

CONDITIONAL DISTRIBUTION OF ORDER STATISTICS AND DISTRIBUTION OF THE REDUCED i th ORDER STATISTIC OF THE EXPONENTIAL MODEL¹

BY ANDRE G. LAURENT

Wayne State University

1. Introduction and summary. In case the underlying distribution of a sample is normal, a substantial literature has been devoted to the distribution of quantities such as $(X_{(i)} - u)/v$ and $(X_{(i)} - u)/w$, where $X_{(i)}$ denotes the i th ordered observation, u and v are location and scale statistics of the sample, or one is a location or scale parameter and w is an independent scale statistic.

The case $i = 1$ or n has been frequently studied in view of the great importance of extreme values in physical phenomena and also with a view to testing outlying observations or the normality of the distribution. Bibliographical references will be found in Savage [10] and, as far as the general problem of testing outliers is concerned, in Ferguson [4]; references to recent literature include Dixon [1], [2], Grubbs [5], Pillai and Tienzo [9].

Thompson [12] has studied the distribution of $(X_i - \bar{X})/s$ where X_i is one observation picked at random among the sample, and this statistic has been used in the study of outliers; Laurent has generalized Thompson's distribution to the case of a subsample picked at random among a sample [7], then to the multivariate case and the general linear hypothesis [8]. Thompson's distribution is not only the marginal distribution of $(X_i - \bar{X})/s$ but its conditional distribution, given the sufficient statistic (\bar{X}, s) , hence it provides the distribution of X_i given \bar{X}, s , and, using the Rao-Blackwell-Lehmann-Scheffé theorem, gives a way of obtaining a minimum variance unbiased estimate of any estimable function of the parameters of a normal distribution for which an unbiased estimate depending on one observation is available, a fact that has been exploited in sampling inspection by variable.

The present paper presents an analogue to Thompson's distribution in case the underlying distribution of a sample is exponential (the exponential model is nowadays widely used in Failure and Queuing Theories). Such a distribution makes it possible to obtain minimum variance unbiased estimates of functions of the parameters of the exponential distribution. Here an estimate is provided for the survival function $P(X > x) = S(x)$ and its powers. As an application of these results the probability distribution of the "reduced" i th ordered observation in a sample and that of the reduced range are derived. For possible applications to testing outliers or exponentiality the reader is invited to refer to the bibliography.

Received November 13, 1961; revised October 15, 1962.

¹ This paper was written with the partial support of the Office of Naval Research and Wayne State University.

2. An exponential analogue to Thompson's distribution. Let $X_{(1)}$ and \bar{X} be the smallest observation and the arithmetic mean of a sample $\mathbf{X} = (X_1, \dots, X_n)$ of n independent observations with probability density

$$(1) \quad (1/c)e^{-(x-m)/c} \quad \text{for } x \geq m; \quad 0 \text{ otherwise.}$$

Let $\bar{\mathbf{X}} = (X_{(1)}, \dots, X_{(n)})$ be the order statistic of the sample.

It is well known [3] that $(\bar{X}, X_{(1)})$ hence $(X_{(1)}, Y)$, where Y denotes $\bar{X} - X_{(1)}$, constitute a complete and sufficient statistic for the distribution of X . Note that $n(\bar{X} - m) = n(X_{(1)} - m) + nY$. Now $2n(\bar{X} - m)/c$ and

$$2n(X_{(1)} - m)/c$$

are chi-square variates with $2n$ and 2 degrees of freedom respectively. Since Y/c is parameter free and independent of $X_{(1)}$, it follows that $2nY/c$ is chi-square with $2(n - 1)$ degrees of freedom and independent of $X_{(1)}$.

Let $\xi = (\xi_1, \dots, \xi_k)$ be a subsample of X with order statistic $\tilde{\xi} = (\xi_{(1)}, \dots, \xi_{(k)})$ and sufficient statistic $(\xi_{(1)}, \bar{\xi} - \xi_{(1)})$, and $\mathbf{X}^* = (X_1^*, \dots, X_{n-k}^*)$ the complementary subsample (so that $\mathbf{X} = (\xi, \mathbf{X}^*)$) with order statistic $\bar{\mathbf{X}}^* = (X_{(1)}^*, \dots, X_{(n-k)}^*)$ and sufficient statistic $(X_{(1)}^*, Y^*)$, where $Y^* = \bar{X}^* - X_{(1)}^*$.

The conditional density $f(\tilde{\xi} | X_{(1)}, Y)$ of $\tilde{\xi}$, given $X_{(1)}, Y$, is parameter free, and according to the Rao-Blackwell-Lehmann-Scheffé theorem the expected value, with respect to f , of any unbiased estimate $\varphi(\tilde{\xi})$ of an estimable function $\psi(m, c)$ of the parameters m, c is the uniformly minimum variance unbiased estimate of ψ . This density is obtained by writing the joint density of $\tilde{\xi}, X_{(1)}^*, Y^*$, making the change of variables $X_{(1)}^* = X_{(1)}^*(X_{(1)}, Y, \tilde{\xi})$, $Y^* = Y^*(X_{(1)}, Y, \tilde{\xi})$, then dividing by the joint density of $X_{(1)}, Y$. The latter is

$$(3) \quad [n^n/c^n \Gamma(n - 1)] Y^{n-2} \exp -(n/c)[(X_{(1)} - m) + Y].$$

The joint density of $\tilde{\xi}, X_{(1)}^*, Y^*$ is

$$(4) \quad [k!(n - k)^{n-k}/c^n \Gamma(n - k - 1)] Y^{*n-k-2} \exp -[k(\bar{\xi} - X_{(1)}^*) + n(X_{(1)}^* - m) + (n - k)Y^*]/c.$$

By definition one has

$$(5) \quad Y^* = nY/(n - k) - [k/(n - k)](\bar{\xi} - X_{(1)}) - (X_{(1)}^*),$$

and there are two cases to consider according to whether $\xi_{(1)}$ is greater than or equal to $X_{(1)}$.

In case $\xi_{(1)} > X_{(1)}$, then $X_{(1)}^* = X_{(1)}$; by substituting the proper values into (4)—the Jacobian of the substitution is

$$J(\tilde{\xi}, X_{(1)}^*, Y^*; \tilde{\xi}, X_{(1)}, Y) = n/(n - k)$$

—and dividing by (3), one obtains the conditional density

$$(6) \quad f(\tilde{\xi} | (X_{(1)}, Y)) = \frac{k!(n - 2)!}{(n - k - 2)!} \left(1 - \frac{k}{n}\right) \left[1 - \frac{k}{n} \frac{\bar{\xi} - X_{(1)}}{Y}\right]^{n-k-2} \frac{1}{(nY)^k}$$

in the domain where the bracket is positive.

In case $\xi_{(1)} = X_{(1)}$, let $\tilde{\xi} = (\xi_{(1)}, \tilde{\xi}^-)$, where $\tilde{\xi}^- = (\xi_{(2)}, \dots, \xi_{(k)})$; using the same technique as above yields the conditional density of $\tilde{\xi}^-$, $X_{(1)}^*$ given $X_{(1)}$, Y . By integrating $X_{(1)}^* - X_{(1)}$ from 0 to $[nY - k(\tilde{\xi}^- - X_{(1)})]/(n - k)$, one obtains

$$(7) \quad f(\tilde{\xi}^- | X_{(1)}, Y) = \frac{k!(n - 2)!}{(n - k - 1)!n!} \left[1 - \frac{k}{n} \frac{\tilde{\xi}^- - X_{(1)}}{Y} \right]^{n-k-1} \frac{1}{n^{k-1}Y^{k-1}}$$

in the domain where the bracket is positive. As the reduced variables $U_{(i)} = (\xi_{(i)} - X_{(1)})/nY$ are independent of $X_{(1)}$ and Y , the densities above provide also the marginal densities of $\tilde{U} = (U_{(1)}, \dots, U_{(k)})$ and $\tilde{U}^- = (U_{(2)}, \dots, U_{(k)})$.

For $k = 1$, one obtains

$$(8) \quad \begin{aligned} f(\xi | X_{(1)}, Y) &= 1/n && \text{if } \xi = X_{(1)}, \\ &= (n - 2) \left(1 - \frac{1}{n} \right) \left[1 - \frac{\xi - X_{(1)}}{Y} \cdot \frac{1}{n} \right]^{n-3} \frac{1}{nY} && \text{if } X_{(1)} < \xi \leq X_{(1)} + nY, \\ &= 0 && \text{otherwise,} \end{aligned}$$

which provides also the marginal density of the reduced variable

$$U = (\xi - X_{(1)})/nY.$$

The corresponding distribution is the analogue for the exponential case of Thompson's distribution for a normal sample.

3. Best estimates of $S(x)$ and $S^r(x)$. Let $S(x)$ denote the "survival function"

$$(9) \quad \begin{aligned} S(x) = P(X > x) &= e^{-(x-m)/c} && \text{for } x \geq m, \\ &= 1 && \text{otherwise.} \end{aligned}$$

An unbiased estimate of $S(x)$ is the characteristic function $I_x(X)$ of the set $(x, +\infty)$, hence the minimum variance unbiased estimate for $S(x)$ is $E[I_x(X) | X_{(1)}, Y]$, namely

$$(10) \quad \begin{aligned} \hat{S}(x) &= \int_x^\infty f(\xi | X_{(1)}, Y) d\xi \\ &= 1 && \text{if } x < X_{(1)} \\ &= \left(1 - \frac{1}{n} \right) \left[1 - \frac{x - X_{(1)}}{nY} \right]^{n-2} && \text{if } X_{(1)} \leq x \leq X_{(1)} + nY \\ &= 0 && \text{if } x > X_{(1)} + nY. \end{aligned}$$

At the same time,

$$\hat{S}(x) = P(\xi > x | X_{(1)}, Y) = P(U > u), \quad \text{with } u = (x - X_{(1)})/nY.$$

This result is consistent with the one given by Tate [11].

Similarly, the minimum variance unbiased estimate for $S^r(x)$ is obtained by

integrating (6), with $k = r$, over the domain $\xi_1 = (x, \infty), \dots, \xi_r = (x, \infty)$. The structures of $S(x)$ and $S^r(x)$, however, suggest trying an estimate $T(x; X_{(1)}, Y)$ obtained by replacing $1 - 1/n$ by some k and Y by Y/r in $\hat{S}(x)$. Taking the expected value of $T(\xi; X_{(1)}, Y)$, (integrate Y from $r(x - X_{(1)})/n$ to $+\infty$ and $X_{(1)}$ from m to $+\infty$) one obtains $k = 1 - r/n$, which yields

$$\begin{aligned}
 [S^r(x)]^* &= 1 && \text{if } x < X_{(1)}, \\
 (11) \quad &= \left(1 - \frac{r}{n}\right) \left[1 - \frac{r}{n} \frac{x - X_{(1)}}{Y}\right]^{n-2} && \text{if } X_{(1)} \leq x \leq X_{(1)} + Yn/r, \\
 &= 0 && \text{if } x > X_{(1)} + Yn/r.
 \end{aligned}$$

4. Distribution of the reduced i th order statistic.

(i) *Lemmas:* It is well known [6] that if $A_1 \dots A_n$ are n events, possible outcomes of a trial, the probability of obtaining at least z of these events is

$$(12) \quad P(Z \geq z) = \sum_{r=z}^n (-1)^{r+z} \binom{r-1}{z-1} \sum P\left(\bigcap_{i=1}^r A_{j_i}\right),$$

where $P(\bigcap_{i=1}^r A_{j_i})$ denotes the probability of obtaining A_{j_1}, \dots, A_{j_r} simultaneously and where the summation extends to the set of $\binom{n}{r}$ combinations of n events r by r .

It is also known that

$$(13) \quad \sum P\left(\bigcap_{i=1}^r A_{j_i}\right) = \frac{1}{r!} E[Z^{[r]}],$$

where $Z^{[r]}$ is the factorial moment of order r of the number of successes Z .

(ii) Let N_x be the number of observations with a value at least equal to x in a sample of size n . The probability distribution of the i th order statistic $X_{(i)}$ is $P(X_{(i)} \leq x) = 1 - P(N_x \geq n - i + 1)$, therefore is given by (12) with $A_i = [X_i \geq x]$ (so that $\sum P(\bigcap_{i=1}^r A_{j_i}) = \binom{n}{r} S^r(x)$) and

$$z = n - i + 1.$$

It follows from (13) that $P(X_{(i)} \leq x)$ admits

$$(14) \quad 1 - \sum_{r=n-i+1}^n (-1)^{r+n-i+1} \binom{r-1}{n-i} N_x^{[r]}/r!$$

as an unbiased estimate and

$$(15) \quad 1 - \sum_{r=n-i+1}^n (-1)^{r+n-i+1} \binom{r-1}{n-i} E\left[\frac{N_x^{[r]}}{r!} \mid X_{(1)}, Y\right]$$

$$(16) \quad = 1 - \sum_{r=n-i+1}^n (-1)^{r+n-i+1} \binom{r-1}{n-i} \sum P\left[\bigcap_{i=1}^r A_{j_i} \mid X_{(1)}, Y\right]$$

$$(17) \quad = P(X_{(i)} \leq x \mid X_{(1)}, Y)$$

as its minimum variance unbiased estimate.

As $N_x^{[r]}/r!$ is an unbiased estimate of $\sum P(\cap_{i=1}^r A_{j_i}) = \binom{n}{r} S^r(x)$ it follows that

$$E \left[\frac{N_x^{[r]}}{r!} \mid X_{(1)}, Y \right] = \sum P \left(\cap_{i=1}^r A_{j_i} \mid X_{(1)}, Y \right)$$

is the minimum variance unbiased estimate $\binom{n}{r} [S^r(x)]^*$ of $\binom{n}{r} S^r(x)$. Making the proper substitution in (16), (17) yields (in view of (11))

$$(18) \quad P(X_{(i)} \leq x \mid X_{(1)}, Y) = 1 - \sum_{r=n-i+1}^n (-1)^{r+n-i+1} \binom{r-1}{n-i} \binom{n}{r} [S^r(x)]^*$$

$$(19) \quad = 1 - \sum_{r=n-i+1}^k (-1)^{r+n-i+1} \binom{r-1}{n-i} \binom{n}{r} \left(1 - \frac{r}{n}\right) \cdot \left(1 - \frac{r}{n} \frac{x - X_{(1)}}{Y}\right)^{n-2}$$

for $x \geq X_{(1)}$ and $k \leq nY/(x - X_{(1)})$, ($[S^r(x)]^* = 0$ for $r > nY/(x - X_{(1)})$).

Since $U_{(i)} = (X_{(i)} - X_{(1)})/nY$ is independent of $X_{(1)}, Y$, one has

$$(20) \quad P(U_{(i)} \leq u) = 1 - \sum_{r=n-i+1}^k (-1)^{r+n-i+1} \binom{n}{r} \binom{r-1}{n-i} \cdot \left(1 - \frac{r}{n}\right) [1 - ru]^{n-2}$$

for $u \geq 0$ and $k \leq 1/u$.

TABLE I
Probability of the reduced range, $P(R \leq u)$

n	$u = .10$.20	.30	.40	.50	.60	.70	.80	.90	1.00
3	.00000	.00000	.00000	.00000	.00000	.20000	.40000	.60000	.80000	1.00000
4	.00000	.00000	.00000	.04000	.25000	.52000	.73000	.88000	.97000	1.00000
5	.00000	.00000	.04800	.18400	.50000	.74400	.89200	.96800	.99600	1.00000
6	.00000	.00000	.05450	.36800	.68750	.87200	.95950	.99200	.99950	1.00000
7	.00000	.00032	.14498	.53824	.81250	.93856	.98542	.99808	.99994	1.00000
8	.00000	.00365	.26244	.67475	.89063	.97133	.99490	.99955	.99999	1.00000
9	.00000	.01724	.38704	.77641	.93750	.98689	.99825	.99990	1.00000	1.00000
10	.00000	.03998	.50476	.84883	.96484	.99410	.99941	.99998	1.00000	1.00000

TABLE II
90% and 99% quantiles of the reduced range

n	3	4	5	6	7	8	9	10	11	12
R.90	.95000	.81743	.70760	.62394	.55907	.50741				
R.99	.99500	.94226	.86428	.78853	.72179	.66440	.61517	.57271	.53584	.51357

In case $i = n$ one obtains the distribution of the reduced range R ,

$$(21) \quad P(R \leq u) = 1 - \sum_{j=i}^k (-1)^{j+1} \binom{n-1}{j} (1 - ju)^{n-2} \quad k \leq 1/u$$

Table I provides for $n = 3$ to 12 and for some values of u the probability of the reduced range. Table II gives, for $n = 3$ to 12, the 99% point of the same statistic.

Alternatively, by means of the transformation $(\bar{X} - X_{(1)}) / (X_{(n)} - X_{(1)}) = 1/nU_{(n)}$, one can use these tables to study the deviation of the smallest value from the mean, studentized by the range. Note also

$$(X_{(n)} - \bar{X}) / (X_{(n)} - X_{(1)}) = 1 - 1/nU_{(n)}$$

and $(X_{(i)} - \bar{X}) / (X_{(n)} - X_{(1)}) = (nU_{(i)} - 1) / nU_{(n)}$.

5. Acknowledgment. The author is grateful to the referee for his suggestions for making the presentation of this paper more concise and lucid.

REFERENCES

[1] DIXON, W. J. (1950). Analysis of extreme values. *Ann. Math. Statist.* **21** 488-506.
 [2] DIXON, W. J. (1951). Ratios involving extreme values. *Ann. Math. Statist.* **22** 68-78.
 [3] EPSTEIN, B. and SOBEL, M. (1954). Some theorems relevant to testing from an exponential distribution. *Ann. Math. Statist.* **25** 373-381.
 [4] FERGUSON, T. S. (1961). Rules for rejection of outliers. *Rev. Inst. Internat. Statist.* **29** 29-43.
 [5] GRUBBS, F. E. (1950). Sample criterion for testing outlying observations. *Ann. Math. Statist.* **21** 27-58.
 [6] GUMBEL, E. J. (1936). La distribution des événements compatibles. *C. R. Acad. Sci. Paris.* **202**, 1637.
 [7] LAURENT, A. G. (1955). Distribution d'échantillons quand la population de référence est Laplace Gaussienne de paramètres inconnus. *J. Soc. Statist. Paris.* **10** 262-296.
 [8] LAURENT, A. G. (1960). Generalization of Thompson's distribution, III. *Ann. Math. Statist.* **31** 526.
 [9] PILLAI, K. C. S. and TIENZO, B. P. (1959). On the distribution of the extreme studentized deviate from the sample mean. *Biometrika* **46** 467-472.
 [10] SAVAGE, R. (1953). Bibliography of nonparametric statistics and related topics. *J. Amer. Statist. Assoc.* **48** 844-906.
 [11] TATE, R. F. (1959). Unbiased estimation: functions of location and scale parameters. *Ann. Math. Statist.* **30** 341-366.
 [12] THOMPSON, W. R. (1935). On a criterion for the rejection of observations and the distribution of the ratio of deviation to sample standard deviation. *Ann. Math. Statist.* **6** 214-219.