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Conditional channel simulation

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In this work we design a specific simulation tool for quantum channels which is based on the use of a control system. This allows us to simulate an average quantum channel which is expressed in terms of an ensemble of channels, even when these channel-components are not jointly teleportation-covariant. This design is also extended to asymptotic simulations, continuous ensembles, and memory channels. As an application, we derive relative-entropy-of-entanglement upper bounds for private communication over various channels, from the amplitude damping channel to non-Gaussian mixtures of bosonic lossy channels. Among other results, we also establish the two-way quantum and private capacity of the so-called "dephrasure" channel.

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I. INTRODUCTION

In quantum information theory [1, 2], the simulation of quantum channels has a long history which dates back to 1996 [3] soon after the introduction of the teleportation protocol [4, 5]. Indeed the first idea of simulating a Pauli channel by teleporting over a two-qubit mixed state was re-visited in various papers (e.g., see Ref. [6]). The most general formulation of channel simulation based on local operation and classical communication (LOCC) has been given in Ref. [7] and allows one to simulate both discrete- and continuous-variable channels [8]. This is also known as LOCC-simulation of a quantum channel (see Ref. [9] for an extensive review on the topic).

Similar ideas were put forward by Nielsen and Chuang [10] in the context of discrete-variable quantum computing. Ref. [10] introduced the notion of quantum programmable gate array (QPGA) where a channel $\mathcal{E}$ is simulated by inserting its input ρ and a program state σ into a unitary operation G so that $\mathcal{E}(\rho) = \mathrm{Tr}_{\mathrm{prog}}[G(\sigma_{\mathrm{prog}} \otimes \rho)]$. For an arbitrary channel $\mathcal{E}$ this is always possible as long as the operation G can be performed over arbitrarily many ancillary systems (i.e., arbitrarily large programs). This can also be understood in the context of port-based teleportation (PBT) [11–14], which allows for perfect simulations in the limit of many ports. Indeed, PBT not only provides a design for the QPGA but also shows that it can be based on a teleportation-like LOCC.

The applications of channel simulations are various. One of the most important is certainly the simplification of adaptive (i.e., feedback-based) quantum protocols into corresponding block (i.e., non feedback) versions. This is achieved by replacing the channels with their simulations and to apply a suitable re-organization of the adaptive operations of the protocol, in such a way to decompose the output state into a tensor product of program states up to a single quantum operation. This adaptive-to-block reduction is also known as (teleportation) stretching of the protocol [7] and can be applied to both discrete- and continuous-variable settings (see Ref. [9] for a review of the various techniques of adaptive-to-block reduction). Combing teleportation stretching with the relative entropy of entanglement (REE) [15–17], Ref. [7] computed the tightest single-letter upper bounds for the secret key capacity of many quantum channels, also establishing the two-way quantum and private capacities of several fundamental ones, including the bosonic lossy channel.

In this work, we consider the general case of a quantum channel which can be expressed as an ensemble of *channel components* with an arbitrary probability distribution. Our aim is to design a LOCC simulation for the average channel in terms of the single simulations associated with the various components. The rationale behind this goal is because these components may have simple simulations (e.g., with program states given by their Choi matrices) while the average channel does not have a simple or known simulation *per se*. For instance the components may be Gaussian channels, while the average channel can be highly non-Gaussian. Furthermore, the channel components do not need to be jointly teleportation covariant, which is the condition that would allow for the direct simulation of the average channel via its Choi matrix.

As we discuss below, this is possible by introducing a system which controls the channel components and, therefore, creates a conditional form of channel simulation. The state of this system will be part of the final program state associated with the average channel. In this way, we can apply teleportation stretching and write single-letter upper bound for the secret key capacity K of the average channel in terms of the REE of the program states associated with the single components.

As an application, we provide the first finite-dimensional simulation of the amplitude damping channel deriving the tightest REE upper bound for its K. We also establish K and all the other two-way assisted capacities of the "dephrasure" channel [18] which is a specific example of erasure pipeline, i.e., a channel followed by the erasure channel. We then extend the conditional channel simulation to bosonic channels, continuous ensembles, and memory channels. In particular, we compute REE upper bounds for various non-Gaussian bosonic channels which can be expressed as mixtures of lossy channels.

II. SIMULATION OF CHANNEL MIXTURES

A. General scenario

Let us consider a mixture of quantum channels $\mathcal{E}_i$ with probability distribution p_i, i.e., the average quantum channel

$$\mathcal{E} = \sum_i p_i \mathcal{E}_i \ . \quad (1)$$

It is clear that the Choi matrix [19] of the average channel $\rho_{\mathcal{E}}$ is equal to the convex combination of the individual Choi matrices $\rho_{\mathcal{E}_i}$, i.e.,

$$\rho_{\mathcal{E}} = \sum_i p_i \rho_{\mathcal{E}_i} \ . \quad (2)$$

Now assume that we know the LOCC simulation of each channel $\mathcal{E}_i$, i.e., we may write [7]

$$\mathcal{E}_i(\rho_T) = \mathcal{L}_i^{PT\to T}(\sigma_P^i \otimes \rho_T), \quad (3)$$

for some trace-preserving LOCC $\mathcal{L}_i$ and some program state σ_P^i of an extra system P which can be further divided in two subsystems A and B. In particular, for a teleportation covariant [20] channel $\mathcal{E}_i$, we know that $\mathcal{L}_i$ is (generalized) teleportation and σ_P^i is given by the Choi matrix of the channel, i.e., $\rho_{\mathcal{E}_i} := \mathcal{I}_A \otimes \mathcal{E}_i^B(\Phi_{AB})$ where $\mathcal{I}_A$ is the identity channel over A and $\Phi_{AB} := |\Phi\rangle\langle\Phi|$, with $|\Phi\rangle$ being the Bell state $|\Phi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$.

In the specific case where $\mathcal{L}_i = \mathcal{L}$ for any i, we call an ensemble $\{\mathcal{E}_i\}$ jointly-simulable. For such an ensemble we may write the joint simulation

$$\mathcal{E}(\rho_T) = \mathcal{L}(\sigma_P \otimes \rho_T), \ \sigma_P := \sum_i p_i \sigma_P^i. \quad (4)$$

In particular, the ensemble is called jointly teleportation-covariant if each $\mathcal{E}_i$ is teleportation-covariant with exactly the same teleportation LOCC $\mathcal{L}_i = \mathcal{L}$. In such a case we may write Eq. (4) where $\mathcal{L}$ is teleportation and the program state becomes $\sigma_P = \sum_i p_i \rho_{\mathcal{E}_i}$.

In general, the previous condition of joint simulability does not hold and it is not known how to simulate the average channel $\mathcal{E}$ starting from the single simulations $\{\sigma_P^i, \mathcal{L}_i\}$ of the components $\mathcal{E}_i$. We now show how this is possible by extending the idea to a control-target scenario, where the simulations are conditional.

B. Conditional channel simulation

Consider the classical state

$$\pi_C := \sum_i p_i |i\rangle_C \langle i| , \quad (5)$$

where $|i\rangle$ is the computational orthonormal basis of a control qudit C whose dimension is equal to the number N of elements in the ensemble $\{\mathcal{E}_i\}$. Let us then introduce the quantum operator [21]

$$M := \sum_i |i\rangle_C \langle i| \otimes \mathcal{E}_i, \quad (6)$$

so that we may write

$$\mathcal{E}(\rho_T) = \mathrm{Tr}_C \left[M(\pi_C \otimes \rho_T) \right]. \quad (7)$$

This is also depicted in Fig. 1(a).

Now, we may replace $\mathcal{E}_i$ with its simulation of Eq. (3) as also shown in Fig. 1(b),

$$M(\rho_{CT}) = \sum_i |i\rangle_C \langle i| \otimes \mathcal{L}_i^{PT\to T}(\sigma_P^i \otimes \rho_{CT}) \ . \quad (8)$$

As a result, inserting the above equation into Eq. (7), we may write

$$\mathcal{E}(\rho_T) = \mathrm{Tr}_C \left[\sum_i p_i |i\rangle_C \langle i| \otimes \mathcal{L}_i^{PT\to T}(\sigma_P^i \otimes \rho_T) \right] \quad (9)$$
$$= \mathcal{L}_{CPT\to T}(\theta_{CP} \otimes \rho_T), \quad (10)$$

where we introduce the "control-program" state

$$\theta_{CP} := \sum_i p_i |i\rangle_C \langle i| \otimes \sigma_P^i, \quad (11)$$

and the "control-program-target" LOCC

$$\mathcal{L}_{CPT\to T}(\rho) := \mathrm{Tr}_C \left[\sum_i |i\rangle_C \langle i| \otimes \mathcal{L}_i^{PT\to T}(\rho) \right]. \quad (12)$$

The final representation of Eq. (10) is also shown in Fig. 1(c).

C. Stretching and single-letter bounds

We may use the channel simulation of Eq. (10) to stretch an adaptive protocol of private communication over the average channel $\mathcal{E} = \sum_i p_i \mathcal{E}_i$. Assuming that Alice and Bob have local registers $\mathbf{a}$ and $\mathbf{b}$, and they perform adaptive LOCCs between each channel transmission, we may apply the procedure of Ref. [7] and write Alice and Bob's n-use output state as

$$\rho_{\mathbf{ab}}^n = \Lambda(\theta_{CP}^{\otimes n}), \quad (13)$$

where Λ is a trace-preserving LOCC including the adaptive LOCCs of the protocol and the simulation LOCCs, while θ_{CP} is the control-program state of Eq. (11).

Using results from Ref. [7], we may bound the key rate achievable by any adaptive protocol of key generation over $\mathcal{E}$. Consider an ε-secure protocol with output $\rho_{\mathbf{ab}}^n$ where $\|\rho_{\mathbf{ab}}^n - \phi_n\| < \varepsilon$ and ϕ_n is a private state with nR_n secret bits. Then, the n-use key rate R_n^ε must satisfy

$$R_n^\varepsilon \leq \frac{E_R(\rho_{\mathbf{ab}}^n) + 2H_2(\varepsilon)}{(1-4\varepsilon\alpha)n}, \quad (14)$$

where α is a constant parameter associated to the dimension of the private state ϕ_n [9] and $H_2(\varepsilon) = -\varepsilon \log_2 \varepsilon - (1-\varepsilon)\log_2(1-\varepsilon)$ is the binary Shannon entropy. The previous bound is simplified by using Eq. (13) and basic

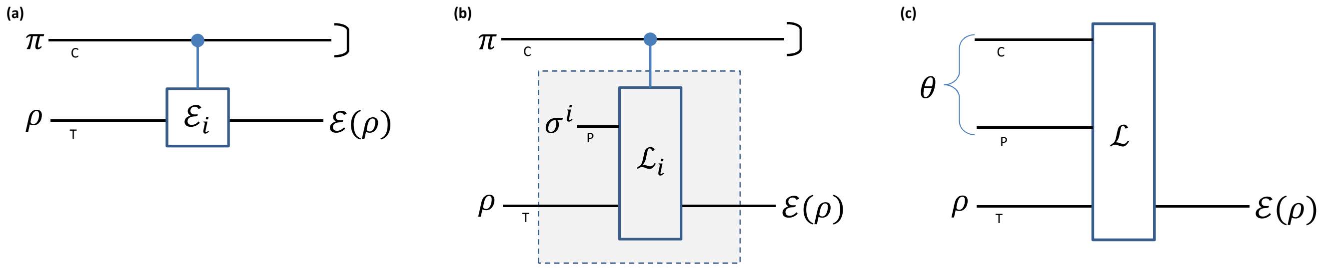

FIG. 1: Steps for conditional channel simulation. See text for explanations.

properties of the REE. In fact, we may write

$$E_R(\rho^n_{\mathbf{ab}}) \overset{(1)}{\leq} E_R(\theta_{CP}^{\otimes n}) \tag{15}$$
$$\overset{(2)}{\leq} nE_R(\theta_{CP}) \tag{16}$$
$$\overset{(3)}{=} nE_R\left(\textstyle\sum_i p_i \left|i\right\rangle_C \left\langle i\right| \otimes \sigma^i_P\right) \tag{17}$$
$$\overset{(4)}{\leq} n\textstyle\sum_i p_i E_R(\left|i\right\rangle_C \left\langle i\right| \otimes \sigma^i_P) \tag{18}$$
$$\overset{(5)}{\leq} n\textstyle\sum_i p_i E_R(\sigma^i_P), \tag{19}$$

where we have used: (1) the monotonicity of the REE under trace-preserving LOCCs as Λ; (2) the subadditivity of the REE over tensor products; (3) the definition of control-program state θ_{CP}; (4) the convexity of the REE over mixtures of states [22]; and (5) the subadditivity of the REE over the tensor product $\left|i\right\rangle_C \left\langle i\right| \otimes \sigma^i_P$ where we may always assume that the separable state $\left|i\right\rangle_C \left\langle i\right|$ belongs to Alice. More precisely, let us set $P = AB$ and denote by $\sigma^{\text{sep}}_{CA|B}$ a state which is separable with respect to the split $CA|B$. Then, in terms of the relative entropy $S(.||.)$, we may write

$$\begin{aligned}
&E_R(\left|i\right\rangle_C \left\langle i\right| \otimes \sigma^i_{AB}) \\
&= \inf_{\sigma^{\text{sep}}_{CA|B}} S(\left|i\right\rangle_C \left\langle i\right| \otimes \sigma^i_{AB} \;||\; \sigma^{\text{sep}}_{CA|B}) \\
&\leq \inf_{\sigma^{\text{sep}}_{A|B}} S(\left|i\right\rangle_C \left\langle i\right| \otimes \sigma^i_{AB} \;||\; \left|i\right\rangle_C \left\langle i\right| \otimes \sigma^{\text{sep}}_{A|B}) \\
&= \inf_{\sigma^{\text{sep}}_{A|B}} S(\sigma^i_P \;||\; \sigma^{\text{sep}}_{A|B}) := E_R(\sigma^i_{AB}).
\end{aligned} \tag{20}$$

By replacing Eq. (19) in Eq. (14), we therefore derive

$$R^\varepsilon_n \leq \frac{\sum_i p_i E_R(\sigma^i_P)}{1-4\varepsilon\alpha} + \frac{2H_2(\varepsilon)}{(1-4\varepsilon\alpha)n}. \tag{21}$$

Now, by taking the limit for large n and small ε (weak converse), we may write

$$\lim_{n,\varepsilon} R^\varepsilon_n \leq \textstyle\sum_i p_i E_R(\sigma^i_P). \tag{22}$$

Finally, by taking the supremum over all adaptive key generation protocols $\mathcal{P}$, we get the secret key capacity of the channel

$$K(\mathcal{E}) = \sup_{\mathcal{P}} \lim_{n,\varepsilon} R^\varepsilon_n \leq \textstyle\sum_i p_i E_R(\sigma^i_P). \tag{23}$$

This is expressed in terms of the program states σ^i_P of the channel components $\mathcal{E}_i$ [23]. Recall that, for an arbitrary channel $\mathcal{E}$, we may write the chain of (in)equalities

$$D_2(\mathcal{E}) = Q_2(\mathcal{E}) \leq P_2(\mathcal{E}) = K(\mathcal{E}), \tag{24}$$

where D_2 is the two-way assisted entanglement distribution capacity, Q_2 is the two-way assisted quantum capacity, and P_2 is the two-way assisted private capacity. Therefore, Eq. (23) is an upper bound for all the capacities in Eq. (24).

III. APPLICATIONS IN FINITE DIMENSION

A. Amplitude damping channel

Here we apply the result to the amplitude damping channel improving the REE bound given in Ref. [7]. Recall that this channel may be represented as

$$\mathcal{E}^{\text{damp}}_p = p\mathcal{E}_0 + (1-p)\mathcal{E}_1, \tag{25}$$

where $\mathcal{E}_0(\rho) := \text{Tr}(\rho) \left|0\right\rangle \left\langle 0\right|$ and $\mathcal{E}_1 = \mathcal{I}$ is the identity channel. The channel $\mathcal{E}_0$ is teleportation covariant and entanglement-breaking, so that it allows for a LOCC simulation with a separable program state and, accordingly, $E_R = 0$. At the same time, $\mathcal{E}_1 = \mathcal{I}$ is teleportation covariant with $E_R(\rho_{\mathcal{I}}) = 1$. Therefore, from Eq. (23), it is easy to compute

$$K(\mathcal{E}^{\text{damp}}_p) \leq 1-p. \tag{26}$$

Note that $\mathcal{E}_0$ are $\mathcal{E}_1$ are not *jointly* teleportation covariant. In fact, given a Pauli operator $P \in \{I, X, Y, Z\}$, this is exactly commuted by the identity, but different is the case for $\mathcal{E}_0$ for which we have

$$\mathcal{E}_0(Z\rho Z^\dagger) = Z\mathcal{E}_0(\rho)Z^\dagger, \quad \mathcal{E}_0(X\rho X^\dagger) = \mathcal{E}_0(\rho). \tag{27}$$

Since the output unitaries become different for the two channel components, these are not jointly teleportation covariant and the amplitude damping channel is not teleportation covariant. For this reason, we cannot write $K(\mathcal{E}_p^{\text{damp}}) \leq E_R(\rho_{\mathcal{E}_p^{\text{damp}}})$.

Nonetheless, since each $\mathcal{E}_i$ in Eq. (25) is individually teleportation covariant, we can use the conditional channel simulation that allows us to write the upper bound of Eq. (23) in terms of the Choi matrices of the components. The resulting bound, firstly presented in this paper, is therefore the only upper bound that can be obtained with a finite-dimensional simulation of the amplitude damping channel. In fact, so far, this is the only known finite-dimensional simulation of this channel.

One may easily check that the result in Eq. (26) outperforms the previously-known REE upper bound of $-\log p$ [7], obtained by simulating the amplitude damping with continuous variable systems, but not the best-known squashed entanglement upper bound, also derived in Ref. [7] (see Fig. 8(b) therein). However, the very simple form of the REE bound in Eq. (26) has the advantage to make it easily extendable to repeater chains and quantum networks [24].

B. Erasure pipeline

Consider an arbitrary qubit channel $\mathcal{N}$ which is followed by an erasure channel $\mathcal{E}_p^{\text{erase}}$ mapping the input state into an orthogonal erasure state $|e\rangle$ with probability p. Explicitly we may write the erasure pipeline $\mathcal{E}_p^{\text{pipe}} := \mathcal{E}_p^{\text{erase}} \circ \mathcal{N}$ as follows

$$\mathcal{E}_p^{\text{pipe}} = (1-p)\mathcal{N} + p\mathcal{E}_e, \tag{28}$$

$$\mathcal{E}_e(\rho) := \text{Tr}(\rho)|e\rangle\langle e| \ . \tag{29}$$

Assume that $\mathcal{N}$ can be LOCC-simulated with a program state $\sigma_{\mathcal{N}}$. We may write a conditional channel simulation for $\mathcal{E}_p^{\text{pipe}}$ and then use Eq. (23) to derive the upper bound

$$K(\mathcal{E}_p^{\text{pipe}}) \leq (1-p)E_R(\sigma_{\mathcal{N}}). \tag{30}$$

Here we use the fact that the channel $\mathcal{E}_e$ is teleportation covariant and entanglement-breaking ($E_R = 0$). It is clear that Eq. (30) also applies to a pipeline of a d-dimensional qudit channel $\mathcal{N}_d$ followed by a d-dimensional erasure channel [whose output is therefore $(d+1)$-dimensional].

C. Dephrasure channel

As an example of erasure pipeline, consider the "dephrasure channel" [18], which is a dephasing channel $\mathcal{E}_q^{\text{deph}}$ with dephasing probability q, followed by an erasure channel $\mathcal{E}_p^{\text{erase}}$. Explicitly we may write the dephrasure channel $\mathcal{E}_{p,q}^{\text{dr}} := \mathcal{E}_p^{\text{erase}} \circ \mathcal{E}_q^{\text{deph}}$ as follows

$$\mathcal{E}_{p,q}^{\text{dr}}(\rho) = (1-p)\left[(1-q)\rho + qZ\rho Z\right] + p\mathcal{E}_e(\rho), \tag{31}$$

where Z is the phase-flip Pauli operator. Note that the channel components $\mathcal{E}_q^{\text{deph}}$ and $\mathcal{E}_e$ are teleportation-covariant but not jointly. Using Eq. (30) with the fact that the dephasing channel is simulable with its Choi matrix $\rho_{\mathcal{E}_q^{\text{deph}}}$, we derive

$$K(\mathcal{E}_p^{\text{pipe}}) \leq (1-p)E_R\left(\rho_{\mathcal{E}_q^{\text{deph}}}\right) = (1-p)[1-H_2(q)], \tag{32}$$

where H_2 is the usual binary Shannon entropy.

Now we prove that the previous relation holds with an equality. In fact, assume that, at the output of the channel, we use a dichotomic measurement with operators $|e\rangle\langle e|$ and $I - |e\rangle\langle e|$. This measurement fully decodes the second (erasure) channel $\mathcal{E}_p^{\text{erase}}$, i.e., with probability $1-p$ we post-select the first (dephasing) channel $\mathcal{E}_q^{\text{deph}}$. It is then known that the two-way entanglement distribution capacity D_2 of $\mathcal{E}_q^{\text{deph}}$ is equal to $1 - H_2(q)$ [7]. As a result, an asymptotically achievable rate for entanglement distribution over a dephrasure channel is equal to

$$D_2(\mathcal{E}_{p,q}^{\text{dr}}) \geq (1-p)[1-H_2(q)]. \tag{33}$$

From Eqs. (32) and (33) we therefore conclude the exact formulas

$$\begin{aligned} Q_2(\mathcal{E}_{p,q}^{\text{dr}}) &= D_2(\mathcal{E}_{p,q}^{\text{dr}}) = P_2(\mathcal{E}_{p,q}^{\text{dr}}) \\ &= K(\mathcal{E}_{p,q}^{\text{dr}}) = (1-p)[1-H_2(q)]. \end{aligned} \tag{34}$$

Note that we cannot achieve the lower bound in Eq. (33) using the reverse coherent information (RCI) of the channel [25]. In fact, let us write the Kraus decomposition of the dephrasure channel, which is

$$\mathcal{E}_{p,q}^{\text{dr}}(\rho) = \sum_{k=0}^{3} E_k \rho E_k^{\dagger}, \tag{35}$$

with operators

$$E_0 = \sqrt{(1-p)(1-q)}(|0\rangle\langle 0| + |1\rangle\langle 1|) \ , \tag{36}$$

$$E_1 = \sqrt{(1-p)q}(|0\rangle\langle 0| - |1\rangle\langle 1|) \ , \tag{37}$$

$$E_2 = \sqrt{p}|e\rangle\langle 0|, \ E_3 = \sqrt{p}|e\rangle\langle 1| \ . \tag{38}$$

We then find its Choi matrix

$$\rho_{\mathcal{E}_{p,q}^{\text{dr}}} = \frac{1-p}{2}|\Phi\rangle\langle\Phi| + \frac{p}{2}(|0e\rangle\langle 0e| + |1e\rangle\langle 1e|) \tag{39}$$

$$- q(|00\rangle\langle 11| + |11\rangle\langle 00|), \tag{40}$$

where $|\Phi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. As a result, we compute the RCI of the dephrasure channel to be

$$I_{\text{RC}}(\mathcal{E}_{p,q}^{\text{dr}}) = (1-p)[1-H_2(q)] - H_2(p) \ . \tag{41}$$

This expression correctly reduces to $1 - p - H_2(p)$ when $q = 0$, which is the RCI of the erasure channel [7]. Because $\mathcal{E}_{p,q}^{\text{dr}}$ is not unital, we have that its RCI is different from its coherent information, which is given by [18]

$$I_{\text{C}}(\mathcal{E}_{p,q}^{\text{dr}}) = (1-p)[1-H_2(q)] - p \ . \tag{42}$$

IV. EXTENSION TO CONTINUOUS VARIABLES

A. Asymptotic simulations

The conditional channel simulation can be extended to ensembles of channels having asymptotic simulations, such as bosonic channels or the amplitude damping channel [7]. This means that we may consider an average channel $\mathcal{E} = \sum_i p_i \mathcal{E}_i$ where each channel component $\mathcal{E}_i$ may have a generally-asymptotic LOCC simulation of the form [7]

$$\mathcal{E}_i(\rho_T) = \lim_\mu \mathcal{L}_{i,\mu}^{PT\to T}(\sigma_P^{i,\mu} \otimes \rho_T), \tag{43}$$

where $\mathcal{L}_{i,\mu}^{PT\to T}$ is a sequence of LOCCs and $\sigma_P^{i,\mu}$ is a sequence of program states. For instance, $\mathcal{E}_i$ may be a teleportation-covariant bosonic channel, so that we may choose a sequence of Choi-approximating program states

$$\sigma_P^{i,\mu} = \rho_{\mathcal{E}_i}^\mu := \mathcal{I} \otimes \mathcal{E}_i(\Phi^\mu), \tag{44}$$

where Φ^μ is a two-mode squeezed vacuum (TMSV) state with variance μ [8].

In general, we may therefore write the following simulation for the average channel

$$\mathcal{E}(\rho_T) = \lim_\mu \mathcal{L}_{CPT\to T}^\mu(\theta_{CP}^\mu \otimes \rho_T), \tag{45}$$

where consider a sequence of control-program states

$$\theta_{CP}^\mu := \sum_i p_i \ \left|i\right\rangle_C \left\langle i\right| \otimes \sigma_P^{i,\mu}, \tag{46}$$

with $|i\rangle$ being orthogonal states, and a sequence of LOCCs

$$\mathcal{L}_{CPT\to T}^\mu(\rho) := \mathrm{Tr}_C \left[\sum_i p_i \ \left|i\right\rangle_C \left\langle i\right| \otimes \mathcal{L}_{i,\mu}^{PT\to T}(\rho)\right]. \tag{47}$$

These equations are a full extension of previous Eqs. (10), (11) and (12). Correspondingly, we may extend the stretching of Eq. (13) and write

$$\rho_{\mathbf{ab}}^n = \lim_\mu \Lambda_\mu(\theta_{CP}^{\mu\otimes n}), \tag{48}$$

for a sequence of LOCCs Λ_μ [28]. Then, repeating the reasonings of Sec. II C and using arguments from Ref. [7], we may write

$$E_R(\rho_{\mathbf{ab}}^n) = \inf_{\sigma_{\mathrm{sep}}} S(\rho_{\mathbf{ab}}^n||\sigma_{\mathrm{sep}}) \tag{49}$$

$$\overset{(1)}{\leq} \inf_{\sigma_{\mathrm{sep}}^\mu} S\left[\lim_\mu \Lambda_\mu(\theta_{CP}^{\mu\otimes n}) \ || \ \lim_\mu \sigma_{\mathrm{sep}}^{\mu\otimes n}\right] \tag{50}$$

$$\overset{(2)}{\leq} \inf_{\sigma_{\mathrm{sep}}^\mu} \liminf_\mu S\left[\Lambda_\mu(\theta_{CP}^{\mu\otimes n}) \ || \ \sigma_{\mathrm{sep}}^{\mu\otimes n}\right] \tag{51}$$

$$\overset{(3)}{\leq} \inf_{\sigma_{\mathrm{sep}}^\mu} \liminf_\mu S\left[\Lambda_\mu(\theta_{CP}^{\mu\otimes n}) \ || \ \Lambda_\mu(\sigma_{\mathrm{sep}}^{\mu\otimes n})\right] \tag{52}$$

$$\overset{(4)}{\leq} \inf_{\sigma_{\mathrm{sep}}^\mu} \liminf_\mu S\left(\theta_{CP}^{\mu\otimes n} \ || \ \sigma_{\mathrm{sep}}^{\mu\otimes n}\right) \tag{53}$$

$$\overset{(5)}{=} n \inf_{\sigma_{\mathrm{sep}}^\mu} \liminf_\mu S\left(\theta_{CP}^\mu \ || \ \sigma_{\mathrm{sep}}^\mu\right) \tag{54}$$

$$\overset{(6)}{\leq} n \sum_i p_i \inf_{\sigma_{\mathrm{sep}}^{i,\mu}} \liminf_\mu S\left(\sigma_P^{i,\mu} \ || \ \sigma_{\mathrm{sep}}^{i,\mu}\right) \tag{55}$$

$$\overset{(7)}{=} n \sum_i p_i \ E_R(\sigma_P^i) \tag{56}$$

where: (1) $\sigma_{\mathrm{sep}}^\mu$ is a sequence of separable states such that $\|\sigma_{\mathrm{sep}} - \sigma_{\mathrm{sep}}^\mu\| \overset{\mu}{\to} 0$ for separable σ_{sep}; (2) we use the lower semi-continuity of the relative entropy S [29]; (3) we use that $\Lambda_\mu(\sigma_{\mathrm{sep}}^{\mu\otimes n})$ are specific types of separable sequences; (4) we use the monotonicity of S under Λ_μ; (5) we use the additivity of S over tensor products; (6) we use the definition of θ_{CP}^μ given in Eq. (46) and the joint convexity of S which can be applied by replacing $\sigma_{\mathrm{sep}}^\mu$ with $\sum_i p_i \ \sigma_{\mathrm{sep}}^{i,\mu}$ [the orthogonal states $\left|i\right\rangle_C \left\langle i\right|$ can be discarded using the same arguments of Eq. (20)]; and (7) we define the REE of an asymptotic state $\sigma := \lim_\mu \sigma^\mu$ as follows [7]

$$E_R(\sigma) := \inf_{\sigma_{\mathrm{sep}}^\mu} \liminf_\mu S\left(\sigma^\mu \ || \ \sigma_{\mathrm{sep}}^\mu\right), \tag{57}$$

with $\|\sigma_{\mathrm{sep}} - \sigma_{\mathrm{sep}}^\mu\| \overset{\mu}{\to} 0$ for separable σ_{sep}.

Using the weaker asymptotic definition of REE of Eq. (57), we may therefore write the upper bound

$$K(\mathcal{E}) \leq \sum_i p_i \ E_R(\sigma_P^i). \tag{58}$$

For computing this upper bound we need to calculate the REE of the program states $\sigma_P^{i,\mu} = \sigma_{AB}^{i,\mu}$ by considering a split between Alice (A) and Bob (B). Typically, one computes a further upper bound which comes from picking a candidate separable state in the minimization of the REE, i.e.,

$$E_R(\sigma_P^i) := \inf_{\sigma_{\mathrm{sep}}^{p,\mu}} \liminf_\mu S\left(\sigma_P^{i,\mu} \ || \ \sigma_{\mathrm{sep}}^{i,\mu}\right) \tag{59}$$

$$\leq \liminf_\mu S(\sigma_P^{i,\mu} \ || \ \tilde{\sigma}_{\mathrm{sep}}^{i,\mu}). \tag{60}$$

If $\sigma_P^{i,\mu}$ and $\tilde{\sigma}_{\mathrm{sep}}^{i,\mu}$ are Gaussian states, then we can use a closed formula for their relative entropy, given in Ref. [7].

Contrary to previous formulations, the formula for the relative entropy between two arbitrary multimode Gaussian states established in Ref. [7] is directly expressed in terms of their statistical moments, without the need of symplectic diagonalizations (for more details see Theorem 6 and Remark 7 of Ref. [9]).

B. Continuous ensembles

Besides asymptotic simulations, we can also extend the tool to continuous ensembles with associated probability densities. This means that we may consider an average channel defined by

$$\mathcal{E} = \int di\ p_i\ \mathcal{E}_i\ , \tag{61}$$

where each channel component $\mathcal{E}_i$ may have a generally-asymptotic LOCC simulation [7], i.e., of the form in Eq. (43). We may extend all the previous formulas with the replacement

$$\textstyle\sum_i p_i\ \rightarrow \int di\ p_i\ . \tag{62}$$

In particular, we may write the simulation of Eq. (45) but with a sequence of control-program states

$$\theta^{\mu}_{CP} := \int di\ p_i\ \left|i\right\rangle_C \left\langle i\right| \otimes \sigma_P^{i,\mu}, \tag{63}$$

where $|i\rangle$ are orthogonal states, and a sequence of LOCCs

$$\mathcal{L}^{\mu}_{CPT\rightarrow T}(\rho) := \mathrm{Tr}_C\left[\int di\ \left|i\right\rangle_C \left\langle i\right| \otimes \mathcal{L}^{PT\rightarrow T}_{i,\mu}(\rho)\right]. \tag{64}$$

This leads again to the stretching of Eq. (48) and then to the following upper bound

$$K(\mathcal{E}) \leq \int di\ p_i\ E_R(\sigma_P^i), \tag{65}$$

where $\sigma_P^i := \lim_{\mu} \sigma_P^{i,\mu}$ and $E_R(\sigma_P^i)$ has the asymptotic expressions in Eqs. (59) and (60).

V. APPLICATIONS TO NON-GAUSSIAN MIXTURES

A. Ensembles of lossy channels

Let us consider the non-Gaussian average channel $\mathcal{E} := \sum_i p_i \mathcal{E}_i$, where $\mathcal{E}_i := \mathcal{E}_{\eta_i}$ is a lossy channel with transmissivity η_i and associated probability p_i. The asymptotic Choi matrix of the average channel $\rho_{\mathcal{E}} = \lim_{\mu} \rho^{\mu}_{\mathcal{E}}$ is defined over the sequence $\rho^{\mu}_{\mathcal{E}} = \mathcal{I} \otimes \mathcal{E}(\Phi^{\mu})$ with Φ^{μ} being a TMSV state. Also note that we may write

$$\rho^{\mu}_{\mathcal{E}} = \sum_i p_i \rho^{\mu}_{\mathcal{E}_i}, \tag{66}$$

where $\rho^{\mu}_{\mathcal{E}_i}$ are the quasi-Choi matrices of the single channel components $\mathcal{E}_i := \mathcal{E}_{\eta_i}$. Each channel component $\mathcal{E}_i$ is teleportation covariant and therefore simulable by teleporting the input over its asymptotic Choi matrix [7]. More precisely, one has the asymptotic simulation in Eq. (43) where $\mathcal{L}^{PT\rightarrow T}_{i,\mu}$ is a generalized Braunstein-Kimble protocol [26] and $\sigma_P^{i,\mu} = \rho^{\mu}_{\mathcal{E}_i}$.

Note that the LOCC $\mathcal{L}^{PT\rightarrow T}_{i,\mu}$ depends on the loss parameter η_i which means that the channel components $\mathcal{E}_i$ are not jointly teleportation-covariant. For this reason, the simulation of the non-Gaussian mixture $\mathcal{E}$ is not via its asymptotic Choi matrix but can be written in the conditional and asymptotic form of Eq. (45) with $\sigma_P^{i,\mu} = \rho^{\mu}_{\mathcal{E}_i}$. Using Eqs. (58) and (60), we compute the upper bound

$$K(\mathcal{E}) \leq \textstyle\sum_i p_i\ \liminf_{\mu} S(\rho^{\mu}_{\mathcal{E}_i} || \tilde{\sigma}^{i,\mu}_{\mathrm{sep}}), \tag{67}$$

for a suitable separable Gaussian state $\tilde{\sigma}^{i,\mu}_{\mathrm{sep}}$. From Ref. [7], we know that the inferior limit provides the PLOB bound $-\log_2(1-\eta_i)$. Therefore, one has [30]

$$K(\mathcal{E}) \leq -\textstyle\sum_i p_i \log_2(1-\eta_i). \tag{68}$$

Let us now derive a lower bound by computing the RCI of the average channel $\mathcal{E}$ in terms of the sequence $\rho^{\mu}_{\mathcal{E}}$

$$I_{RC}(\mathcal{E}) = \lim_{\mu} I(A\rangle B)_{\rho^{\mu}_{\mathcal{E}}}, \tag{69}$$

$$I(A\rangle B)_{\rho^{\mu}_{\mathcal{E}}} = S(\rho^{\mu}_A) - S(\rho^{\mu}_{\mathcal{E}}), \tag{70}$$

where $S(.)$ is the von Neumann entropy and we have set $\rho^{\mu}_A = \mathrm{Tr}_B \rho^{\mu}_{\mathcal{E}}$. Note that for any $\rho = \sum_i p_i \rho_i$ we may use the concavity properties [27]

$$\textstyle\sum_i p_i S(\rho_i) \leq S(\rho) \leq \sum_i p_i S(\rho_i) + H(\{p_i\}), \tag{71}$$

where $H(\{p_i\}) := -\sum_i p_i \log p_i$ is the Shannon entropy. Therefore, from Eq. (66), we may write

$$\begin{aligned} I(A\rangle B)_{\rho^{\mu}_{\mathcal{E}}} &= S(\mathrm{Tr}_B \rho^{\mu}_{\mathcal{E}}) - S(\rho^{\mu}_{\mathcal{E}}) \\ &= S\left(\textstyle\sum_i p_i \mathrm{Tr}_B \rho^{\mu}_{\mathcal{E}_i}\right) - S\left(\textstyle\sum_i p_i \rho^{\mu}_{\mathcal{E}_i}\right) \\ &\geq \textstyle\sum_i p_i S(\mathrm{Tr}_B \rho^{\mu}_{\mathcal{E}_i}) - \sum_i p_i S(\rho^{\mu}_{\mathcal{E}_i}) - H(\{p_i\}) \\ &= \textstyle\sum_i p_i I(A\rangle B)_{\rho^{\mu}_{\mathcal{E}_i}} - H(\{p_i\}). \end{aligned} \tag{72}$$

Therefore, from Eq. (69) we get

$$\begin{aligned} I_{RC}(\mathcal{E}) &= \lim_{\mu} \textstyle\sum_i p_i I(A\rangle B)_{\rho^{\mu}_{\mathcal{E}_i}} - H(\{p_i\}) \\ &= \textstyle\sum_i p_i I_{RC}(\mathcal{E}_i) - H(\{p_i\}) \\ &= -\textstyle\sum_i p_i \log_2(1-\eta_i) - H(\{p_i\}), \end{aligned} \tag{73}$$

where we have used the fact that the RCI of the lossy channel $\mathcal{E}_i := \mathcal{E}_{\eta_i}$ is simply $I_{RC}(\mathcal{E}_i) = -\log_2(1-\eta_i)$ [7]. As a result, we may write the sandwich

$$\begin{aligned} &-\textstyle\sum_i p_i \log_2(1-\eta_i) - H(\{p_i\}) \leq Q_2(\mathcal{E}) \\ &\leq K(\mathcal{E}) \leq -\textstyle\sum_i p_i \log_2(1-\eta_i). \end{aligned} \tag{74}$$

B. Continuous ensembles of lossy channels

Note that we may also consider a continuous ensemble of lossy channels with different transmissivities, i.e., the non-Gaussian channel

$$\mathcal{E} := \int d\eta\ p_\eta \mathcal{E}_\eta, \tag{75}$$

for some suitable probability density p_η. It is easy to repeat previous steps and write the upper bound

$$K(\mathcal{E}) \leq -\int d\eta\ p_\eta \log_2(1-\eta). \tag{76}$$

Another continuous ensemble of lossy channels can be created by considering a beam splitter operation between the system and the environment

$$\tilde{\mathcal{E}}_\eta(\rho) := \mathrm{Tr}_E[U_\eta^{\mathrm{BS}}(\rho\otimes\sigma_E)U_\eta^{\mathrm{BS}\dagger}]\ , \tag{77}$$

where in the above definition η is the transmissivity and σ_E is a reference state of the environment. For the bosonic lossy channel σ_E is the vacuum state, while in the thermal-loss channel σ_E is a thermal state. In general, one can write any Gaussian and non-Gaussian state using the Glauber P-representation

$$\sigma_E = \int d^2\gamma\ p_\gamma \left|\gamma\right\rangle\left\langle\gamma\right|\ , \tag{78}$$

where $|\gamma\rangle$ is a coherent state with amplitude γ. If the state σ_E is classical, then p_γ is a classical probability density, and we can easily show that the non-Gaussian channel $\tilde{\mathcal{E}}_\eta(\rho)$ is represented by the average

$$\tilde{\mathcal{E}}_\eta(\rho) = \int d^2\gamma\ p_\gamma\, \mathcal{E}_{\eta,\gamma}(\rho)\ , \tag{79}$$

where $\mathcal{E}_{\eta,\gamma}$ is a displaced lossy channel

$$\mathcal{E}_{\eta,\gamma}(\rho) = \mathrm{Tr}_E[U_\eta^{\mathrm{BS}}(\rho\otimes\left|\gamma\right\rangle_E\left\langle\gamma\right|)U_\eta^{\mathrm{BS}\dagger}] \tag{80}$$

$$= D(\gamma\sqrt{1-\eta^2})\,\mathcal{E}_{\eta,0}(\rho)\,D^\dagger(\gamma\sqrt{1-\eta^2}), \tag{81}$$

with $D(\alpha)=\exp(\alpha a^\dagger-\alpha^* a)$ being the displacement operator in terms of the ladder operators a and $a^\dagger$.

Let us write the beam-splitter action

$$U_\eta^{\mathrm{BS}} a^\dagger U_\eta^{\mathrm{BS}\dagger} = \cos\theta a^\dagger - \sin\theta a_E^\dagger, \tag{82}$$

where $\cos^2\theta=\eta$ and $a^\dagger$ ($a_E^\dagger$) is the creation operator acting on the system (environment). We may show that the non-Gaussian channel $\tilde{\mathcal{E}}_\eta$ is teleportation covariant. In fact, we have

$$\mathcal{E}_{\eta,\gamma}[D(z)\,\rho\,D(-z)] = D(z\cos\theta)\,\mathcal{E}_{\eta,\gamma}(\rho)\ D(-z\cos\theta). \tag{83}$$

Since the correction unitary $D(z\cos\theta)$ does not depend on γ, we have that the channels $\mathcal{E}_{\eta,\gamma}$ are jointly teleportation covariant with respect to γ. As a result, $\tilde{\mathcal{E}}_\eta$ is teleportation covariant and simulable with its asymptotic Choi matrix $\rho_{\tilde{\mathcal{E}}_\eta}=\lim_\mu \rho^\mu_{\tilde{\mathcal{E}}_\eta}$ where $\rho^\mu_{\tilde{\mathcal{E}}_\eta}=\mathcal{I}\otimes\tilde{\mathcal{E}}_\eta(\Phi^\mu)$. Therefore, we may write the upper bound

$$K(\tilde{\mathcal{E}}_\eta) \leq \liminf_\mu S(\rho^\mu_{\tilde{\mathcal{E}}_\eta}||\tilde{\sigma}^{\eta,\mu}_{\mathrm{sep}}), \tag{84}$$

for some suitable separable state $\tilde{\sigma}^{\eta,\mu}_{\mathrm{sep}}$. Note that the quasi-Choi matrix takes the form

$$\rho^\mu_{\tilde{\mathcal{E}}_\eta} = \int d^2\gamma\ p_\gamma\,\rho^\mu_{\mathcal{E}_{\eta,\gamma}} \tag{85}$$

$$= \int d^2\gamma\ p_\gamma\ [I\otimes D(\gamma\sin\theta)]\,\rho^\mu_{\mathcal{E}_{\eta,0}}[I\otimes D(-\gamma\sin\theta)]. \tag{86}$$

Since the relative entropy does not depend on displacements, we may write

$$K(\tilde{\mathcal{E}}_\eta) \leq \liminf_\mu S(\rho^\mu_{\mathcal{E}_{\eta,0}}||\tilde{\sigma}^{\eta,\mu}_{\mathrm{sep}}) = -\log(1-\eta), \tag{87}$$

so that the PLOB bound applies to the non-Gaussian channel $\tilde{\mathcal{E}}_\eta$ for any classical state σ_E of the environment.

VI. EXTENSION TO MEMORY CHANNELS

The conditional channel simulation can also be used to represent memory quantum channels. Let us consider M channel ensembles simultaneously acting on M quantum systems, i.e.,

$$\mathcal{E}_{\mathbf{i}} = \mathcal{E}^1_{i_1}\otimes\mathcal{E}^2_{i_2}\otimes\cdots\otimes\mathcal{E}^M_{i_M}, \tag{88}$$

where the instance $\mathbf{i}=i_1,i_2,\cdots,i_M$ occurs with joint probability $p_{\mathbf{i}}$. The process is memoryless if and only if the probability is factorized as $p_{\mathbf{i}}=p_{i_1}p_{i_2}\cdots p_{i_M}$, otherwise there is a classical memory among the channels.

Consider the average M-system channel

$$\mathcal{E} = \textstyle\sum_{\mathbf{i}} p_{\mathbf{i}}\mathcal{E}_{\mathbf{i}}\ . \tag{89}$$

In order to write its conditional simulation, we extend the formulas of Sec. II B by means of the replacement $i\to\mathbf{i}$. Therefore, we may write Eq. (7) where

$$\pi_C := \textstyle\sum_{\mathbf{i}} p_{\mathbf{i}}\left|\mathbf{i}\right\rangle_C\left\langle\mathbf{i}\right|,\quad M := \sum_{\mathbf{i}}\left|\mathbf{i}\right\rangle_C\left\langle\mathbf{i}\right|\otimes\mathcal{E}_{\mathbf{i}}, \tag{90}$$

with $|\mathbf{i}\rangle=|i_1\rangle|i_2\rangle\otimes\cdots\otimes|i_M\rangle$ being the computational orthonormal basis of a control system C. Let us replace $\mathcal{E}^k_{i_k}$ by its simulation with program state σ_P^{k,i_k}. Then, we may write Eq. (10) with the "control-program" state

$$\theta_{CP} := \textstyle\sum_{\mathbf{i}} p_{\mathbf{i}}\left|\mathbf{i}\right\rangle_C\left\langle\mathbf{i}\right|\otimes\bigotimes_{k=1}^M \sigma_P^{k,i_k}. \tag{91}$$

Assuming an adaptive protocol over n uses of $\mathcal{E}$, we may write the stretching of the output state $\rho^n_{\mathbf{ab}}$ as in Eq. (13) and derive

$$K(\mathcal{E}) \leq \textstyle\sum_{\mathbf{i}} p_{\mathbf{i}} E_R\left(\bigotimes_{k=1}^M\sigma_P^{k,i_k}\right) \tag{92}$$

$$\leq \textstyle\sum_{\mathbf{i}} p_{\mathbf{i}}\sum_{k=1}^M E_R\left(\sigma_P^{k,i_k}\right), \tag{93}$$

with suitable extensions to asymptotic simulations and continuous ensembles.

VII. CONCLUSIONS

In this work we have designed a tool for channel simulation which is particularly helpful for mixtures of channels. This simulation is based on the use of a control system which generates the probability distribution associated with the channel ensemble; the state of this control system is then included in the final program state. In this way we can handle mixtures of teleportation-covariant channels which are not jointly covariant, we can simulate the amplitude damping channel without resorting to continuous variables, and we can also simulate non-Gaussian channels and memory channels.

The conditional channel simulation can be exploited in the stretching of adaptive protocols, so that we may bound the two-way quantum and private capacities in terms of the REE. This allowed us to derive the tightest REE upper bound for the amplitude damping channel, and to establish all the two-way capacities of the recently introduced "dephrasure" channel. We have also derived bounds for various non-Gaussian channels that can be described in terms of ensembles of lossy channels.

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[19] Recall that the Choi matrix of a quantum channel $\mathcal{E}$ is defined by propagating part of a maximally entangled state through the channel. For a qubit channel, we therefore have $\rho_{\mathcal{E}} := \mathcal{I} \otimes \mathcal{E}(\Phi)$ where $\mathcal{I}$ is the identity channel and $\Phi := |\Phi\rangle \langle\Phi|$, with $|\Phi\rangle$ being a Bell state.

[20] Recall that a quantum channel $\mathcal{E}_i$ is called *teleportation covariant* if, for any teleportation unitary U (namely for any unitary matrix from the Weyl-Heisenberg group), there exists some unitary V_i such that [7]

$$\mathcal{E}_i(U\rho U^\dagger) = V_i \mathcal{E}_i(\rho) V_i^\dagger \ .$$

[21] Note that, alternatively, we may consider the quantum operation $M := \sum_i \mathcal{C}_i \otimes \mathcal{E}_i$, where $\mathcal{C}_i(\pi) = \sum_i |i\rangle_C \langle i|\pi|i\rangle_C \langle i|$. In such a case, Eq. (8) takes the form $M(\rho_{CT}) = \sum_i \mathcal{C}_i^{C\to C} \otimes \mathcal{L}_i^{PT\to T}(\sigma_P^i \otimes \rho_{CT})$, but the final representation of Eq. (10) remains exactly the same.

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[23] While the finite-size result in Eq. (21) never appeared in the literature to our knowledge, the asymptotic result in Eq. (23) can also be obtained from a probabilistic argument where a channel component $\mathcal{E}_i$ with probability p_i appears np_i times in an asymptotic protocol with large n. Therefore, this component gives rise to np_i copies of the program state σ_i in the stretching of the adaptive protocol. Overall, one has the output state in the form $\Lambda(\otimes_i \sigma_i^{np_i})$ [7] which leads to the same final expression as in Eq. (23).

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[25] Recall that the RCI $I_{\mathrm{RC}}(\mathcal{E})$ of a quantum channel $\mathcal{E}$ is defined as the reverse coherent information when the input state is a maximally entangled state, so that it is computed over the channel's Choi matrix [7].

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