq 64$, and seek a contradiction. The assumption that $\chi(\ell) = 1$ for all primes ℓ below X with $\ell \nmid q$ gives

$$S(X,\chi) = \sum_{n\leq X}\Lambda(n)\log(X/n) - \sum_{\substack{n\leq X\\(n,q)>1}}\Lambda(n)\log(X/n).$$

Using Lemma 2.1 and Lemma 3.1 (recall that $\chi(n) = 1$ for all $n \leq X$) we obtain the lower bound

$$\mathrm{Re}(S(X,\chi)) \geq X - \sqrt{X}\left(2|B| + 2\omega(q)\frac{(\log\log q)^2}{\log q}\right) - \log(2\pi)\log X - 1 - 2|B|, \tag{3.1}$$

where we used that $X \geq (\log q)^2$ and so $(\log X)^2/\sqrt{X} \leq 4(\log\log q)^2/\log q$.

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Next we work on the upper bound for $\mathrm{Re}(S(X,\chi))$. Note that

$$S(X,\chi) - S(X,\tilde{\chi}) = \theta \sum_{\substack{n\le X\\ (n,q/\tilde{q})>1}} \Lambda(n)\log(X/n) = \theta \frac{\omega(q/\tilde{q})}{2}(\log X)^2. \tag{3.2}$$

Using Lemma 2.2 for $\tilde{\chi}$, and since $\tilde{E}_{\mathfrak{a}}(X) \le -11/4$ for $X \ge 64$, we find

$$\mathrm{Re}(S(X,\tilde{\chi})) \le (2\sqrt{X}+2+\log X)|\mathrm{Re}(B(\tilde{\chi}))| + \frac{1}{2}\Big(\log\frac{\tilde{q}}{\pi}\Big)\log X - \frac{11}{4}. \tag{3.3}$$

We shall bound $|\mathrm{Re}(B(\tilde{\chi}))|$ above using Lemma 2.3. First note that by Lemmas 2.4 and 3.1, we have

$$\begin{aligned}\sum_{n\le X}\frac{\Lambda(n)\tilde{\chi}(n)}{n}\Big(1-\frac{n}{X}\Big) &= \sum_{n\le X}\frac{\Lambda(n)}{n}\Big(1-\frac{n}{X}\Big) - \sum_{\substack{n\le X\\ (n,\tilde{q})>1}}\frac{\Lambda(n)}{n}\Big(1-\frac{n}{X}\Big)\\ &\ge \max\Big(0, \log(X) - \frac{8}{5} - \sum_{p|q}\frac{\log p}{p-1}\Big)\\ &\ge \max\Big(0, 2\log\log q - \frac{8}{5} - \sum_{p|q}\frac{\log p}{p-1}\Big) = A(q).\end{aligned}$$

For $X \ge 64$, we have that $(1-\frac{1}{\sqrt{X}})^{-2} \le 1 + \frac{5}{2\sqrt{X}}$ and $E_{\mathfrak{a}}(X) < -1/4$. Using this in Lemma 2.3, we get

$$|\mathrm{Re}(B(\tilde{\chi}))| \le \Big(1-\frac{1}{\sqrt{X}}\Big)^{-2}\frac{1}{2}\log\frac{\tilde{q}}{\pi} - A(q) + E_{\mathfrak{a}}(X) \le \frac{\log\tilde{q}}{2} + \frac{2}{5} - A(q).$$

We use this bound in (3.3) and combine that with (3.2) to obtain an upper bound for $\mathrm{Re}(S(X,\chi))$. Since $\omega(q/\tilde{q}) \le (\log(q/\tilde{q}))/\log 2$, the resulting upper bound is largest when $\tilde{q}=q$. Thus

$$\begin{aligned}\mathrm{Re}(S(X,\chi)) &\le (2\sqrt{X}+2+\log X)\Big(\frac{\log q}{2}+\frac{2}{5}-A(q)\Big) + \frac{\log X}{2}\Big(\log\frac{q}{\pi}\Big) - \frac{11}{4}\\ &\le \sqrt{X}\Big(\log q + \frac{4}{5} - 2A(q) + \frac{\log X}{\sqrt{X}}\log q + \frac{\log q}{\sqrt{X}}\Big) - \frac{39}{20}\\ &\le \sqrt{X}\Big(\log q + 2\log\log q + \frac{9}{5} - 2A(q)\Big) - \frac{39}{20}.\end{aligned}$$

Comparing the upper bound above with the lower bound (3.1) gives the desired contradiction.

3.2. **Proof of Part 2.** Here we suppose that $X \ge (\log q)^2$ and seek a contradiction. The proof follows the same lines as Part 1, with simplifications due to the assumption that q has no prime factors below X, and we take a little more care with constants. Thus using Lemma 2.1 we have the lower bound

$$\mathrm{Re}\, S(X,\chi) = \sum_{n\le X}\Lambda(n)\log(X/n) \ge X - 2|B|(\sqrt{X}+1) - \log(2\pi)\log X - 1. \tag{3.4}$$

Now we turn to the upper bound. Since q has no prime factors below X we have $S(X,\chi) = S(X,\tilde{\chi})$. From Lemma 2.3 and Lemma 2.4 we have that

$$|\mathrm{Re}(B(\tilde{\chi}))| \le \Big(1-\frac{1}{\sqrt{X}}\Big)^{-2}\Big(\frac{1}{2}\log\frac{\tilde{q}}{\pi} - \log X + \frac{8}{5} + E_{\mathfrak{a}}(X)\Big).$$

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For $X \geq 64$ we have that $(1-1/\sqrt{X})^{-2} \leq (1+5/(2\sqrt{X}))$, and we may check that, for $\mathfrak{a} = 0$ or 1,

$$-\frac{1}{2}\log \pi - \log X + \frac{8}{5} + E_{\mathfrak{a}}(X) \leq -\log X + \frac{7}{6}.$$

Therefore, as $X \geq (\log q)^2$, and $\tilde{q} \leq q$,

$$\begin{aligned} |\mathrm{Re}(B(\tilde{\chi}))| &\leq \Big(1+\frac{5}{2\sqrt{X}}\Big)\Big(\frac{1}{2}\log \tilde{q} - \log X + \frac{7}{6}\Big) \\ &\leq \frac{\log q}{2} + \frac{5}{4} + \Big(1+\frac{5}{2\sqrt{X}}\Big)\Big(\frac{7}{6} - \log X\Big) \\ &\leq \frac{\log q}{2} - \frac{\log X}{2} - \frac{4}{7}, \end{aligned}$$

where the last bound follows upon using $X \geq 64$ together with a little calculus. Using this in Lemma 2.2, and noting that $\tilde{E}_{\mathfrak{a}}(x) < 0$ for $\mathfrak{a} = 0$ or 1 and $x \geq 3$, we get (recall $X \geq (\log q)^2$)

$$\begin{aligned} \mathrm{Re}\, S(X,\chi) &\leq (2+2\sqrt{X}+\log X)\Big(\frac{1}{2}\log q - \frac{1}{2}\log X - \frac{4}{7}\Big) + \frac{1}{2}\Big(\log\frac{q}{\pi}\Big)\log X \\ &\leq \sqrt{X}\log q + \log X\Big(-\frac{1}{2}\log(\pi X) - \frac{11}{7}\Big) - \frac{\sqrt{X}}{7} - \frac{8}{7} \\ &\leq \sqrt{X}\log q - \frac{\sqrt{X}}{7} - 2\log X - \frac{8}{7}. \end{aligned} \tag{3.5}$$

Comparing the bounds (3.4) and (3.5) gives a contradiction, which proves the claimed result.

4. The least prime in a given coset

As before, let $\tilde{H}$ denote the group of characters $\chi \pmod q$ with $\chi(n) = 1$ for all $n \in H$. Recall that $|\tilde{H}| = [G : H] = h$, and given a coset aH we have the orthogonality relation

$$\frac{1}{h}\sum_{\chi \in \tilde{H}} \overline{\chi(a)}\chi(n) = \begin{cases} 1 & \text{if } n \in aH, \\ 0 & \text{if } n \notin aH. \end{cases} \tag{4.1}$$

Note that $q \geq 20000$. Let X be such that no prime below X lies in the coset aH, and we assume below that $X \geq \max(10^9, ((h-1)\log q)^2)$.

Set $S(X,\chi) = \sum_{n \leq X} \Lambda(n)\chi(n)\log(X/n)$, so that

$$\sum_{\substack{n \leq X \\ n \in aH}} \Lambda(n)\log(X/n) = \frac{1}{h}\sum_{\chi \in \tilde{H}} \overline{\chi(a)} S(X,\chi). \tag{4.2}$$

Our strategy is again to obtain upper and lower bounds for the quantity above, and then to derive a contradiction.

4.1. **Preliminary bounds.** First, consider the principal character $\chi_0 \pmod q$ which certainly belongs to the group $\tilde{H}$. Since

$$S(X,\chi_0) = \sum_{n \leq X} \Lambda(n)\log(X/n) - \sum_{\substack{n \leq X \\ (n,q)>1}} \Lambda(n)\log(X/n),$$

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using Lemmas 2.1 and 3.1, we obtain that for $X \geq \max((h-1)^2 \log^2 q, 10^9)$,

$$|S(X, \chi_0) - X| \leq 2|B|(\sqrt{X}+1) + (\log 2\pi)\log X + 1 + \frac{\omega(q)}{2}(\log X)^2$$

$$\leq \frac{\sqrt{X}}{20} + \frac{\omega(q)}{2}(\log X)^2. \tag{4.3}$$

For a non-principal character $\chi \in \tilde{H}$, let $\tilde{\chi}$ (mod $\tilde{q}$) denote the primitive character that induces χ. By Lemma 2.2 we find that

$$|S(X, \tilde{\chi})| \leq (2\sqrt{X}+2)|\mathrm{Re}(B(\tilde{\chi}))| + \Big|\frac{\xi'}{\xi}(0, \tilde{\chi})\Big| \log X + \Big|\frac{1}{2}\Big(\log \frac{\tilde{q}}{\pi}\Big)\log X + \tilde{E}_{\mathfrak{a}}(X)\Big|. \tag{4.4}$$

Now for $X \geq 10^9$ we know that $\tilde{E}_{\mathfrak{a}}(X) < 0$, and examining the definition of $\tilde{E}_{\mathfrak{a}}$ we find that

$$\Big|\frac{1}{2}\Big(\log \frac{\tilde{q}}{\pi}\Big)\log X + \tilde{E}_{\mathfrak{a}}(X)\Big| \leq \frac{1}{2}(\log X)\log\max\Big(\frac{q}{\pi}, 2X\Big). \tag{4.5}$$

Recall from (3.2) that

$$|S(X, \chi)| \leq |S(X, \tilde{\chi})| + \omega(q/\tilde{q})\frac{(\log X)^2}{2}. \tag{4.6}$$

We now invoke Lemma 2.3, taking there $x = 100$. Since $|\xi'/\xi(0, \tilde{\chi})| = |\xi'/\xi(0, \overline{\tilde{\chi}})| \geq |\mathrm{Re}(B(\tilde{\chi}))|$ we obtain

$$\Big|\frac{\xi'}{\xi}(0, \tilde{\chi})\Big|\Big(1 - \frac{2}{10} - \frac{1}{100}\Big) \leq \frac{1}{2}\frac{99}{100}\log\frac{\tilde{q}}{\pi} + \sum_{n \leq 100} \frac{\Lambda(n)}{n}\Big(1 - \frac{n}{100}\Big).$$

It follows with a little computation that

$$\Big|\frac{\xi'}{\xi}(0, \tilde{\chi})\Big| \leq \frac{2}{3}\log\frac{q}{\pi} + 4. \tag{4.7}$$

Further, using Lemma 2.3 (taking there $x = X$, and since $E_{\mathfrak{a}}(X) \leq -2/7$ for $X \geq 10^9$), we find that

$$|\mathrm{Re}(B(\tilde{\chi}))| \leq \Big(1 - \frac{1}{\sqrt{X}}\Big)^{-2}\Big(\frac{1}{2}\Big(1 - \frac{1}{X}\Big)\log\frac{\tilde{q}}{\pi} - \mathrm{Re}\sum_{n \leq X}\frac{\Lambda(n)}{n}\tilde{\chi}(n)\Big(1 - \frac{n}{q}\Big) - \frac{2}{7}\Big).$$

Further,

$$\begin{aligned}\mathrm{Re}\sum_{n \leq X}\frac{\Lambda(n)}{n}\tilde{\chi}(n)\Big(1 - \frac{n}{X}\Big) &\geq \mathrm{Re}\sum_{n \leq X}\frac{\Lambda(n)}{n}\chi(n)\Big(1 - \frac{n}{X}\Big) - \sum_{\substack{n \leq X \\ (n, q/\tilde{q}) > 1}}\frac{\Lambda(n)}{n}\Big(1 - \frac{n}{X}\Big) \\ &\geq \mathrm{Re}\sum_{n \leq X}\frac{\Lambda(n)}{n}\chi(n)\Big(1 - \frac{n}{X}\Big) - \Big(1 - \frac{1}{X}\Big)\sum_{p | (q/\tilde{q})}\frac{\log p}{p-1}.\end{aligned}$$

Therefore, we find that for $X \geq 10^9$, the quantity $(2\sqrt{X}+2)|\mathrm{Re}(B(\tilde{\chi}))|$ is bounded by

$$\leq \Big(\sqrt{X} + \frac{19}{6}\Big)\Big(\log\frac{\tilde{q}}{\pi} + 2\sum_{p | (q/\tilde{q})}\frac{\log p}{p-1} - 2\mathrm{Re}\sum_{n \leq X}\frac{\Lambda(n)}{n}\chi(n)\Big(1 - \frac{n}{X}\Big) - \frac{4}{7}\Big).$$

Consider the quantity above together with the term $\omega(q/\tilde{q})(\log X)^2/2$ appearing in (4.6). Since $\log\tilde{q} \leq \log q - \sum_{p | (q/\tilde{q})}\log p$, and $X \geq 10^9$, we may check that the sum

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of these two quantities is largest when $q/\tilde{q}$ is 6. Putting in this worst case bound, we find that

$$(2\sqrt{X}+2)|\mathrm{Re}(B(\tilde{\chi}))| + \omega(q/\tilde{q})\frac{(\log X)^2}{2} \le (\log X)^2 \\ + \Big(\sqrt{X}+\frac{19}{6}\Big)\Big(\log\frac{q}{\pi}+\log 2-\frac{4}{7}-2\mathrm{Re}\sum_{n\le X}\frac{\Lambda(n)}{n}\chi(n)\Big(1-\frac{n}{X}\Big)\Big).$$

Combining this estimate with (4.4), (4.5), (4.6), and (4.7) we conclude that

$$|S(X,\chi)| \le \Big(\sqrt{X}+\frac{19}{6}\Big)\Big(\log q-\frac{51}{50}-2\mathrm{Re}\sum_{n\le X}\frac{\Lambda(n)}{n}\chi(n)\Big(1-\frac{n}{X}\Big)\Big)+(\log X)^2 \\ + \Big(\frac{2}{3}\log\frac{q}{\pi}+4\Big)(\log X)+\frac{(\log X)}{2}\log\max\Big(\frac{q}{\pi},2X\Big). \tag{4.8}$$

We sum the above over the $h-1$ non-principal characters of $\tilde{H}$. Note that, using the orthogonality relations and Lemma 2.4,

$$-\mathrm{Re}\sum_{\substack{\chi\in\tilde{H}\\ \chi\ne\chi_0}}\sum_{n\le X}\frac{\Lambda(n)}{n}\chi(n)\Big(1-\frac{n}{X}\Big)\le\sum_{n\le X}\frac{\Lambda(n)}{n}\Big(1-\frac{n}{X}\Big)\le\log X-\frac{3}{2}.$$

Thus, we conclude that

$$\sum_{\substack{\chi\in\tilde{H}\\ \chi\ne\chi_0}}|S(X,\chi)| \le \Big(\sqrt{X}+\frac{19}{6}\Big)\Big((h-1)\log q-(h+2)+2\log X\Big)+(h-1)(\log X)^2 \\ +(h-1)\log X\Big(\frac{2}{3}\log\frac{q}{\pi}+4+\frac{1}{2}\log\max\Big(\frac{q}{\pi},2X\Big)\Big).$$

Using that $q\ge 20000$ and $X\ge\max(10^9,((h-1)\log q)^2)$, we may simplify the above bound, and obtain

$$\sum_{\substack{\chi\in\tilde{H}\\ \chi\ne\chi_0}}|S(X,\chi)| \le \sqrt{X}\Big((h-1)\log q-h+\frac{6}{5}+3\log X\Big)+(h-1)(\log X)^2 \\ +(h-1)\frac{\log X}{2}\log\max\Big(\frac{q}{\pi},2X\Big). \tag{4.9}$$

Combining this with (4.3) we obtain a lower bound for the LHS of (4.2).

4.2. **Upper bound for (4.2).** Now we turn to the upper bound. By assumption there are no primes $p\le X$ with $p\in aH$. Therefore,

$$\sum_{\substack{n\le X\\ n\in aH}}\Lambda(n)\log(X/n)\le\sum_{\substack{p^{2k}\le X\\ p^{2k}\in aH}}(\log p)\log(X/p^{2k})+\sum_{p^{2k+1}\le X}(\log p)\log(X/p^{2k+1}).$$

The second term above is, using Lemma 2.1, and a little computation using $X\ge 10^9$,

$$\le\sum_{k\le\log X/(2\log 2)-1/2}(2k+1)\sum_{p\le X^{1/(2k+1)}}(\log p)\log(X^{1/(2k+1)}/p) \\ \le\sum_{k\le\log X/(2\log 2)-1/2}(2k+1)\Big(X^{1/(2k+1)}+\frac{X^{1/(4k+2)}}{20}\Big)\le\frac{\sqrt{X}}{7},$$

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so that

$$\sum_{\substack{n\le X\\ n\in aH}} \Lambda(n)\log(X/n) \le \sum_{\substack{p^{2k}\le X\\ p^{2k}\in aH}} (\log p)\log(X/p^{2k}) + \frac{\sqrt{X}}{7}. \tag{4.10}$$

4.3. **The least prime in an arithmetic progression.** Using a computer we checked Corollary 1.2 when $q \le 20000$. Suppose now that $q > 20000$ and let $a \pmod q$ be an arithmetic progression with $(a,q)=1$ such that no prime below $X = (\phi(q)\log q)^2$ is $\equiv a \pmod q$.

If q has at least six distinct prime factors then $\phi(q) \ge \phi(2\cdot3\cdot5\cdot7\cdot11\cdot13) = 5760$, while if q has at most five prime factors then $\phi(q) \ge q\phi(2\cdot3\cdot5\cdot7\cdot11)/(2\cdot3\cdot5\cdot7\cdot11) > 4155$. In either case, $\phi(q)\ge 4156$. By similar elementary arguments we may check that for $q > 20000$ we have $2^{\omega(q)} \le q^{3/7}$, and that $\phi(q) \ge q^{5/6}$.

Thus $X = (\phi(q)\log q)^2 \ge 10^9$, and we may use our work above; note that here H is the trivial subgroup consisting of just the identity, and $h = \phi(q)$. First we work out the lower bound for the quantity in (4.2). Using (4.3) and (4.9), together with the bounds $2^{\omega(q)} \le q^{3/7}$ and $X \le (q\log q)^2$, we find that

$$\phi(q) \sum_{\substack{n\le x\\ n\equiv a \pmod q}} \Lambda(n)\log(X/n) \ge X - \sqrt{X}\Big((\phi(q)-1)\log q - \frac{24}{25}\phi(q) + \frac{3}{2} + 3(\log X)\Big).$$

To obtain a corresponding upper bound, it remains to estimate the first term in (4.10). Note that the number of square roots of $a \pmod q$ is bounded by $2^{\omega(q)+1} \le 2q^{3/7}$. Since $(\log p)(\log X/p^{2k}) \le (\log X)^2/8$ we find that

$$\sum_{\substack{p^{2k}\le X\\ p^{2k}\in aH}} (\log p)\log(X/p^{2k}) \le \frac{(\log X)^2}{8} \sum_{\substack{n\le\sqrt{X}\\ n^2\equiv a \pmod q}} 1 \le q^{3/7}\frac{(\log X)^2}{4}\Big(\frac{\sqrt{X}}{q}+1\Big).$$

Therefore,

$$\sum_{\substack{n\le x\\ n\equiv a \pmod q}} \Lambda(n)\log(X/n) \le \frac{\sqrt{X}}{7} + q^{3/7}\frac{(\log X)^2}{4}\Big(\frac{\sqrt{X}}{q}+1\Big).$$

Using our bounds $\phi(q)\ge q^{5/6}$, $q > 20000$ and $(q\log q)^2 \ge X \ge 10^9$, a little computation shows that the upper and lower bounds derived above give a contradiction.

4.4. **The general case.** In general we bound the first term in (4.10) crudely by

$$\le 2\sum_{n\le\sqrt{X}} \Lambda(n)\log\frac{\sqrt{X}}{n} \le 2\Big(\sqrt{X} + \frac{X^{1/4}+1}{20}\Big),$$

so that

$$\sum_{\substack{n\le X\\ n\in aH}} \Lambda(n)\log(X/n) \le \frac{11}{5}\sqrt{X}.$$

On the other hand, using (4.3) and (4.9) together with the bounds $2^{\omega(q)} \le q^{3/7}$ and $X \ge \max(10^9, ((h-1)\log q)^2)$, we obtain

$$h\sum_{\substack{n\le X\\ n\in aH}} \Lambda(n)\log(X/n) \ge X - \sqrt{X}\Big((h-1)\log q - \frac{24}{25}h + \frac{5}{4} + \frac{7}{2}(\log X) + \frac{(\log X)^2}{3(h-1)}\Big).$$

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Comparing this with our upper bound, we must have

$$\sqrt{X} \le (h-1)\log q + \frac{5}{4}h + \frac{5}{4} + \frac{7}{2}(\log X) + \frac{(\log X)^2}{3(h-1)}.$$

Since $X \ge 10^9$, we have $\frac{7}{2}\log X + \frac{(\log X)^2}{3} \le \sqrt{X}/100$, so that from the above estimate we may first derive that $\sqrt{X} \le 2(h-1)\log q$. Now inserting this bound into our estimate, we obtain the refined bound

$$\sqrt{X} \le (h-1)\log q + \frac{5}{4}(h+1) + 7\log(2(h-1)\log q) + \frac{4(\log(2(h-1)\log q))^2}{3(h-1)}.$$

The bound stated in Theorem 1.4 follows from this with a little calculation.

5. Explicit bounds for $|L(1,\chi)|$: Proof of Theorem 1.5

5.1. **Upper bounds for $L(1,\chi)$.** Let $q \ge 10^{10}$ be a positive integer and $x \ge 100$ be a real number to be chosen later. Lemma 2.5 gives

$$\log|L(1,\chi)| \le \operatorname{Re}\sum_{n\le x}\frac{\chi(n)\Lambda(n)}{n\log n}\frac{\log(x/n)}{\log x} + \frac{1}{2\log x}\Big(\log\frac{q}{\pi} + \psi_0\Big(\frac{1+\mathfrak{a}}{2}\Big)\Big)$$
$$- \Big(\frac{1}{\log x} - \frac{2}{\sqrt{x}(\log x)^2}\Big)|\operatorname{Re}(B(\chi))| + \frac{2}{x(\log x)^2}.$$

Invoking Lemma 2.3 (with the same value of x above), we have

$$|\operatorname{Re}(B(\chi))| \ge \Big(1+\frac{1}{\sqrt{x}}\Big)^{-2}\Big(\frac{1}{2}\Big(1-\frac{1}{x}\Big)\log\frac{q}{\pi} - \operatorname{Re}\sum_{n\le x}\frac{\Lambda(n)\chi(n)}{n}\Big(1-\frac{n}{x}\Big) + E_{\mathfrak{a}}(x)\Big).$$

Now for $x \ge 100$,

$$-E_{\mathfrak{a}}(x)\Big(1+\frac{1}{\sqrt{x}}\Big)^{-2}\Big(\frac{1}{\log x} - \frac{2}{\sqrt{x}(\log x)^2}\Big) + \frac{1}{2\log x}\psi_0\Big(\frac{1+\mathfrak{a}}{2}\Big) + \frac{2}{x(\log x)^2} \le 0,$$

and therefore

$$\log|L(1,\chi)| \le \operatorname{Re}\sum_{n\le x}\frac{\chi(n)\Lambda(n)}{n\log n}\frac{\log(x/n)}{\log x} + \frac{\log(q/\pi)}{(\sqrt{x}+1)\log x}\Big(1+\frac{1}{\log x}\Big)$$
$$+ \Big(\frac{1}{\log x} - \frac{2}{\sqrt{x}(\log x)^2}\Big)\Big(1+\frac{1}{\sqrt{x}}\Big)^{-2}\operatorname{Re}\sum_{n\le x}\frac{\Lambda(n)\chi(n)}{n}\Big(1-\frac{n}{x}\Big).$$

The right hand side above is largest when $\chi(p) = 1$ for all $p \le x$, and so

$$\log|L(1,\chi)| \le \sum_{n\le x}\frac{\Lambda(n)}{n\log n}\frac{\log(x/n)}{\log x} + \frac{1}{\log x}\sum_{n\le x}\frac{\Lambda(n)}{n}\Big(1-\frac{n}{x}\Big) + \frac{\log q}{(\sqrt{x}+1)\log x}\Big(1+\frac{1}{\log x}\Big).$$

Appealing now to Lemmas 2.4 and 2.6 we conclude that

$$\log|L(1,\chi)| \le \log\log x + \gamma - \frac{1}{\log x} + \frac{\log q}{\sqrt{x}\log x}\Big(1+\frac{1}{\log x}\Big).$$

Choosing $x = (\log q)^2/4$, so that $x \ge 130$ for $q \ge 10^{10}$, we deduce that

$$|L(1,\chi)| \le e^{\gamma}\Big(\log x + 1 + \frac{3.1}{\log x}\Big) \le 2e^{\gamma}\Big(\log\log q - \log 2 + \frac{1}{2} + \frac{1}{\log\log q}\Big).$$

This proves the stated upper bound for $|L(1,\chi)|$.

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5.2. **Lower bounds for $|L(1,\chi)|$.** The argument proceeds similarly to the one for upper bounds. Let $q \ge 10^{10}$ be a positive integer. As before choose $x = (\log q)^2/4$ so that $x \ge 132$. Lemma 2.5 gives

$$\log|L(1,\chi)| \ge \operatorname{Re}\sum_{n\le x}\frac{\Lambda(n)\chi(n)}{n\log n}\frac{\log(x/n)}{\log x} + \frac{1}{2\log x}\Big(\log\frac{q}{\pi} + \psi_0\Big(\frac{1+\mathfrak{a}}{2}\Big)\Big) - \Big(\frac{1}{\log x} + \frac{2}{\sqrt{x}(\log x)^2}\Big)|\operatorname{Re}(B(\chi))| - \frac{2}{x(\log x)^2}.$$

From Lemma 2.3 (with the same value of x above) we find that

$$|\operatorname{Re}(B(\chi))| \le \Big(1-\frac{1}{\sqrt{x}}\Big)^{-2}\Big(\frac{1}{2}\Big(1-\frac{1}{x}\Big)\log\frac{q}{\pi} - \operatorname{Re}\sum_{n\le x}\frac{\Lambda(n)\chi(n)}{n}\Big(1-\frac{n}{x}\Big) + E_{\mathfrak{a}}(x)\Big).$$

Now for $x \ge 100$,

$$-E_{\mathfrak{a}}(x)\Big(1-\frac{1}{\sqrt{x}}\Big)^{-2}\Big(\frac{1}{\log x}+\frac{2}{\sqrt{x}(\log x)^2}\Big) + \frac{1}{2\log x}\psi_0\Big(\frac{1+\mathfrak{a}}{2}\Big) - \frac{2}{x(\log x)^2} \ge 0,$$

and therefore,

$$\log|L(1,\chi)| \ge \operatorname{Re}\sum_{n\le x}\frac{\Lambda(n)\chi(n)}{n\log n}\frac{\log(x/n)}{\log x} - \frac{\log(q/\pi)}{(\sqrt{x}-1)\log x}\Big(1+\frac{1+1/\sqrt{x}}{\log x}\Big) + \Big(1-\frac{1}{\sqrt{x}}\Big)^{-2}\Big(\frac{1}{\log x}+\frac{2}{\sqrt{x}(\log x)^2}\Big)\operatorname{Re}\sum_{n\le x}\frac{\Lambda(n)\chi(n)}{n}\Big(1-\frac{n}{x}\Big). \tag{5.1}$$

From Lemma 2.4 it follows that, for $x \ge 132$,

$$\operatorname{Re}\sum_{n\le x}\frac{\Lambda(n)\chi(n)}{n}\Big(1-\frac{n}{x}\Big) \ge -\log x + 1,$$

and therefore, with a little calculation, we get

$$\Big(\Big(1-\frac{1}{\sqrt{x}}\Big)^{-2}\Big(\frac{1}{\log x}+\frac{2}{\sqrt{x}(\log x)^2}\Big) - \frac{1}{\log x}\Big)\operatorname{Re}\sum_{n\le x}\frac{\Lambda(n)\chi(n)}{n}\Big(1-\frac{n}{x}\Big) \ge -\frac{2}{\sqrt{x}}.$$

Using this bound in (5.1) we obtain

$$\log|L(1,\chi)| \ge \operatorname{Re}\sum_{n\le x}\Lambda(n)\chi(n)\Big(\frac{1}{n\log n}-\frac{1}{x\log x}\Big) - \frac{\log(q/\pi)}{(\sqrt{x}-1)\log x}\Big(1+\frac{1+1/\sqrt{x}}{\log x}\Big) - \frac{2}{\sqrt{x}}. \tag{5.2}$$

The next lemma shows that the sum over n above is smallest when $\chi(p) = -1$ for all $p \le x$.

Lemma 5.1. *For $x \ge 100$ we have*

$$\operatorname{Re}\sum_{n\le x}\Lambda(n)\chi(n)\Big(\frac{1}{n\log n}-\frac{1}{x\log x}\Big) \ge \sum_{p^k\le x}\Lambda(p^k)(-1)^k\Big(\frac{1}{p^k\log p^k}-\frac{1}{x\log x}\Big).$$

Proof. Consider the contribution of the powers of a single prime $p \le x$ to both sides of the inequality above; we claim that the contribution to the left hand side is at least as large as the contribution to the right hand side. If $\chi(p) = 0$ then the contribution to the left hand side is zero, and the contribution to the right hand

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side is negative. If $\chi(p) \neq 0$ then writing $\chi(p) = -e(\theta)$, we see that the difference in the contributions to the left hand side and right hand side equals

$$(\log p) \sum_{k \leq \log x/\log p} (-1)^{k-1}(1-\cos(k\theta))\Big(\frac{1}{p^k \log p^k} - \frac{1}{x \log x}\Big).$$

If $p \geq 3$ then using $(1-\cos(k\theta)) \leq k^2(1-\cos\theta)$ we see that the quantity above is

$$\geq (\log p)(1-\cos\theta)\Big(\frac{1}{p\log p} - \frac{1}{x\log x} - \sum_{j=1}^{\infty} \frac{(2j)^2}{p^{2j}\log p^{2j}}\Big) \geq 0.$$

For $p = 2$ we use the bound $0 \leq (1-\cos(k\theta)) \leq k^2(1-\cos\theta)$ for $k \geq 6$ and compute explicitly the trigonometric polynomial arising from the first five terms. With a little computer calculation the lemma follows in this case. □

Note that

$$\sum_{p^k \leq x} \Lambda(p^k)(-1)^k\Big(\frac{1}{p^k\log p^k} - \frac{1}{x\log x}\Big) = -\sum_{n\leq x} \frac{\Lambda(n)}{n\log n}\frac{\log(x/n)}{\log x}$$
$$- \frac{1}{\log x}\sum_{n\leq x}\frac{\Lambda(n)}{n}\Big(1-\frac{n}{x}\Big) + 2\sum_{m^2\leq x}\Lambda(m)\Big(\frac{1}{m^2\log m^2} - \frac{1}{x\log x}\Big). \tag{5.3}$$

The third term above is

$$\log\zeta(2) - \sum_{m^2 > x}\frac{\Lambda(m)}{m^2\log m} - 2\sum_{m^2\leq x}\frac{\Lambda(m)}{x\log x} \geq \log\zeta(2) - \frac{3}{2\sqrt{x}},$$

where we bound the sums over prime powers trivially by replacing them with sums over odd numbers and powers of 2. The first two terms in (5.3) are handled using Lemmas 2.4 and 2.6. Putting these together, and using $x \geq 132$, we conclude that

$$\sum_{p^k\leq x}\Lambda(p^k)(-1)^k\Big(\frac{1}{p^k\log p^k} - \frac{1}{x\log x}\Big) \geq -\log\log x - \gamma + \log\zeta(2) + \frac{1}{\log x} - \frac{8}{5\sqrt{x}}.$$

Inserting this estimate and Lemma 5.1 into (5.2), and noting that $\log(q/\pi) \leq 2\sqrt{x} - 1$, we obtain with a little calculation

$$\log|L(1,\chi)| \geq -\log\log x - \gamma + \log\zeta(2) - \frac{1}{\log x} - \frac{2}{(\log x)^2} - \frac{4}{\sqrt{x}}.$$

Exponentiating this, and using $x \geq 132$, we obtain the stated lower bound.

6. Asymptotic bounds

Let δ be a fixed positive real number, and let $K(s)$ denote an even function holomorphic in a region containing $-\frac{1}{2} - \delta \leq \mathrm{Re}(s) \leq \frac{1}{2} + \delta$. Further, suppose that for all s in this region we have $|K(s)| \ll 1/(1+|s|^2)$. For $\xi > 0$ define the Mellin transform

$$\widetilde{K}(\xi) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} K(s)\xi^s ds,$$

where the integral is over any vertical line with $-\frac{1}{2} - \delta \leq c \leq \frac{1}{2} + \delta$. Since K is even it follows that $\widetilde{K}(\xi) = \widetilde{K}(1/\xi)$. Further, note that for any $-\frac{1}{2} - \delta \leq \sigma \leq \frac{1}{2} + \delta$, since $|K(s)| \ll 1/(1+|s|^2)$, we have $|\widetilde{K}(\xi)| \ll \xi^\sigma$; thus

$$|\widetilde{K}(\xi)| \ll \min(\xi, 1/\xi)^{\frac{1}{2}+\delta}. \tag{6.1}$$

Finally, we assume that K is such that $\widetilde{K}(\xi) > 0$ for all $\xi > 0$.

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As before, let $G = (\mathbb{Z}/q\mathbb{Z})^*$ and H be a proper subgroup of G. Let X be such that all primes $\ell \nmid q$ with $\ell \le X$ lie in the subgroup H. We may assume that $X \gg (\log q)^2$. Further, let $\tilde{H}$ denote the group of characters $\chi \pmod q$ with $\chi(n) = 1$ for all $n \in H$. Recall that $|\tilde{H}| = [G : H] = h$, and that the orthogonality relation (4.1) holds.

Lemma 6.1. *Assume GRH. Let $\chi \pmod q$ be in $\tilde{H}$. If χ is the principal character then*

$$\sum_n \frac{\Lambda(n)\chi(n)}{\sqrt{n}}\tilde{K}(x/n) = K(1/2)\frac{\sqrt{x}}{2} + O\Big(1 + \log q \frac{\log x}{\sqrt{x}}\Big).$$

If χ is non-principal then

$$\mathrm{Re}\sum_n \frac{\Lambda(n)\chi(n)}{\sqrt{n}}\tilde{K}(x/n) = \theta(1+o(1))\frac{\log q}{2\pi}\int_{-\infty}^{\infty}|K(it)|dt + O\Big(1 + \log q \frac{\log x}{\sqrt{x}}\Big).$$

Proof. Consider first the case when χ is non-principal. Let $\tilde{\chi} \pmod{\tilde{q}}$ denote the primitive character that induces χ and let $\xi(s, \tilde{\chi})$ denote the corresponding completed L-function. We start with

$$I = \frac{1}{2\pi i}\int_{(1/2+\delta)} -\frac{\xi'}{\xi}(s + \tfrac{1}{2}, \tilde{\chi})K(s)(x^s + x^{-s})ds, \tag{6.2}$$

which we evaluate, as usual, in two ways. Note that

$$\frac{1}{2\pi i}\int_{(1/2+\delta)}\Big(-\frac{1}{2}\log\frac{\tilde{q}}{\pi}\Big)K(s)(x^s + x^{-s})ds = -\log\frac{\tilde{q}}{\pi}\tilde{K}(x) \ll \frac{\log q}{\sqrt{x}}.$$

Further, by moving the line of integration to $\mathrm{Re}(s) = 0$, we find that

$$\frac{1}{2\pi i}\int_{(1/2+\delta)} -\frac{1}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{1/2 + s + \mathfrak{a}}{2}\Big)K(s)(x^s + x^{-s})ds \ll \int_{-\infty}^{\infty}\log(2+|t|)|K(it)|dt \ll 1.$$

Now by expanding $-L'/L$ into its Dirichlet series we have

$$\frac{1}{2\pi i}\int_{(1/2+\delta)} -\frac{L'}{L}(\tfrac{1}{2} + s, \tilde{\chi})K(s)(x^s + x^{-s})ds = \sum_n \frac{\Lambda(n)\tilde{\chi}(n)}{\sqrt{n}}(\tilde{K}(x/n) + \tilde{K}(xn)).$$

Note that

$$\sum_n \frac{\Lambda(n)\tilde{\chi}(n)}{\sqrt{n}}\tilde{K}(xn) \ll \sum_n \frac{\Lambda(n)}{\sqrt{n}}(xn)^{-\frac{1}{2}-\delta} \ll \frac{1}{\sqrt{x}}.$$

Moreover,

$$\sum_n \frac{\Lambda(n)\tilde{\chi}(n)}{\sqrt{n}}\tilde{K}(x/n) - \sum_n \frac{\Lambda(n)\chi(n)}{\sqrt{n}}\tilde{K}(x/n) \ll \sum_{(n,q)>1}\frac{\Lambda(n)}{\sqrt{n}}|\tilde{K}(x/n)|,$$

and using (6.1) this is

$$\ll \sum_{p|q}(\log p)\sum_{k=1}^{\infty}\frac{1}{p^{k/2}}\min\Big(\frac{\sqrt{x}}{p^{k/2}}, \frac{p^{k/2}}{\sqrt{x}}\Big) \ll \sum_{p|q}(\log p)\Big(\frac{1}{\sqrt{x}} + \frac{\log x}{\sqrt{x}\log p}\Big) \ll (\log q)\frac{\log x}{\sqrt{x}}.$$

From these observations, we conclude that

$$I = \sum_n \frac{\Lambda(n)\chi(n)}{\sqrt{n}}\tilde{K}(x/n) + O\Big(1 + \log q\frac{\log x}{\sqrt{x}}\Big). \tag{6.3}$$

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Now evaluate the integral in (6.2) by shifting the line of integration to $\mathrm{Re}(s) = -\frac{1}{2} - \delta$. Thus, with γ running over the ordinates of zeros of $\xi(s, \tilde{\chi})$,

$$I = -\sum_{\gamma} K(i\gamma)(2\cos(\gamma \log x)) + \frac{1}{2\pi i}\int_{(-\frac{1}{2}-\delta)} -\frac{\xi'}{\xi}(\tfrac{1}{2}+s, \tilde{\chi})K(s)(x^s + x^{-s})ds.$$

Using now the functional equation for ξ, and a change of variables, we find that the integral on the right hand side above equals $-\overline{I}$. Thus

$$\mathrm{Re}(I) = \theta \sum_{\gamma} |K(i\gamma)| = \theta(1+o(1))\frac{\log \tilde{q}}{2\pi}\int_{-\infty}^{\infty} |K(it)|dt,$$

where the final estimate follows from an application of the explict formula (see Theorem 5.12 of [11]). This establishes our lemma for non-principal characters. The argument for the principal character is similar, the only difference being the pole of $\zeta(s)$ at $s = 1$. We omit the details. □

Proposition 6.1. *Assume GRH. Keep the notations above, and recall that X is such that all primes $\ell \nmid q$ with $\ell \le X$ lie in the subgroup H. Then, for any $\lambda > 0$ we have*

$$\Big(h\int_0^{\lambda} \tilde{K}(u)\frac{du}{\sqrt{u}} - \frac{K(1/2)}{2}\Big)\sqrt{X} \le (1+o(1))\sqrt{\lambda}(h-1)(\log q)\Big(\frac{1}{2\pi}\int_{-\infty}^{\infty}|K(it)|dt\Big).$$

Proof. We may assume that $X \gg (\log q)^2$. We shall use Lemma 6.1 for each character $\chi \in \tilde{H}$, and taking $x = X/\lambda$. Thus we obtain that

$$\sum_{\chi \in \tilde{H}} \mathrm{Re}\sum_n \frac{\Lambda(n)\chi(n)}{\sqrt{n}}\tilde{K}(x/n) \le K(1/2)\frac{\sqrt{x}}{2} + (1+o(1))(h-1)\log q\Big(\frac{1}{2\pi}\int_{-\infty}^{\infty}|K(it)|dt\Big).$$

From (4.1), and since $\tilde{K}(x/n) \ge 0$ for all n, we see that the left hand side above is

$$\ge h\sum_{n \le X}\frac{\Lambda(n)}{\sqrt{n}}\tilde{K}(x/n) + O\Big(h\sum_{(n,q)>1}\frac{\Lambda(n)}{\sqrt{n}}\tilde{K}(x/n)\Big) = h\int_1^X \tilde{K}(x/t)\frac{dt}{\sqrt{t}} + o(h\log q).$$

Since $\tilde{K}(u) = \tilde{K}(1/u)$, after a change of variables $u = t/x$, the above becomes

$$h\sqrt{x}\int_0^{\lambda}\tilde{K}(u)\frac{du}{\sqrt{u}} + o(h\log q).$$

The proposition follows with a little rearrangement. □

6.1. **Limitations of the method.** By Mellin inversion we have that $K(1/2) = \int_0^{\infty}\tilde{K}(u)du/\sqrt{u}$, and note also that $\tilde{K}(u) \le \frac{1}{2\pi}\int_{-\infty}^{\infty}|K(it)|dt$. Therefore,

$$h\int_0^{\lambda}\tilde{K}(u)\frac{du}{\sqrt{u}} - \frac{K(1/2)}{2} \le (h-1/2)\int_0^{\lambda}\tilde{K}(u)\frac{du}{\sqrt{u}} \le (2h-1)\sqrt{\lambda}\frac{1}{2\pi}\int_{-\infty}^{\infty}|K(it)|dt.$$

Hence Proposition 6.1 cannot lead to a bound for X that is better than $(\frac{h-1}{2h-1} + o(1))^2(\log q)^2$.

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6.2. **Bounds for large h.** Take $K(s) = (\frac{(e^{\alpha s}-e^{-\alpha s})}{s})^2$ and $\lambda = 1$. Note that the Mellin transform for K is $\tilde{K}(u) = \max(0, 2\alpha - |\log u|)$. Now

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |K(it)|dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\frac{2\sin(\alpha t)}{t}\Big)^2 dt = 2\alpha$$

and

$$\int_0^1 \tilde{K}(u)\frac{du}{\sqrt{u}} = \int_{e^{-2\alpha}}^1 \frac{2\alpha+\log u}{\sqrt{u}}du = 4\alpha - 4 + 4e^{-\alpha}.$$

Thus Proposition 6.1 implies that

$$\Big(h(4\alpha - 4 + 4e^{-\alpha}) - 2(e^{\alpha/2} - e^{-\alpha/2})^2\Big)\sqrt{X} \leq (1+o(1))(2\alpha(h-1))(\log q).$$

Selecting $\alpha = \frac{1}{2}\log(2h)$, it follows that for $h \geq 4$,

$$X \leq \Big(\frac{1}{4} + o(1)\Big)\Big(1 - \frac{1}{h}\Big)^2\Big(\frac{\log(2h)}{\log(2h)-2}\Big)^2(\log q)^2.$$

For large h, the above bound is about $(1/4 + o(1))(\log q)^2$ which is of the same quality as the limit of the method, but the convergence to this limit is quite slow.

6.3. **Bounds for small h.** For smaller values of h, good estimates may be obtained using the kernel $K(s) = -(\Gamma(s) + \Gamma(-s))$. Note that $K(s)$ is even and holomorphic in the region $-1 < \sigma < 1$. Further, for any $0 < c < 1$,

$$\begin{aligned}\tilde{K}(u) &= \frac{1}{2\pi i}\int_{(c)} -(\Gamma(s)+\Gamma(-s))u^s ds = -e^{-1/u} - \frac{1}{2\pi i}\int_{(c)}\Gamma(-s)u^s ds \\ &= -e^{-1/u} - \frac{1}{2\pi i}\int_{(-c)}\Gamma(s)u^{-s}ds = 1 - e^{-1/u} - e^{-u}.\end{aligned}$$

Thus $\tilde{K}(u) > 0$ for all $u > 0$, and Proposition 6.1 applies. A computer calculation shows that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|K(it)|dt = \frac{1}{\pi}\int_{-\infty}^{\infty}|\mathrm{Re}\ (\Gamma(it))|dt = 0.291\ldots.$$

Further, $K(1/2) = \sqrt{\pi}$. Thus Proposition 6.1 yields

$$\sqrt{X}\Big(h\int_0^\lambda \frac{\tilde{K}(u)}{\sqrt{u}}du - \frac{\sqrt{\pi}}{2}\Big) \leq (0.291\ldots + o(1))\sqrt{\lambda}(h-1)\log q.$$

When $h = 2$, we choose $\lambda = 8.35$, which is more or less optimal, and find that $X < (0.42 + o(1))(\log q)^2$. For $h = 3$, we choose $\lambda = 6.55$, and find that $X < (0.49 + o(1))(\log q)^2$. In general, since the ratio increases as h increases for $\lambda < 4$, we find that $X < (0.51 + o(1))(\log q)^2$ for all h, by letting $h \to \infty$ and $\lambda = 3.9$. These are essentially optimal choices for $h > 3$.

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