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CONDITIONAL ESTIMATION IN EXPONENTIAL MODELS

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Abstract. A two-sided conditional confidence interval for the parameter of an exponential probability distribution is constructed. The construction relies on a decision following a preliminary test of significance for the equality of two exponential population means. The coverage probability, the expected length together with the coefficient of variation of this interval are studied. A shrinkage version of the interval is also proposed. Furthermore, a numerical study on the accuracy of the interval estimator is performed.

1. Introduction

Preliminary test of significance estimators initiated in Bancroft [3] are widely used in practice to improve efficiency of estimators. In this paper, we consider the problem of interval estimation for the parameter θ_1 of an exponential population when it is suspected that θ_1 is greater or equal than the parameter θ_2 of another exponential population. We estimate then θ_1 after performing a preliminary test on equality of the two population parameters. The conditional interval estimation of the exponential scale and location parameters following rejection of a preliminary test is investigated in and Chiou and Han [4,5] in the case of a single sample. In Chiou and Han [4] a conditional confidence interval for the scale parameter θ of a two-parameter exponential distribution is constructed following a rejection of a pre-test with significance level α for $H_0 : \theta = \theta_0$ versus $H_a : \theta \neq \theta_0$. This interval is compared in terms of length and coverage probability to a corresponding unconditional interval with same targeted coverage probability. The inference is based on a type II censored sample. The authors have shown that the conditional interval performs fairly well when $\psi = \hat{\theta}/\theta_0$ is close to one and when α is not small, say $\alpha \geq .4$. $\hat{\theta}$ is the minimum variance unbiased estimator of θ. However, if one has prior knowledge that ψ is not close to one, then it is recommended to use the unconditional confidence interval. Similar results are obtained in Chiou and Han [5] for the case of conditional estimation of the exponential location parameter based on a non censored sample. For a detailed account on the use of preliminary

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test procedures, see, for instance, Jugde and Bock [8], Mahdi [10], Giles et al. [7], and more recently, Rai and Srivastava [14]. It is worth nothing that preliminary test procedures require high computations which lead to analytically unmanageable formulas. Therefore the use of computer programming and numerical solutions are unavoidable to study the properties of pre-test estimators, as pointed out in Mahdi [9]. We organize this paper as follows. In section 2 we state the considered problem and in sections 3 and 4 we respectively derive the coverage probability and the expected length of the conditional interval estimator. In section 5, we study the coefficient of variation for the length of the constructed interval. In section 6, we introduce a shrinkage version of the conditional confidence interval. A numerical application and simulation results are given in sections 7 and 8, respectively. We conclude in section 9. A table and figures illustrating the numerical results are presented in Appendix.

2. Statement of the problem

Suppose that $X_1, \cdots, X_{n_1}$ and $Y_1, \cdots, Y_{n_2}$ are two independent random samples of n_1 and n_2 observations from the exponential distributions $\exp(\theta_1)$ and $\exp(\theta_2)$, respectively. It is suspected that $\theta_1 \geq \theta_2$. A two sided confidence interval for θ_1 is desired.

Let $U_1 = \sum_{i=1}^{n_1} X_i$ and $U_2 = \sum_{j=1}^{n_2} Y_j$. It is well known that $\frac{2U_i}{\theta_i}$, $i = 1, 2$, are independent and distributed as chi-square variables with $2n_i$ degrees of freedom, respectively (see, e.g., Takada [15] or Epstein [6]). Thus $U_i \sim \frac{\theta_i}{2}\chi^2_{2n_i}$ for $i = 1, 2$. If $\theta_1 > \theta_2$, a $(1-\alpha)100\%$ confidence interval for θ_1 is given by

$$I_1 = \left[\frac{2U_1}{\chi^2_{2n_1,\alpha/2}}; \quad \frac{2U_1}{\chi^2_{2n_1,(1-\alpha/2)}}\right], \tag{1}$$

where $\chi^2_{n,\alpha}$ designates the $100(1-\alpha)\%$ percentile point of the chi-squared distribution with n degrees of freedom. However, if $\theta_1 = \theta_2$, we can pool the two samples and use the confidence interval

$$I_2 = \left[\frac{2(U_1+U_2)}{\chi^2_{2(n_1+n_2),\alpha/2}}; \quad \frac{2(U_1+U_2)}{\chi^2_{2(n_1+n_2),(1-\alpha/2)}}\right], \tag{2}$$

for estimating θ_1. But since θ_1 and θ_2 are unknown, the question arises as whether I_1 or I_2 is to be used as a confidence interval for θ_1. To answer this question, we first test the null hypothesis H_0: $\theta_1 = \theta_2$ against the alternative H_1: $\theta_1 > \theta_2$.

Under the null hypothesis H_0, we have

$$\eta = \frac{n_2 U_1}{n_1 U_2} \sim F_{(2n_1, 2n_2)} \tag{3}$$

where $F_{(2n_1,2n_2)}$ represents the Fisher variable with $2n_1$ and $2n_2$ degrees of freedom. Therefore, we do not reject H_0 at the level of significance ϵ, if $\eta \leq F_{(2n_1,2n_2,\epsilon)}$, and reject H_0 otherwise. For simplification, the $100(1-\epsilon)\%$ percentile of the Fisher distribution with $2n_1$ and $2n_2$ degrees of freedom, $F_{(2n_1,2n_2,\epsilon)}$, will be denoted by F_ϵ, thereafter.

We suggest to estimate the parameter θ_1 by the following conditional confidence interval

$$I = \begin{cases} I_1, & \text{if } \eta > F_\epsilon \\ I_2, & \text{if } \eta \le F_\epsilon \end{cases} \tag{4}$$

based on the response of the preliminary test of significance. The coverage probability of the interval I together with its expected length are studied below.

3. Coverage probability of the conditional confidence interval

Since $\frac{2U_i}{\theta_i}$, $i = 1,2$, are independently distributed as chi-squared variables with $2n_i$ degrees of freedom, respectively, the joint probability density function of (U_1, U_2) is given by

$$f(u_1,u_2) = \begin{cases} K u_1^{n_1-1} u_2^{n_2-1} \exp -[\frac{u_1}{\theta_1} + \frac{u_2}{\theta_2}], & \text{if } (u_1,u_2) \in [0,\infty)^2, \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

where

$$K = [\Gamma(n_1)\Gamma(n_2)\theta_1^{n_1}\theta_2^{n_2}]^{-1}. \tag{6}$$

The coverage probability of the confidence interval I is given by $P = P_1 + P_2$, such that

$$P_1 = Prob\left[\frac{2U_1}{\chi^2_{2n_1,\alpha/2}} \le \theta_1 \le \frac{2U_1}{\chi^2_{2n_1,(1-\alpha/2)}}, \eta > F_\epsilon\right] \tag{7}$$

and

$$P_2 = Prob\left[\frac{2(U_1+U_2)}{\chi^2_{2(n_1+n_2),\alpha/2}} \le \theta_1 \le \frac{2(U_1+U_2)}{\chi^2_{2(n_1+n_2),(1-\alpha/2)}}, \eta \le F_\epsilon\right]. \tag{8}$$

3.1. Evaluation of the probability P_1. To evaluate P_1, we integrate the joint probability density function (5) over the u_1u_2 plane region D_1 given by

$$D_1 = \left\{(u_1,u_2) \text{ such that } \frac{b}{2}\theta_1 \le u_1 \le \frac{a}{2}\theta_1 \text{ and } 0 \le u_2 \le \lambda u_1\right\},$$

where $a = \chi^2_{2n_1,\alpha/2}$, $b = \chi^2_{2n_1,(1-\alpha/2)}$ and $\lambda = \frac{n_2}{n_1 F_\epsilon}$. Thus,

$$\begin{aligned} P_1 &= K\int_{u_1=\frac{b}{2}\theta_1}^{\frac{a}{2}\theta_1} u_1^{n_1-1}\exp(-\frac{u_1}{\theta_1})\left[\int_{u_2=0}^{\lambda u_1} u_2^{n_2-1}\exp(-\frac{u_2}{\theta_2})\,du_2\right]du_1 \\ &= K\theta_2^{n_2}\Big\{\Gamma(n_2)\Gamma(n_1)\theta_1^{n_1} I_\Gamma(n_1,\frac{b}{2},\frac{a}{2}) - \Big[\sum_{j=1}^{n_2}\frac{\Gamma(n_2)\Gamma(n_1+n_2-j)}{\Gamma(n_2-j+1)}(\frac{\lambda}{\theta_2})^{n_2-j} \\ &\times (\frac{\theta_1\theta_2}{\theta_2+\lambda\theta_1})^{n_1+n_2-j} \times I_\Gamma(n_1+n_2-j,\frac{b}{2}(\frac{\theta_2+\lambda\theta_1}{\theta_2}),\frac{a}{2\theta_2}(\theta_2+\lambda\theta_1))\Big]\Big\}, \end{aligned} \tag{9}$$

where I_Γ represents the incomplete gamma function, that is,

$$I_\Gamma(n, x_0, x_1) = \frac{1}{\Gamma(n)} \int_{x_0}^{x_1} z^{n-1} \exp[-z]\, dz, \tag{10}$$

as defined in Pearson [12], for instance.

Now substituting the value $\phi = \dfrac{\theta_2}{\theta_1}$ into equation (9), we get

$$P_1 = I_\Gamma(n_1, \frac{b}{2}, \frac{a}{2}) - \frac{1}{\Gamma(n_1)}\Big\{\sum_{j=1}^{n_2} \frac{\Gamma(n_1+n_2-j)}{\Gamma(n_2-j+1)}(\frac{\phi}{\lambda+\phi})^{n_1}(\frac{\lambda}{\lambda+\phi})^{n_2-j}$$
$$\times I_\Gamma(n_1+n_2-j, \frac{b}{2}(\frac{\phi+\lambda}{\phi}), \frac{a}{2}(\frac{\phi+\lambda}{\phi}))\Big\}. \tag{11}$$

3.2. Evaluation of the probability P_2. The probability P_2 is evaluated by integrating $f(u_1, u_2)$ over the plane region D_2 given by

$$D_2 = \Big\{(u_1, u_2) \text{ such that } \frac{2(u_1+u_2)}{c} < \theta_1 < \frac{2(u_1+u_2)}{d} \text{ and } \frac{u_1}{u_2} < \frac{1}{\lambda}\Big\}$$

where $c = \chi^2_{2(n_1+n_2),\alpha/2}$ and $d = \chi^2_{2(n_1+n_2),1-\alpha/2}$. In order to evaluate P_2, we subdivide D into two disjoint regions D_{21} and D_{22} such that

$$D_{21} = \Big\{(u_1, u_2) \text{ such that } 0 < u_1 < \frac{d\theta_1}{2(1+\lambda)} \text{ and } \frac{d\theta_1}{2} - u_1 < u_2 < \frac{c\theta_1}{2} - u_1\Big\}$$

and

$$D_{22} = \Big\{(u_1, u_2) \text{ such that } \frac{d\theta_1}{2(1+\lambda)} < u_1 < \frac{c\theta_1}{2(1+\lambda)} \text{ and } \lambda u_1 < u_2 < \frac{c\theta_1}{2} - u_1\Big\}.$$

Let us denote ID_{21} the integral of $f(u_1, u_2)$ over D_{21} and ID_{22} the integral of $f(u_1, u_2)$ over D_{22}. To evaluate these integrals, we need to consider separately the case $\phi \neq 1$ and the case $\phi = 1$. In the case $\phi \neq 1$, we get after several steps of integration

$$ID_{21} = K \int_{u_1=0}^{\frac{d\theta_1}{2(1+\lambda)}} u_1^{n_1-1} \exp[-\frac{u_1}{\theta_1}] \Big[\int_{u_2=\frac{d\theta_1}{2}-u_1}^{\frac{c\theta_1}{2}-u_1} u_2^{n_2-1} \exp[-\frac{u_2}{\theta_2}]\, du_2\Big]\, du_1$$
$$= \frac{1}{\Gamma(n_1)\Gamma(n_2)}\Big\{-\exp[-\frac{c}{2\phi}]\{\sum_{j=1}^{n_2} \frac{\Gamma(n_2)}{\Gamma(n_2-j+1)}\{\sum_{k=0}^{n_2-j}\binom{n_2-j}{k}(\frac{c}{2\phi})^{n_2-j-k}$$
$$\times (-1)^k \frac{1}{\phi^k}[\frac{\phi}{\phi-1}]^{n_1+k}\{\sum_{l=1}^{n_1+k} \frac{\Gamma(n_1+k)}{\Gamma(n_1+k-l+1)}(-1)^{l+1}$$
$$\times \left(\frac{d(\phi-1)}{2(1+\lambda)\phi}\right)^{n_1+k-l} \exp[\frac{d(\phi-1)}{2(1+\lambda)\phi}] - (-1)^{n_1+k+1}(n_1+k-1)!\}\}]$$
$$+ [\exp[-\frac{d}{2\phi}]\{\sum_{j=1}^{n_2} \frac{\Gamma(n_2)}{\Gamma(n_2-j+1)}\{\sum_{k=0}^{n_2-j}\binom{n_2-j}{k}(\frac{d}{2\phi})^{n_2-j-k}(-1)^k(\frac{1}{\phi^k})$$

$$\times [(\frac{\phi}{\phi-1})^{n_1+k}\{\sum_{l=1}^{n_1+k}\frac{\Gamma(n_1+k)}{\Gamma(n_1+k-l+1)}(-1)^{l+1}\left(\frac{d(\phi-1)}{2(1+\lambda)\phi}\right)^{n_1+k-l}$$

$$\times \exp[\frac{d(\phi-1)}{2(1+\lambda)\phi}] - (-1)^{n_1+k+1}(n_1+k-1)!\}\}]\Big\}, \quad (12)$$

and in the case $\phi = 1$, we have

$$ID_{21} = \frac{1}{\Gamma(n_1)\Gamma(n_2)}\Big[-\exp[-\frac{c}{2\phi}]\{\sum_{j=1}^{n_2}\frac{\Gamma(n_2)}{\Gamma(n_2-j+1)}\{\sum_{k=0}^{n_2-j}\binom{n_2-j}{k}(\frac{c}{2\phi})^{n_2-j-k}$$

$$\times (-1)^k(\frac{1}{\phi^k})(\frac{1}{n_1+k})(\frac{d}{2(1+\lambda)})^{n_1+k}\}\}$$

$$+ [\exp[-\frac{d}{2\phi}]\{\sum_{j=1}^{n_2}\frac{\Gamma(n_2)}{\Gamma(n_2-j+1)}\{\sum_{k=0}^{n_2-j}\binom{n_2-j}{k}(\frac{d}{2\phi})^{n_2-j-k}(-1)^k$$

$$\times (\frac{1}{\phi^k})(\frac{1}{n_1+k})(\frac{d}{2(1+\lambda)})^{n_1+k}\}\}]\Big]. \quad (13)$$

Similarly, by integrating carefully $f(u_1, u_2)$ over the domain D_{22}, we get

$$ID_{22} = K\int_{\frac{d\theta_1}{2(1+\lambda)}}^{\frac{c\theta_1}{2(1+\lambda)}} u_1^{n_1-1}\exp[-\frac{u_1}{\theta_1}]\left(\int_{u_2=\lambda u_1}^{\frac{c\theta_1}{2}-u_1} u_2^{n_2-1}\exp[-\frac{u_2}{\theta_2}]\,du_2\right)du_1$$

$$= \frac{1}{\Gamma(n_1)}\sum_{j=1}^{n_2}\frac{\Gamma(n_2)}{\Gamma(n_2-j+1)}(\frac{\lambda}{\phi})^{n_2-j}[\sum_{k=1}^{n_1+n_2-j}\frac{\Gamma(n_1+n_2-j)}{\Gamma(n_1+n_2-j-k+1)}(\frac{\phi}{\lambda+\phi})^k$$

$$\times \{(\frac{d}{2(1+\lambda)})^{n_1+n_2-j-k}\exp[-\frac{d(\lambda+\phi)}{2(1+\lambda)\phi}] - (\frac{c}{2(1+\lambda)})^{n_1+n_2-j-k}$$

$$\times \exp[-\frac{c(\lambda+\phi)}{2(1+\lambda)\phi}]\}] - \frac{1}{\Gamma(n_1)}\sum_{j=1}^{n_2}\frac{1}{\Gamma(n_2-j+1)}\{\sum_{l=0}^{n_2-j}\frac{1}{\phi^{n_2-j-l}}$$

$$\times \binom{n_2-j}{l}(-1)^l(\frac{c}{2\phi})^l\exp[-\frac{c}{2\phi}](\frac{c}{2(1+\lambda)})^{n_1+n_2-j-l}$$

$$\times \{\sum_{k=1}^{n_1+n_2-j-l}\frac{\Gamma(n_1+n_2-j-l)}{\Gamma(n_1+n_2-j-l-k+1)}(-1)^{k+1}$$

$$\times [(\frac{2\phi(1+\lambda)}{c(1-\phi)})^k\exp[\frac{c(1-\phi)}{2\phi(1+\lambda)}] - (\frac{2\phi(1+\lambda)}{d(1-\phi)})^k\exp[\frac{d(1-\phi)}{2\phi(1+\lambda)}]]\}\}, \quad (14)$$

in the case $\phi \neq 1$. For $\phi = 1$, we have

$$ID_{22} = \frac{1}{\Gamma(n_1)}\sum_{j=1}^{n_2}\frac{1}{\Gamma(n_2-j+1)}\{-[\sum_{l=0}^{n_2-j}(\frac{c}{2})^l(-1)^l\binom{n_2-j}{l}\exp[-\frac{c}{2}]\frac{1}{n_1+n_2-j-l}$$

$$\times [(\frac{c}{2(1+\lambda)})^{n_1+n_2-j-l} - (\frac{d}{2(1+\lambda)})^{n_1+n_2-j-l}]] + \lambda^{n_2-j}\frac{1}{n_1+n_2-j}$$

$$\times [(\frac{c}{2(1+\lambda)})^{n_1+n_2-j} - (\frac{d}{2(1+\lambda)})^{n_1+n_2-j}]\}. \quad (15)$$

Thus,

$$P_2 = ID_{21} + ID_{22}. \tag{16}$$

As a partial check of the above results, we consider the particular cases $F_\epsilon = 0$ and $F_\epsilon = 1$. When $F_\epsilon \to 0$, that is, $\lambda \to \infty$, both ID_{21} and ID_{22} tend to zero for any value of ϕ. Thus P_2 tends to zero as well. This agrees with the decision of non pooling the two samples. Furthermore, when we let $\lambda \to \infty$ in equation (11), we get

$$P_1 = I_\Gamma(n_1, \frac{b}{2}, \frac{a}{2}) = \frac{1}{\Gamma(n_1)} \int_{\frac{b}{2}}^{\frac{a}{2}} z^{n_1-1} \exp[-z]\, dz. \tag{17}$$

Now by using the change of variable $\xi = 2z$ in equation (17), we find

$$P_1 = \frac{1}{2^{2n_1/2}} \frac{1}{\Gamma(2n_1/2)} \int_b^a \xi^{\frac{2n_1-2}{2}} \exp[-\frac{\xi}{2}]\, d\xi = 1 - \alpha \tag{18}$$

which is the coverage probability of the confidence interval I_1 based only on the sample $X_1, \cdots, X_{n_1}$. On the other hand, when we substitute $\lambda = 0$, that is, we let $F_\epsilon \to \infty$ in equation (11), we get $P_1 = I_\Gamma(n_1, \frac{b}{2}, \frac{a}{2}) - I_\Gamma(n_1, \frac{b}{2}, \frac{a}{2}) = 0$ which also agrees with the decision of pooling the two samples. So, the whole confidence coefficient is carried by P_2 in such a case.

4. Expected length of the conditional confidence interval

The length of the conditional confidence interval is expressed by the random variable

$$L = \begin{cases} 2(\frac{1}{b} - \frac{1}{a})U_1, & \text{if } \eta > F_\epsilon \\ 2(\frac{1}{d} - \frac{1}{c})(U_1 + U_2), & \text{if } \eta \le F_\epsilon \end{cases} \tag{19}$$

The expected value of the random variable L is given by

$$E(L) = E(L \mid \eta > F_\epsilon)P(\eta > F_\epsilon) + E(L \mid \eta \le F_\epsilon)P(\eta \le F_\epsilon). \tag{20}$$

THEOREM 1. *The expected length of the conditional confidence interval, expressed as a fraction of θ_1, is given by*

$$\begin{aligned} \frac{E(L)}{\theta_1} &= 2(\frac{1}{d} - \frac{1}{c})[n_2\phi I\beta_{\frac{n_1\phi F_\epsilon}{n_2}}(n_1, n_2+1) + n_1 I\beta_{\frac{n_1\phi F_\epsilon}{n_2}}(n_1+1, n_2)] \\ &\quad + 2(\frac{1}{b} - \frac{1}{a})\, n_1 \times I\beta_{\frac{n_2}{n_1\phi F_\epsilon}}(n_2, n_1+1), \end{aligned} \tag{21}$$

where $I\beta$ represents the Euler Incomplete Beta function (25).

To prove Theorem 1, we evaluate separately $E(L|\eta > F_\epsilon)$ and $E(L|\eta \le F_\epsilon)$ below.

4.1. Evaluation of $E(L|\eta > F_\epsilon)$. To evaluate $E(L|\eta > F_{\epsilon})$, let us make the change of variables $x = u_1$ and $y = \frac{n_2 u_1}{n_1 u_2}$ in equation (5). The Jacobian for this transformation is $\frac{n_2 u_1}{n_1 u_2^2}$ and, the ranges of x and y are $[0, \infty)$ and $[F_\epsilon, \infty)$, respectively. The probability density function of the random vector (X, Y) is given by

$$g(x,y) = K(\frac{n_2}{n_1})^{n_2} \frac{x^{n_1+n_2-1}}{y^{n_2+1}} \exp[-x(\frac{1}{\theta_1} + \frac{n_2}{n_1\theta_2 y})]. \tag{22}$$

Thus, the conditional expectation $E(L|\eta > F_\epsilon)$ is given by

$$E(L|\eta > F_\epsilon) = \frac{2(\frac{1}{b} - \frac{1}{a})}{P(\eta > F_\epsilon)} \int_{x=0}^{\infty} \int_{y=F_\epsilon}^{\infty} x g(x,y)\, dy\, dx. \tag{23}$$

After integration and simplification, we get

$$E(L|\eta > F_\epsilon) = 2\frac{1}{P(\eta > F_\epsilon)}(\frac{1}{b} - \frac{1}{a})\, \theta_1 n_1 \times I\beta_{\frac{n_2}{n_1 \phi F_\epsilon}}(n_2, n_1 + 1) \tag{24}$$

where

$$I\beta_{x_0}(m,n) = \frac{1}{\beta(m,n)} \int_0^{x_0} \frac{t^{m-1}}{(1+t)^{m+n}}\, dt, \tag{25}$$

as defined in Pearson [13].

4.2. Evaluation of $E(L|\eta \le F_\epsilon)$. Similarly, to evaluate $E(L|\eta \le F_\epsilon)$, we make the change of variables $x = u_1 + u_2$ and $y = \frac{n_2 u_1}{n_1 u_2}$ in equation (5). The Jacobian for this transformation is $\frac{(n_1 y + n_2)^2}{n_1 n_2 x}$. This leads to the joint probability density function of (X, Y) given by

$$h(x,y) = K\frac{n_1^{n_1} n_2^{n_2}}{(n_2 + n_1 y)^{n_1+n_2}} x^{n_1+n_2-1} y^{n_1-1} \exp[-(\frac{1}{n_2 + n_1 y})(\frac{n_1 xy}{\theta_1} + \frac{n_2 x}{\theta_2})] \tag{26}$$

with $0 \le x < \infty$ and $0 \le y \le F_\epsilon$.

Thus,

$$E(L|\eta \le F_\epsilon) = \frac{1}{P(\eta \le F_\epsilon)} 2(\frac{1}{d} - \frac{1}{c}) \int_{x=0}^{\infty} \int_{y=0}^{F_\epsilon} x\, h(x,y)\, dy\, dx. \tag{27}$$

After integration and simplification, we get

$$E(L|\eta \le F_\epsilon) = 2(\frac{1}{d} - \frac{1}{c})\frac{1}{P(\eta \le F_\epsilon)}\theta_1[n_2 \phi I\beta_\delta(n_1, n_2+1) + n_1 I\beta_\delta(n_1+1, n_2)], \tag{28}$$

where $\delta = \frac{n_1 \phi F_\epsilon}{n_2}$. Substituting terms from equations (24) and (28) into equation (20), we get the result stated in Theorem 1.

As a partial check of the above results, we first let $F_\epsilon = 0$ in equation (21), that is, we always reject H_0 and use the confidence bounds based only on the sample $X_1, \cdots, X_{n_1}$. In this case, $\frac{n_1 \phi F_\epsilon}{n_2} = 0$ and $\frac{n_2}{n_1 \phi F_\epsilon} \to \infty$. This gives $E(L) = 2(\frac{1}{b} - \frac{1}{a}) n_1 \theta_1$ which is the average length of the confidence interval I_1. On the other hand, if we let $F_\epsilon \to \infty$ in equation (21), that is we always pool the two samples, we get $E(L) = 2(\frac{1}{d} - \frac{1}{c})[n_1\theta_1 + n_2\theta_2]$ which the average length of the confidence interval I_2 as expected.

COROLLARY 1. *When $F_\epsilon \to \infty$, the expected length of the conditional confidence interval increases monotonically as ϕ increases.*

Proof. When $F_\epsilon \to \infty$, the average length of the conditional confidence interval is given by

$$E(L) = 2(\frac{1}{d} - \frac{1}{c})[n_1 + n_2\phi]\theta_1 \tag{29}$$

which is an increasing function of ϕ. The maximum value corresponds to the average length of the confidence interval I_2 when $\theta_1 = \theta_2$.

THEOREM 2. *The average length of the conditional confidence interval is maximum for $F_{\epsilon'} = (\frac{n_2}{n_1})^2 [\frac{1/d - 1/c}{1/b - 1/a - 1/d + 1/c}]$ and, decreases monotonically as we move from this optimum in both direction.*

Proof. If we differentiate the expression (21) with respect to F_ϵ, we get after simplifications

$$\frac{dE(L)}{dF_\epsilon} = \frac{n_1^{n_1} n_2^{n_2} \phi^{n_1+1} F_\epsilon^{n_1-1}}{(n_2 + n_1\phi F_\epsilon)^{n_1+n_2+1}} \left[n_1^2(k_2 - k_1)F_\epsilon + n_2^2 k_2\right] \tag{30}$$

where $k_1 = 2\,(\frac{1}{b} - \frac{1}{a})$ and $k_2 = 2\,(\frac{1}{d} - \frac{1}{c})$. The equation, $\frac{dE(L)}{dF_\epsilon} = 0$, admits the roots $F_\epsilon = 0$, $F_\epsilon = \infty$ and $F_\epsilon = F_{\epsilon'} = \frac{n_2^2}{n_1^2} \frac{k_2}{k_1 - k_2} = (\frac{n_2}{n_1})^2 [\frac{1/d - 1/c}{1/b - 1/a - 1/d + 1/c}]$. It is easy to see that $k_2 - k_1 < 0$, $k_2 > 0$ and $F_{\epsilon'} > 0$, since that, percentile values of chi-squared distributions increase as the number of degrees of freedom increases. Furthermore, $\frac{dE(L)}{dF_\epsilon} > 0$ for $F_\epsilon < F_{\epsilon'}$ and, $\frac{dE(L)}{dF_\epsilon} < 0$ for $F_\epsilon > F_{\epsilon'}$. Therefore, $F_\epsilon = F_{\epsilon'}$ is a maximum value for $E(L)$. This result suggests to search the best preliminary test significance level ϵ only by numerical studies. However, we have to consider large values of F_ϵ, that is, small values of ϵ.

5. Coefficient of variation of the length of the conditional confidence interval

The sensitivity of the length of the conditional confidence interval can be measured by the length variance and more accurately by its coefficient of variation. To evaluate these two parameters, we need first to compute the second order moment

$E(L^2)$ of L. The value of $E(L^2)$ follows from the mathematical routines previously applied to evaluate $E(L)$. After integration, we get

$$\frac{E(L^2)}{\theta_1^2} = 4\Big[(\frac{1}{d} - \frac{1}{c})^2[n_2(n_2+1)\phi^2 I\beta_{\frac{n_1\phi F_\epsilon}{n_2}}(n_1, n_2+2) \\ + n_1(n_1+1)I\beta_{\frac{n_1\phi F_\epsilon}{n_2}}(n_1+2, n_2) + 2n_1n_2\phi I\beta_{\frac{n_1\phi F_\epsilon}{n_2}}(n_1+1, n_2+1)] \\ + (\frac{1}{b} - \frac{1}{a})^2 n_1(n_1+1)I\beta_{\frac{n_2}{n_1\phi F_\epsilon}}(n_2, n_1+2)\Big]. \quad (31)$$

The variance and the coefficient of variation of the length L is easily obtained by combining equations (21) and (31). However, the obtained expressions are difficult to handle analytically. Nevertheless, in the case, $F_\epsilon \to \infty$, we have

$$Var(L) = 4(\frac{1}{d} - \frac{1}{c})^2\theta_1^2[n_2\phi^2 + n_1] \quad (32)$$

and

$$CV(L) = \frac{\sqrt{Var(L)}}{E(L)} = \sqrt{\frac{n_2\phi^2 + n_1}{(n_1 + n_2\phi)^2}}. \quad (33)$$

The partial differentiation of the coefficient of variation of L with respect to ϕ gives

$$\frac{dCV(L)}{d\phi} = -\frac{n_1n_2(1-\phi)}{(n_1+n_2\phi)^3}\left[\frac{n_2\phi^2 + n_1}{(n_1+n_2\phi)^2}\right]^{-\frac{1}{2}} < 0. \quad (34)$$

This shows that, for large values of F_ϵ, the coefficient of variation of L decreases when ϕ increases from 0 to 1.

6. Shrinkage interval estimator

Although preliminary test point estimators often provide better results than non conditional estimators, they have been found to possess high risks, see, for instance, Ahmed [1]. Therefore, shrinkage version of preliminary test estimators have been proposed which dominate, in terms of bias and efficiency, the usual preliminary test estimators, see, e.g., Ahmed [1] and, Ahmed and Badahdah [2]. Accordingly, we propose the following shrinkage interval estimator

$$I^S = \begin{cases} I_1, & \text{if } \eta > F_\epsilon \\ I_2^S, & \text{if } \eta \le F_\epsilon \end{cases} \quad (35)$$

where

$$I_2^S = \begin{cases} I_1 & \text{with probability } \gamma \\ I_2 & \text{with probability } 1-\gamma. \end{cases} \quad (36)$$

The probability $1-\gamma$ reflects the degree of confidence in H_0 when this hypothesis is not rejected. However, the prior choice of γ is very ambiguous and the experimenter has often to rely on data at hand to fix γ. Note that for $\gamma = 0$, $I^S = I$ and for $\gamma = 1$, $I^S = I_1$. In absence of any prior information, we suggest to use $\gamma = 1 - Pv(\eta)$

where $Pv(\eta)$ is the P-value corresponding to the observed η. Then, more $Pv(\eta)$ is large, more we use $I_2^S = I_2$, as it should be. Analytical study on interval I^S is not attempted here but numerical investigations are conducted. Figure 2 in Appendix **B** displays simulation results on the performance of I^S when $\gamma = 1 - Pv(\eta)$ and $\epsilon = .0001$.

7. Numerical application

As a numerical application of formulas (11), (16) and (21), we considered the cases of $n_1 = n_2 = 25$, $\alpha = 5\%$, $\phi = .2, .5, .8, .9, 1$ and $\epsilon = 0, 1, .001, \epsilon^{'}$. In this case, $\epsilon^{'} = .99$ and corresponds to $F_{\epsilon'} = .516$. The obtained results are summarized in Table 1 of appendix **A**. The last four columns of Table 1 represent respectively: the coverage probability of I ($P.C.I.$), the expected length of I ($E.L.I$) which is expressed as a fraction of θ_1, the expected length ($E.L.I_1$) of I_1, with same coverage probability as I and also expressed as a fraction of θ_1, and, the reduction rate (RR) in average length of I_1 due to conditional estimating . The reduction rate is evaluated as $RR = [E.L.I_1 - E.L.I]/E.L.I_1$. From column 2, we see that the coverage probability of I is always .95 when $\epsilon = 1$ and when both $\epsilon = 0$ and $\phi = 1$. This agrees with the extreme cases of always rejecting H_0 and always accepting H_0, respectively. In all other cases, the coverage probability of I is less than or equal to .95. From column 3, we see that the expected length of I is maximum for $\epsilon = \epsilon^{'}$ and it is always less than or equal to the average length of I_1 which has a 95% coverage probability, expect when $\epsilon = \epsilon^{'}$. However, when $\epsilon = \epsilon^{'}$, the expected length of I is close to the expected length of I_1 since $\epsilon^{'} = .99 \simeq 1$. Furthermore, we see that for $\epsilon = 0$, the expected length given in column 3 increases as ϕ increases from 0 to 1. This agrees with Corollary 1. Now from column 5, we remark that for $\epsilon = .001$ the reduction rate RR increases significantly as ϕ increases and reaches the maximum value .32 for large values of ϕ. Moreover, the associated coverage probability is fairly large especially when ϕ is large. Therefore, for large values of ϕ, the interval I based on $\epsilon = .001$ performs better than I_1 in this situation.

8. Simulation results

To see the performance of intervals I and I^S over interval I_1, we conducted a simulation study in the cases of $5 \leq n_1 \leq 20$, $5 \leq n_2 \leq 20$, $\alpha = .05$, $\phi = .1$ to 1, and, $\epsilon = 1, 10^{-1}, 10^{-4}, 0, \epsilon'$; ($\epsilon' \simeq 1$). To this end, we randomly drew 10^6 pairs (n_1, n_2) and for each pair we generated 10^5 random samples $X_1, \cdots, X_{n_1}$ and $Y_1, \cdots, Y_{n_2}$ from $\exp(\theta_1)$ and $\exp(\theta_2)$ distributions, respectively. Then, for each situation of the parameter space, we estimated empirically the average coverage probability, the expected length, the median length and the coefficient of variation of I, I_1 and I^S. The results are illustrated in figures 1 to 4 of Appendix **B**. The simulation showed that interval I_1 always maintains a 95% coverage probability. In the particular case $\epsilon = 1$, I and I_1 have on average same length and coverage probability. Figure 1 shows that for $\epsilon = 10^{-1}$, I and I_1 have almost the same average length and coverage probability when $\phi \leq .2$. However, when $.2 < \phi < .5$, both coverage probability and average length of I decrease. For $\phi \geq .5$, the coverage probability of I increases and reaches a maximum of 96% of coverage probability of I_1 while the average length of I continues to decrease to reach about 72% of average length of I_1. Figure 2 shows that for $\epsilon = 10^{-4}$ the coverage probability of I decreases when $\phi \leq .3$ and increases when $\phi > .3$. This coverage probability gets close to .95 when $\phi > .75$. Furthermore, the average length of I decreases when $\phi < .4$ and increases slowly for $\phi > .4$. Nevertheless, it keeps a value less than or equal to 64% of length of I_1 for $.4 < \phi \leq 1$. Figure 2 also shows that, for any value ϕ, I^s has almost a 95% confidence level but a very large average length in comparison to I. Note that I^S is computed using the probability γ =1 - p-value corresponding to η. Figure 3 shows that, for $\epsilon = 0$, the average length of I increases uniformly for $\phi > 0$ and takes on values much smaller than the average length of I_1. Figure 4 shows that, for $\epsilon = \epsilon'$, I and I_1 have very similar average length and coverage probability for $\phi \leq .8$. However, when $\phi > .8$, I has a slightly larger length and coverage probability than I_1. On the other hand, the simulations have shown globally that the median lengths have same trend as average lengths and, that I has smaller average length coefficients of variation than I_1. Consequently, $\epsilon = 10^{-4}$ provides a better interval estimator I for θ_1 when $\phi \geq .75$. Although I^S performs globally better than I_1, it has less performance than I for large values of ϕ.

9. Conclusion

We have proposed a conditional confidence interval for the parameter θ_1 of an exponential distribution. The interval is based on a decision following a pre-test on equality of two exponential parameters θ_1 and θ_2. Formulas for the expected length and probability coverage are derived. A shrinkage version of the interval is also developed. Furthermore, numerical studies on the accuracy of the estimators are also performed. Simulations results showed that, roughly, when $\phi = \theta_2/\theta_1 > .75$, the conditional interval obtained with a pre-test significance level around .0001 performs better than the other estimators. However for small values of ϕ the coverage probability of the conditional interval is very small and therefore it is better to use one of the other estimators in such situations. Consequently, one has to rely on a prior knowledge about the parameter ϕ, whenever this is possible, to make a better decision. Finally, it is worth noting that Chiou and Han [4,5] have considered conditional interval estimation based on one sample in the two-parameter exponential model. In Chiou and Han [4] the conditional interval estimation of the exponential scale parameter θ following rejection of a preliminary test is investigated. The inference is based on a two censored type II sample of size r from a two-parameter exponential distribution. After rejection of a two sided preliminary test with significance level α for the null hypothesis $H_0 : \theta = \theta_o$, a conditional and an unconditional intervals with a same confidence level are constructed. The α level critical region K of the preliminary test is given by $K = \{T : 2T/\theta_0 < \chi_{\alpha/2} \text{ or } 2T/\theta_0 > \chi_{1-\alpha/2}\}$ where T is the total test time. The conditional probability density function of T given $T \in K$ is an excised gamma distribution with the boundary points at the critical values of the preliminary test. This conditional distribution depends on α and $\Psi = \theta/\theta_0$. When $\Psi \to 0$ or ∞ or when $\alpha \to 1$, the conditional distribution of T converges to the unconditional distribution of T. The conditional confidence interval is constructed following the procedure set forth by Meeks and D'Agostino [11] while the unconditional one is the usual confidence interval for an exponential scale parameter with unknown location parameter. The bisection method has been used to compute the bounds of the conditional interval. The actual coverage probability that is provided at the fixed nominal level of the unconditional interval is computed using the conditional probability distribution of T since this interval is defined only after rejection of H_0. A Table giving the ratio of the length of a conditional 90 percent confidence interval to the length of an unconditional 90 percent confidence interval as function of $\hat{\Psi} = \hat{\theta}/\theta_0$ in the case $r = 8$ and $\alpha = .10$ is presented. Another table displaying the coverage probability of 90 percent nominal unconditional confidence intervals in the cases $r = 8$, $\alpha = 0.05, 0.10, 0.20, 0.40$ and $\Psi = 0.1(0.1)4.0$ is also given. It has been remarked that the conditional interval is shorter than the unconditional interval within a region of $\hat{\Psi}$ near $\chi^2_{\alpha/2}/2(r-1)$ and $\chi^2_{1-\alpha/2}/2(r-1)$ and that the ratio of lengths increases significantly as $\hat{\Psi}$ moves away from these bounds. This ratio reaches the value 1 from above when $\hat{\Psi}$ becomes very small or very large. This result agrees with the fact that the conditional distribution of T given $T \in K$ and the unconditional distribution of T gets closer to each other as $\hat{\Psi}$ increases to ∞ or decreases to 0. On the other hand, the conditional coverage probability of the unconditional interval exceeds the nominal level over much of the region of Ψ but it is much smaller than the nominal level for values of Ψ around 1 unless α is very large. This also agrees with the fact that the unconditional distribution of T and the corresponding unconditional distribution resemble each other when $\Psi \to 0$ or ∞ or when $\alpha \to 1$. It is then recommended to use the unconditional interval following the rejection of H_0 only if one has a prior knowledge that Ψ is not close to one and the direct use of the unconditional interval following rejection without this knowledge is arguable.

In Chiou and Han [5] a conditional interval estimation of the location parameter η in the exponential model with two parameters is considered. The estimation is based on a random sample of size n and the construction of the confidence intervals follows the rejection of a two sided preliminary test of the null $H_0 : \eta = \eta_0$. The α critical region of the test $H_0 : \eta = \eta_0$ against $H_a : \eta \neq \eta_0$ is given by $\mathcal{K} = \{X_{(1)} : 2n(X_{(1)} - \eta_0)/\theta < \chi^2_{\alpha/2}$ or $2n(X_{(1)} - \eta_0)/\theta > \chi^2_{1-\alpha/2}\}$ and depends upon the first order statistic $X_{(1)}$. The conditional confidence interval is constructed following the procedure set forth by Meeks and D'Agostino [11] and the unconditional one is the usual confidence interval for the exponential location parameter. The actual coverage probability of the unconditional interval is computed using the conditional probability distribution of $X_{(1)}$ given that $X_{(1)} \in \mathcal{K}$. The cases of known scale parameter and unknown scale parameter are separately considered. Tables for the ratio of the length of the two intervals and for the conditional coverage probability of the unconditional interval in the case of known and unknown scale parameters are provided. A performed numerical study has shown that length of the conditional confidence interval is shorter than that of the unconditional interval over the region $x_0 = 2n(X_{(1)} - \eta_0)/\theta < \chi^2_{\alpha/2}$ and within a small region near $x_0 = 2n(X_{(1)} - \eta_0)/\theta > \chi^2_{1-\alpha/2}$ and that the ratio of lengths increases as x_0 moves from the interval bounded by these values. On the other hand, the conditional coverage probability of the unconditional interval maintains a high level over much of the region of $\Psi = 2n(\eta - \eta_0)/\theta$ but for a considerable set of values of Ψ around zero the conditional coverage is far less than the nominal level unless α is large. Therefore it has been concluded that if one has prior knowledge that Ψ is not far away from zero, then the conditional confidence interval following rejection of H_0 may be preferable to the unconditional one.

In this paper we have considered conditional and unconditional intervals estimations of the scale parameter, in one-parameter exponential model, following a preliminary test on equality of two exponential scale parameters. It is worth noting that the scale parameter is the most important one in practice. The inference procedure that we used is the one initiated by Bancroft [3] and it is quite different from the conditional estimation following rejection of the preliminary hypothesis used by Chiou and Han [4,5] although they are closely related. This difference is pointed out in the introduction sections of Chiou and Han [4,5]. In our case, the average length and the probability coverage of the unconditional confidence interval do not depend on the significance level and the result of the preliminary test. Only, the conditional interval which is a mixture of the unconditional interval based on the main sample and the unconditional interval based on the pooled sample depends on the preliminary test significance level and its rejection or non-rejection. As in the finding of Chiou and Han [4,5], none of these two intervals totally outperforms the other in the whole parametric space. Therefore, the always use of unconditional confidence intervals independently of the preliminary test result may lead to less accurate estimations.

The numerical studies have been carried out with Mathematica [16] and Gauss (Version 3.2.42).

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Appendix

A. Numerical Application

TABLE 1. Coverage probabilities, expected lengths and reduction rates in the case $n_1 = n_2 = 25$

B. Figures

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