CONDITIONAL EXPECTATION IN AN OPERATOR ALGEBRA, I

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(Received January 25, 1956)

1. Introduction. The theory of integration on a measure space has been generalized to a W*-algebra by Segal [10] and Dixmier [2] as a non-commutative extension of it. Applying their theory, some parts of the probability theory may be described in a certain W*-algebra. In the paper of Dixmier [2], he has proved the existence of a mapping x-xP defined op, aa semi-finite W*-algebra A acting on a Hilbert space H into its W*-subalgebra Al with the similar properties of the Dixmier's trace (=natural mapping) in the finite W*-algebra, A being semi-finite provided every non-zero proje-ction in A contains a non-zero finite projection in A (cf. [5]). In the previous paper [11], we have discussed for a o-finite finite W*-algebra A (with the faithful normal trace with (I)=1) that the mapping x-is defined on L1(A) and valued on L1(A1) and it has the likewise properties with the c)nditional expectation in the usual probability space, and we have also cLlled it the conditional expectation relative to the W*-subalgebra A1, where L'(A) being a Banach space of all integrable operators on H in the sense of Segal (cf. [10]) which coincides with that in the sense of Dixmier (cf. [2]) as Banach space. Nat amura-Turumaru have also given a very simple proof of the characterization theorem of the conditional expectation in A (cf. [S]). If A is a commutative W*-algebra with a faithful normal trace 1a, then there exists a probability space (f2, B, v) such that, considering the space B of all bounded random variables as the multiplication algebra on a Hilbert space L'(f2, B, v), B is isomorphic with A by the canonical mapping satis-

frying, (x) f(1(x))(co)dv(co)-for every x E A. Conversely, let (i1, B, v) be
sz

a probability space. Then the multiplication algebra B is a W*-algebra on L2(12, B, v) and, defined by the above equation, is a faithful normal trace on it. Furthermore, the canonical mapping 4 defines an isomorphism between L (A) and L"((2, B, v) as Banach spaces (1), 11(A) being the Banach space defined by Dixmier (cf. [2]). For any W*-subalgebra Al of A, there corres-ponds a cr-subfield B, of B, and A1, Lr(A1) are isomorphic with B1, Lry(Q, B1, v) respectively, where B1 being the multiplication algebra of the bounded random variables on (1l, B1, v). The conditional expectation defined for the commutative algebra A (relative to the A1) is transformed to the one defined for the corresponding probability space (f2, B, v) (relative to the B1) by the canonical mapping (cf. [7] and [11]).
In the probability theory, the martingales have been investigated by many authors, particularly by Doob, Levy and Ville (cf. [3]), which is defined.

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by a linear system of the conditional expectations. The concept of the martingale can be extended to a non-commutative W*-algebra as the gene-ralized conditional expectation.
In the present paper, we shall begin with a characterization theorem
of the Dixmier mapping in a semi-finite W*-algebra (cf. Theorem 1 of 2). This is a generalization of the characterization theorem of the conditional expectation of Moy (cf. Theorem 3 of [7], also cf. Nakamura-Turumaru [8]), and we shall call the Dixmier mapping to be the conditional expectation (cf. 2 below). In 3, we shall give the definition of the M -net in a semi-finite or finite W*-algebra A with respect to a given gage i (cf. [10] or [2], and cf. 2 below). If A is commutative and is a faithful normal trace, then any M--net is transformed to a martingale in the corresponding probability space (12, B, v). In Theorem 2 and its Corollary, for a o--finite finite W*-algebra A with a faithful normal trace 1t we shall prove that for an M -net to be simple (cf. Def. in 3) and to converge in Lt-mean, are equivalent to the weak conditional compactness of it, or to the uniform c-integrabilities of the real and imaginary parts and Lt-uniform boundedness, and moreover that if an M-net is uniformly bounded then it is simple and converges strongly to a bounded operator in A. If the directed set D is decreasing (cf. 3), then any M-net with finite integral in semi-finite A necessarily converges strongly, and if the, M-net belongs to L2(A) then it converges in the V-mean. These facts can be applied to a convergence of a sequence of bounded operators (cf. In h or IL-factor, Theorem 6 of [9]), which was introduced by von Neumann and we can show that it is a simple M -net. I want to thank Mr. Sakai for his valuable remarks.

2. Let A be a semi-finite W*-algebra on a Flilbert space H with a regular gage in the sense of Segal (cf. [10]) which is considered as the

"normale, fidele, essentielle et maximale" trace in the sense of Dixmier (cf.
[2]). Let L'(A) and L2(A) be the space of all integrable and square integrable operators with respect to in the sense of Segal respectively (cf. [10] and. [2]). Denote the set of all-integrable operators belonging to A by J which is a two-sided ideal of A and is dense in L1(A) and L2(A) relative to the respective norm 1 (=L1-norm) and j 1(=V-norm). Dixmier has proved the following theorem (cf. Theoreme 8 of [2]).

THEOREM D1>Let Al be a W*-subalgebra of A. Then there exist a maximal central projection p1. in Al and a linear mapping x-*x'e from A into itself such that the range Ac=puA1 and for all x E A
(D.1);x6.<Jj x, xbeing operator bound.
(D. 2) xee=xe and x*cX<(x*x).
1) Dixmier has proved more strict conditions i. e. (D. 8):xe J),.J x j,. for x E
Jl. (r 1) and (D. 9) (D. 9) holds for x E J11r1 and y E (J (1 A1)11 (1/ri1/rz=1), where the power 1/r and the norm 1. are notations of him (cf. [21). But we can see their equivalences such as (D. 9) implies (D.8)', (D. 8)' implies (D. 9) (by (D. 5) and by Holder's inequality of Dixmier, (cf. [2] and Proposition 5 of [2]), and (D. 9)' implies (D. 9) by (D. 7).

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(D.3) x 0 implies x>0. (D. 3) x*e=xe*.
(D.4) x>0 and xe=0 imply pp=0.
(D.5) (yx&)=yxey' and (p xp)e=xe for y, y' E A1.
(D.6) (xy)e=(yx)e for every y E A fl Ai.
(D.7) The mapping x--*x is strongest and weakest continuous.
(D.8)) xeGI; x(1 and j xe CI2 xz for every x E J.
(D.9) (xye), (xey) for every x E A and y E J.
It is clear that Ie=, and the mapping x--x isextensible uniquely onto LI(A) and L2(A) by (D.8). In the case that the gage is finite, regular and normal i. e. /L(I)=1(we shall call such a to be a faithful normal trace), the Dixmier's mapping x-* xe satisfies Ie=I. In the previous paper [11], we called such a mapping x--xe from L1(A) into L1(A1) with respect to the faithful normal trace to be a conditional expectation relative to A1. If A is commutative, it coincides with the conditional expectation in the usual probability sense on the corresponding probability space (S, B, v). In the present paper, we shall also call the Dixmier's mapping x--xe from LI(A) (or L2(A)) into itself to be a conditional expectation relative to A1, and xe denotes it.

Firstly we shall prove a characterization theorem of the conditional expectation in the semi-finite case. Let A be a semi-finite W*-algebra and let µ be a regular gage on A. Then

THEOREM 1. Let x-xG be a linear mapping from A into itself satisfying
the conditions (D.2), (D.3), (D.9) and P<I. Then for any x E A, xe coincides with the conditional expectation x relative to the W*-subalgebra Al which is the direct sum of A {xe; x E A} and {X(I-Ie): X complex numbers}.

PROOF. The linearity of xe and (D.3) imply obviously (D.3). Since for any x E A, 0<x*x x*x 1, by (D.2), (D.3) we have

(1) 0 E x*x<(x*x)S x*x l I

and, xe Ie x(so we have (D.1). Let {xy}D c A be a uni-formly; -bounded directed set converging weakly to x E A, D being a directed set, then p(xyy)-+p(xy) for any y E I. By (D. 9), JF cJ and

(2) jit(xyy)(xyye) "apt xye)/L(xey)
for every y E J. Since {x} is uniformly jI, -bounded, (2) implies the weak convergence of xy to xe on H by Dixmier's Theorem (cf. Corollary 2 of [2]). We shall now prove that AE is a weakly closed self-adjoit subalgebra*)
of A. If x E A then xE*=x' E A&, i. e. AE and similarly f are self-adjoint. While, by (D. 2), for any x E Je
(3) x*x=x*exe<(x*x)E.
As P<I, for any x E J
) In this paper, by a weakly closed self-adjoint algebra we mean a-algebra which is closed in the weak operator topology, not necessarily having the identity operator; and by a W*-algebra we mean a weakly closed *-algebraa which has the identity operator.

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(4) μ(x2)=μ(xIe)≦μ(x).

Therefore, putting y=(x*x)F-x*x, since y E J+ and by (3), (4)
0, (y)=((x*x)F)--a(x*x)<0.
This implies y=0 and x*x=(x*x) which belongs to J For any x, y E J, xy

can be expressed by; Xzz; for some z; E JE and complex numbers X; (j=

1, 2, 3, 4). Consequently,

(xy)=X (zjzs)e=X, z*z, =xy

and xy E J. Therefore J is a self-adjoint subalgebra of A. Next we shall prove AE=F. For x E J, there is {xy}D C Je such that xy<JJ x and xY converges weakly to x by a Kaplansky's Theorem (cf. [6]). Hence xy=xY con-verges weakly to x=xF and x E A, i. e. P c: Ae. Conversely, since J=A, for x E Ae we can take {xy}D c j crnverging weakly to x and xy <;x, and obtain that xy canverges weakly to xF=x, i. e. x E J'. Therefore AF=JF, and AE is a self-adjoint weakly closed subalgebra of A.
Further, we prove that P is a self-adjoint unit element in the algebra AF. For any x E JI and y E AE, since xy E J by the above fact, (5),(yxIE)=C(yx)eI)=1a(yx).
Hence, for any complex number h and for any y E AE, ((y+XI)xP)=((y+X1)x).
This implies zxL)=zx) for every x E Je and z E Al. Therefore we have x P=x and similarly=Px for every x E J. Since P=P is clear and since Af=P, P is a self-adjoint unit element in AF. Consequently Al is the direct sum of A and {X(I-IC); X complex numbers}, and P is a maximal central projection in A1.
Finally, in order to prove x=x for all x E A, x being the conditional expectation relative to A1, we show J=J n A1. Each x E J fl Al is expressed by x+X(I--Ie) for some x E J and X, and hence for every y E J se(xy)=((x'+X(1+P))y)=1c(x'y)+Xa((I-P)y)=
Since x E J and x' E J,, u(x(y+X'I))=p(x'(y+X'I)) for every y E J and complex numbers X'. This implies easily a(xz)=1a(x'z) for all z E Al and x=x' which belongs to J. Since J c J fl Al is clear, we obtain J=J fl Al. Therefore, for every x E A and y E J fl Al (=JE),
(x'y)=, (xy)=(xy)=
i. e. xF=xe for all x E A.

Using a method of Nakamura and Turumaru (cf. Cor. of [3]), the

2) For any subset S in A, S denote the weak closure (as operator on Hj of S which coincides with the strong closure when S is convex, S+ denotes the set of all non-negative operators in S.

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conditional expectation satisfying the condition P=I can be characterized as the following

COROLLARY 1.1. Let x--t be a linear mapping from A into itself satisfying (D.2), (D.8), 1=I and

(6) (xy) (xyE)<+c for every x E A and y E J}
Then the range AE is a W*-subalgebra and xE coincides with the conditional expectation relative to AF.

PROOF. Taking y E J+such that 4y)=1 and putting a(x) (xye) for
x E A, then by (6) and (D. 2) 6(xE) tx (x) for every x E At Since P=I and o J(1)=(ye)=1, by the proof of the Nakamura-Turumaru's Theorem we have that o (x)=a-(x) for every x E A. This implies that (x y)=(xye) for every y E J+and hence for every y E J. Further, since the strong con-tinuity of xe (on bounded part) is followed from (b) (cf. Remark 1.2 below), and since (D.9) holds for x, y E J (by(6)), we obtain (4) and complete the proof.

REMARK 1. 1. We shall give in the last section (cf. 4) an example of the conditonal expectation satisfying P=I in a semi-finite W*-algebra. When is a faithful normal trace, i. e. A is a o--finite, finite W*-algebra, Theorem
I holds and Corollary 1.1 also characterizes the conditional expectation.
These characterizations are analytical and somewhat simple when compared
with the Theorem 2 in the preceding paper (cf. [11]).

REMARK 1. 2. In Corollary 1.1, if P<I then x--xe is strongly continuous
on the unit sphere of A and AE is a self-adjoint weakly closed subalgebra of A. The first part will follow from the fact that J CJ (by (6)) and for every x C A and yE1
<((x*x)(yy*)e) (7) xey iz=(y*x*exey)<(y*(x*x)ey)=((x*x)eyy*)

The second part follows by the similar way of the proof of Theorem 1. Further we remark that if is a faithful normal trace and the mapping x-*x satisfies P=I and a weaker condition than (6):

(8) (xe)<(x) for every x E A,
then A is a W*-subalgebra of A. Indeed, for the present A and, J=A and

xe'(x'xe)<((x*x)e)<(x*x) I; x: for every x E A.
This implies the strong continuity of x-xF on the unit sphere, and hence by the similar way of the first part in this Remark, we obtain the required
ones.

Let A be again a semi-finite W*-algebra and let be a regular gage on
A. For any W*-subalgebra Al of A, we shall also denote the contracted gage on Ar by. Then the space Y(A1) is considered as a closed subspace of Lr(A), r=1, 2. For any self-adjoint operator x in Y(A) (r=1,
3) In this case we have assumed (D.3) and (D.5) but not (D.3).

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2), let x=f X dEA(x) be the spectral resolution of x. Then each(xY

belongs to A. Denote by W(x) the W*subalgebra of A generated by {EA(x); X}. If x is not self-adjoint, then it can be uniquely expressed by x= x1>+ix where x and xt2 are the real and imaginary parts respectively. Then there correspond the W*-subalgebras W(x) and W(x2) to x and x2 respectively. Let W(x) be the W*-subalgebra generated by W(x(i)) and. W(xtj>). Then W(x) is a minimal W*-subalgebra of A containing the reso-lutions of identities EA(x1)) and EA(xJ)) of x(1> and x2 respectively. Under these notations we have

PROPOSITION. For any subset S of L(A) (r=1 or 2 rest.) there corresponds uniquely a minimal W*-subalgebra W(S) of A such that S c L'(W(S)). The operation S-k W(S) has the properties that, W(Lr(W(S)))=W(S); S e A implies W(S)=S4; Sl c S2 implies W(S1) c W(S2); and further for Si and S having' the same closed linear hull in D(A) (r=1, or 2 resp., W(S1)=W(S).
PROOF. Let S e L(A) (r=1 or 2 resp.) and let W(S) be a W*-subalgebra
of A generated by {W(x); x E S}. Since for any x E S xci>, x>E L(A) and the projections EA(x>), E(x2) belongs to W(S), x and xtj> are measurable with respect to W(S) in the sense of Segal (cf. [10]) and hence they belong to L(W(S)). For a W*, subalgebra W of A such that S c L(W), W(S) c W follows from [10]. Hence W(S) is minimal and uniquely determined by S. Now we prove W(L(W(S)))=W(S). Since S c Lr(W(S)), W(S) c W(L'(W(S))) (because S1 c S implies clearly W(S1) c W(S2)). Conversely, for x=x* in. Lr(W(S)), EA(x) E W(S) and hence W(D(W(S))) c W(S). The other parts in this proposition will easily follow from these facts.
The following corollary contains a generalization of a halt part of a theorem of Bahadur (cf. [1]).

COROLLARY 1.2. (10) Let A be a semi-finite W'-algebra with a regular
gage a, and put L=D(A). Let x-* x be a projection in L such that x*E=xe*. Then the following conditions are equivalent:
(1)x--x coincides with a conditional expectation xe (on L) relative to a certain W*subalgebra Al of A.
(2') L=Lr(W(L)).
(2) If is a faithful norm2l trace, then (1') and (2') are equivalent to the following each condition:
(3) LF contains a set adjoint subalgebra B of A such that I E B and B is L2-dense in L.
(4') AE=W(LF).

PROOF. (1). (1')-+(2'): Since Le=L2(A1) and Je is dense in Ii, by Theorem
1, Je=f Al n J fl W(f) Je and Je=J(W(f). Further, by the preceding Proposition, W(Je)=W(L). Hence we obtain V=L(W(L)) and (2') holds.

4) For any subset S of bounded operators or.. H, S' denotes the set of all bounded operators on H which with all operators in e S. S denotes (S')'. S" is a W*-algebra generated by S.

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(2')(1'):<ni> denotes the inner product in L. Since x-xE is a proje-ction in L, for x, y E L

(xfy)=<x, y*>=<x, y*F>=<x, y*>=
Let xe be the conditional expectation relative to W(LE). Then, for every x E L, xEe=x= and xeE=xe, and for every y E J
(9) (xey)=(xeey)=(xeyn)=(xy)=(xye)=(xey)
This implies that x= xe for every x E L. (2) will be followed immediately from (1) as its corollary.
By a slight modification of the proof of Cor.1.2 it also holds that: In Cor. 1.2, (1), put L=L1(A), and let x--} xE be a bounded linear mapping from L into itself satisfying x=xE and (D. 9) for every x, y E J. Then the condition (1') is equivalent to
(2) V=L1(W(D)) and J' C A.

3. As the preceding section we shall consider a semi-finite W*-algebra
A on a Hilbert space H, with a regular gage a. Let {xa, cr E D} be a family of operators in A (or 12(A) or L2(A) resp.), D being directed set. Let Aa be W*-subalgebra of A generated by {W(xy);y C a}, W(x) being the W*- subalgebra given in the Proposition in 2. Then Aa Ag if and only if a Q. If {xa, a E D} satisfies the conditions xa=xa for every a, E D (a c (3), where xea denotes the conditional expectation relative to Aa, then we shall call the family of operators {xa, a E D} to be an M-net (with respect to the gage, u), and {Aa, a E D} the family of W*-subalgebras associated to the M-net. We shall call an M-net to be increasing or decreasing, whenever: for any a,, & E D there exists'y E D such that a,<ry or 7<a, respe-ctively.
An example of M-net is given such as:Let {Ba, a E D} be a family of W*-subalgebras of A and suppose that Ba C B if and only if a /3. Let {xa, a E D} be a family of operators in L1(A) or L2(A) such that
(10) xa=xa for every a, E D (a /3)
where xEa denotes the conditional expectation relative to Ba. Then {xa, a E D} is an M nets>. We denote such an M-net by {xa, Ba, a E D}. Further, for any x E L1(A) or L2(A) putting xa=xEa (a ED), {xa, Ba, a E D} is also an M-net. Such an M-net {x,, B,, a E D} is called to be simple. Any finite M net is clearly simple. The sequence of bounded operators in L or IL-factor given by von Neumann (cf. p. 118 of [9] and cf. 4 in this paper) is an example of simple M-net.
If A is a or-finite commutative W*-algebra with faithful normal trace,
then any M-net {xa, a E D} in L1(A) is transformed to a martingale on the corresponding probability space by the canonical mapping.
By the definition of M-net and the properties of the conditional expec-tations the following conditions are equivalent for a given family of operators

5) That is, taking the corresponding family of W*-subalgebras {Aa, a E D}, it satisfies that xa=xa for every a, E D (a<).

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{xa, a E D} in/2(A) or L2(A):
(i) {xa, a E D} is an M-net.
(ii), (yxa)=,(yxp) for every a, & E D (a) and y E J fl Aa, Aa being
the W*-subalgebra given at the first paragraph in this section.
(iii) xa=x ra) for every a, f, y E D such that y a _/3, where e(y, a) denotes the conditional expectation relative to the W*-subalgebra W(xy, xa). If {xa, Aa, a E D} and {ya, Aa, a E D} are M--nets, then {xa, Aa, a E D}, {Xxa, Aa, a E D} and {xa+ya, Aa, a E D} are also M--nets, X being any com-plex number. We shall say an , M-net {xa, a E D} to be real or positive if xa=xa for every a E D or xa 0 for every a E D. Any M-net {xa, a E D} can be decomposed into two reat M-nets in an obvious way, that is,

{xa), a E D} and {x), a E D} where x)=-1-- (xa+x*) and xc=)=1: (xa-x) a a 2 a a 22 a

In an M -net {xa, a E D}, for any directed subset D' of D, {xa, a E D} is also an M--net.

Besides, we shall define a subset S C L'(A) to be uniformly a-integrable
if for any (9> 0 there is a positive number > 0 such that , a(p)<(p being projection in L'(A)) implies c(p j x I)<(9 for all x E S.
With these terminologies, w e shall prove

THEOREM 2." Let A be a o--finite, finite W*-algebra on a Hilbert space H with a faithful normal trace, and let D be an increasing directed set. Then, for a given M-net {xa, a E D}, the following conditions are equivalent:
(2.1) Both {x), a E D}, and {xa'), a E D} are uniformly -integrable and uniformly bounded in L'-norm.
(2.2) {xa, a E D} is weakly* conditional compact in L'(A).
(2.3) {xa, a E D} is simple.
(2.4) There exists x E L'(A) such that xa-x-+0.

PROOF. If the M -net {xa, a E D} is finite, the proof is trivial. Hence we consider the case that it is infinite. Let {A,, a E D} be the family of the

W*-subalgebras of A associated to {xa, a E D}. Let Ao =LJAa and let
aeD

AI be the weak closure of A0. Let zea be the conditional expectation of z E L'(A) relative to A.
Firstly we prove that (2.1)-} (2.2): Each xa is uniquely expressed by xa -'x such that x', xa E L(A)a x, x>0 and x' x"=0 for every a E D.

a a a a=a s

Since {xal), a E D} is uniformly t-integrable and x,) (=xa+x, for any (9>0 there exists a>0 such that, a(p)<implies for all a E D

(11) (pxa)+(pxa)=(p(xa+xa))=(p I xa I)<5/2.
Since (pxa), c(pxa)>0, both <(9/2 for all a E D, and hence {xa, a E D} and {xa, a E D} are uniformly c-integrable. Putting o(y)=tc(yxa) and o' (y) =(yx) for all y E A and a E D, y a and O-(belong to the conjugate

6) This theorem contains a generalization of L'-mean convergence of a martingale
in a probability space (cf. Theorem 1.4 of [3D.

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Banach space A of A. Let T' and T" be the weak closures of {o-a, a E D} and {o-, a E D} as functional on A respectively. Let 6o- E T' be a limiting point of {o-'a, a E D} which is a positive linear functional on A. We take a sequence of projections {p, j 1, 2,....} in A such that pLL (j*k). For

5>0, taking >0 as (11), there exists an integer ko>0 such that p

and since p is a projection in A, by (11)
J=ko

σα(Σpj)=μ(Σpj)x'α<ε/2.

Choosing o-a such that

σ'(Σρj)-σα(Σpj)<ε/2

J=kp

then for any integer k>kJ

σ'(∑pj)≦σ'(Σpj=ko)≦P(∑[k)|+σα(Σpj)

<<9/2+<9/2=<9.

Therefore a- is countably additive and by Dye-Radon-Nikodym's Theorem (cf. [4]), there exists x' E L'(A) such that x'>0 and

(12) a-(y)=(yx') for all y E A. Hence every a-in T' (and similarily every a-" in T") are represented as
(12) (and 6"(y)=p yx") for some x" E L'(A)). x' and x" are uniquely deter-mined by a-and a-in L'(A) respectively. Therefore the weak* closures of {xa, a E D} and {x,, a E D} in L1(A) are weak* compact in L'(A) and so is the weak* closure of {xa-xa, a E D}, i. e. a E D} is weakly* conditio-nally compact. Since for {xa'), a E D} the same fact can be proved by the same way and xa=xa>+a E D} is weakly* conditionally compact. Secondly we prove that (2.2)-(2.3): Put S={a-, a E D} a-a(y)=
and Sand Sthe weak* closures in Ai of S with respect to AI and Ao respectively, that is, the closures with respect to the weak* topologies on 12(A1) defined by the neighborhoods:
U(x0; zi,.....z12, 9>0){x E Ll (A1); I a((xo-x)z,) I <<9, j=1, 2....., n}
z; belonging to Al or A0 respectively. Then by (2.2) Sis weakly* compact. Since the weak* topology on Ll(A1) with respect to Al is stronger than the one with respect to A0, the canonical mapping from S1 to S is continuous, and it is also one-to-one. For, since Aa is strongly dense in Al (as operator on H), p x, z)=p x2z) for x1, x2 E Ll(A1) and for all z E A0 thus a fortiori, for all z E A1. Therefore S'1 is compact (and hence closed) in Shy, and S c Sr1

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mplies So C Slvo=S1. Further, by the definition of the M -net, lira, a(yxa)

(=v(y) say) always exists for every y E A0, which belongs to Sand hence S1. Consequently, there is an x E L'(A1) such that a-(y) (yx) for every y E A0. Since (yx)=lim (yxa) for every y E A, for any fixed cr E D and for any y E Aa,

(yx)=h mm 1(yxp)=li m 1i(yxa) (c<)=(yxa). Hence we obtain xa=xa for every a E D.
Next we shall show the equivalence of (2.3) and (2.4). Assume (2.3). For any z E A, putting za=zPa for all a E D and zt=zel, {za, a E D} is a simple M-net satisfying zea IJz<1 eel 2 and
(14) za-z1=(z*z1)-c(z*za)-0
and hence i za-zt za-zt h 0. Let x E L'(A) be xa=xea (cr E D) and take {z7j e Al such that xe'-z,,<1/3n (n=1, 2,....). Moreover for each n taking an E D such that z, z-zyia' t<1/3n for all a E D, (a, z a),
xel--a t<i xe z. (t+z,,-zPna t+(z-x)ea <1/n
and xel-xa t-+O. This implies (2.4). Conversely, assuming (2.4), lim (yxa) =(yx) for all y E A, and if y E Aa, then t(yxao)=(yx) for all a E D (a0
a). These facts imply that 1a(yxao)=(yx) for all y E Aaa.
Finally we shall prove that (2.3) and (2.4) imply (2.1). Let x E L'(A) be xa=xea and xa-x1-}0. The expressions x=x+ix(2) and x)=x-x(x',

x>0 and x'x"=0) are unique. Putting xa=xea and xa=x1 ea (a E D), xa

=xa- xa and {xa, a E D} and {xa, a E D} are positive simple M-sets
satisfying xa-x zi I xa-x" t-0. If {xaL), a E D} is not uniformly -inte-grable, then so is at least one {x,, a E D} or {x, a E D}. Indeed, if both are uniformly c-integrable, then let I xa (=vax being canonical decomposition, va being partially isometric operator,
I xaj) I) G(pvaxal))=c(pvaxa)-/(vaxa)
C (xa)1/(vavaxa)i 1+p(f xa)11(vavaxa)t 12
x (pxa)+x I tl 1a(Pxa)112,
because xa=x'a and xa=xea. This implies the uniformly c-integrability of {xa, a E D}. Now if {x,, a E D} i s not uniformly -integrable, then there exist an E>0, sequences of projections {pn} c A and indices {an} c D such that a(p)<1/n, c(mX) and an<a1,,+1 (n=1, 2,....). Let B be a W*-subal-gebra of A generated by {Aa,, 1 n=1, 2,....} and let x'e be the conditional

expectation of x' relative to B. Then j xan-x'6 11-*0 (n-+cx). Therefore </j7ax'n) C I fG(n(xan7a--x'e))I+(o(mx/e)) (x'an.-x'e'I1-1a(p x'e)-+0(n-o0). This is a contradiction. The uniform 1a integrability of {x/, a E D} also follows in the same way as for case of {xa'), a E D}. Q. E. D. For the M-nets in L2(A) and A we have the following

COROLLARY 2.1. Let A and 1a have the same meanings as Theorem 2. Let

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{xa, a E D} be an increasing M-net in L2(A) (in A, rest.). Then the following three conditions are equivalent:
(2.1) {xa, a E D} is uniformly bounded in L'-norm (-norm rest.).
(2.3)' {xa, a E D} is simple for x E L2(A) (for x E A rest.).
(2.4)' There exists an x E L2(A) (x E A rest.) such that xa converges to x
in the L2-mean (in the strong operator topology on H rest..

In the proof of this corollary, we shall use the notations in the proof of Theorem 1.

PROOF FOR THE L2(A)-CASE. (2.1)--(2.3): Since x,= jJ xal) iz--xa)
(cr E D) and for any projection p E A
(i(p J x) (f); gy(p)'/2(xa)*xa))hl'-C, (t)'/2 ii x, cJ) L (1=1, 2),
Theorem 1 (2.1) is satisfied. Hence {x, a E D} are simple in L1(A), i. e. xa')=x)ea and ((x)-XU) (-0 for some x) E LE(A), and

(yxj))I (y*y)l, 2(x(i))1/2*xa,))
for every y E A. Consequenly,) (yx) f<sup i'2 jy 2 andd xU) E L 2(A)

(j=1, 2). (2.3)-+(2.4) As in the proof of Theorem 2 (cf. the part (2.3 (2.4)), taking {z,, a E D} C A and z1 E A, (2.4)' holds (see (14)). Since A is dense in L2(A), we can show xa-x'1 2-0 by the same method of the proof of Theorem 1, if we take the L2 norm instead of V-norm. (2.4)'-+(2.3)' and (2.3)'--+(2.1)' are obvious by Theorem 1 and by the fact that the x belonging to L2(A).
PROOF FOR THE A-CASE. (2.1)'-+(2.3)' Since (xaxa)<(xa (,,(2.1)' holds for L2-norm and (2.3)' holds for x E L2(A). Hence, by Cor. 2.1. (2.4)', for every YEA

ii xy 2=p(x*xyy*)=1im tc(x*xyy*)<sup (xay2

and x E A. (2.3)-(2.4)': We can assume that x E A satisfies xa=xa and ((xa-xlJ-0. Whence for any y E A JJ (xa-x)y;(-0. Since J xa J c x (and A is dense in L2(A), xa--x strongly on the Hilbert space L2(A). This fact implies the strong convergence of xa to x on H. It is clear that (2.4)'-} (2.3)' and (2.3)'-*(2.1)' for A.

REMARK 2.1. In Theorem 1, the condition (2.1) implies (2.2) for arbitrary
set S in LE(A) and the converse case holds for S consisting of positive operators in LE(A). Indeed, the former follows from the proof of Theorem 1, and the latter will be obtained by the last part of its proof, because we can take the weak* convergence in the place of the L'-mean convergence in that part of the proof.
As the final part in this section, we shall discuss a decreasing M -nets
in a semi-finite W*-algebra A on H with a regular gage p:

THEOREM 3. Let {x,, a E D} be a decreasing M-net in L2(A) (in L2(A) n
A rest.). Then xa converges to an operator x E L2(A) (in L2(A) n A rest.) in L2-mean (strongly as operator on H rest.). In particular, if A is o--finite finite

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and/is a faithful normal trace and {x, a E D} c L'(A), then xa converges to an x E L'(A) in L1-mean.

PROOF. Let {Aa, a E D} be the family of the W*-subalgebras associated
to {xa, a ED} (cf. The first paragraph of 3). Let A1= n aD Aa which is a W*-subalgebra of A and let yel be the conditional expectation of y relative to A1.

L2-CASE: Since for any a,, E D (a<5), 0<xaxa x *j xa (x8)ea, 0<(xaxa)t((xxp)ea)<tC(xxp)
and Em (xaxa) exists (uniquely, =X say). Therefore

C xa-xp i=, ((xa--xp)*(xa-xp))=(xx)-(xaxa)+X- X=0,
and there exists an x E LL(A) such that (1 xa x;(2 -*0.

L2 n A CASE: Since {xa, a E D} c L2(A), it converges to x E L2(A) in the L2-mean. While for any y E J and any fixed a0 E D, (c(xy)f=lim, 4x, y)
a<ao

< xa0 (y 1i which implies x E A. Hence x E LL(A) f A and (x,--x)y 1
y' I xa =-x ii-0 for every y E I, and I((x,-x)y 1 -+0 for every y E LL(A), because j is dense in L2(A). Therefore xa converges strongly to x on H. Finally we prove the last part. For fixed aQ E D, taking {yn} c Aa o such that yn-x 0 -0 (n-cc),
yna-ynp 1'Z yna-Y, j.2-+0
and

--xa I, 1<1.Yn-xa o 11-*0 (n- oo, a<a0)
Therefore for any 9>0 there are a~ and n such that for every a, <a, ao lixa-xp(lc xc-y, a"r+11yna-Y 1i+l, ynp X 11
and xa converges to some x E L1(A) in the L'-mean.

REMARK 2.2. In the above proof, each limit operator belongs to L2(A1), L2(A) fl AT or Ll(A1) respectively. For, let x be the limit operator, then by the above proof, we find c(yx)=tc(yx,) for every y E J fl Al and for every a E D.

4. In this section we shall show that a sequence of bounded operators defined by von Neumann (cf. p. 118 of [9]) is a simple M-set, and apply the preceding consideration to the convergence theorem of i t (cf. Theorem 6 of [9]). Firstly, we show a lemma:

LEMMA 1. (Misonou). Let W be a W*-algebra on H and let p be a projection
in W. For any x E W, but
(15) x1=pxp+(1-p)x(1-p)7)
Then the range W of the mapping x-+xl p is a W*-subalgebra of W and the mapping is linear and satisfies the conditions (D.1)-(D.5), (D.7) (in

7) These notations were introduced by von Neumann (cf. [9;p. 118J).

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Theorem D) and P=I.

PROOF. We prove only (D. 5), since the others are almost obvious. For any x, yEW

(16) (xT y) I p=((pxp+(1--p)x(1--p))y) I P=pxpyp+(1--p)x(1--p)y(1--p)
and x1y1p=(pxp+(1-p)x(1-p)) (pyp+(1-p)y(1--p)) which equals to the right side of (16). This implies (x1y)1 p=x1 y1' and similarly=(xy1 p)1 p. Since yIP=y for every y E WI2, (yxy')IP y(xy')Ip=yxl "y' for y, y' E WI P.
For any projections in W of finite number pl,, p, we denote (XIP1)11)2, ((XIP1)12'2)Ipa and ((....((x1P1)1P2)1P2)....)IPn by XIP1IP2, XIPIIP21p3, and X hIP2...IPn respectively.

LEMMA 2 (von Neumann [9]). If the projections p1i ....p72 in W commute
with each others, then for any permutation (1', 2', ..., n') of (1, 2,....,n)
xl pl'I p2' In=xf 1.1V21.. Ipn

This lemma wass proved by von Neumann for the h or IL factor (cf.
[9]), which is valid for the present case.
Let W=A be a semi-finite W*-algebra on H and let be a regular gage. Let {p2} be a finite or infinite sequence of projections in S commuting with each others. For any x E A, put

(17) xEn=xl X11 p2... it n=1, 2....
Under these notations we obtain

THEOREM 4. A={xFn; x E A} is a W*-subal gebra of A for each n=
1, 2,...., and the mapping-+xn transforms A onto A, a and is the condi-
tional expectation relative to An satisfying Ian=I. Putting x72=x' for each n, {xn, n=1, 2,....} is a decreasing simple M-net.

PROOF. By Lemmas 1 and 2, each A,, is obviously a W*-subalgebra satisfying

(18) A1.....
For any fixed projection p E A and for x E J, xlp belongs to J n A and satisfying

(19) (xl p)=(pxp+(1-p)x(1-p))=,(px)+((1-p)x)=(x).
Hence by Lemma 1 (y1 x)=((y1px)1 p)=((yxl')12')=, (yx1 p) for every y E A and x E.I, and by Theorem 1 the mapping x-+x1 p is the conditional ex-pectation relative to AID'. Similarly, (xl11IP2)=a(x) and by Lemmas 1 and 2 (xl llp2y)I1hhI 2xIp1IP2y1'1IP2 holds. Hence by the same way for xl2, the mapping x-.*XI P1IP2 is the conditional expectation relative to AIP1IPZ (=A2). By the inductive method and by (18) these facts hold for every E,. It follows from the definition of M-net and r=I that {x 1, n=1, 2.....} is a decreasing simple, M-set.
For x E L2(A) n A and each n =1, 2.....

(x*-nx_n)=(x* nxn)<((x*x)en)=((x*x)

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and hence x9a E L2(A) n An. Then by Theorems 3 and 4, x,converges strongly to an operator x in L2(A) fl A which is also a limit in L2-mean,

where A=QAn. This implies the Theorem of van Neumann:

THEOREM 5. For any x E L2(A) n A, {xFn} belongs to L(A) fl A, and converges strongly as operator on H and in L2-mean to an operator x in L2(A) n A.

Put x=for x E L2(A) fl A. Since; xFn iz=x, z c x (2, we have

(20) IJ xF J2<xa for every x E L2(A) fl A.
While for every x, y E L2(A) fl A, x*F=llm x'n=llm xen*=xF* and

(xEy)F=lien (x"y)'n=lira xFyn=xey'=lxm x'nyE=(xy)F,

where the limit is that with respect to the weak operator topology, and for every x, y E J

fL(xy)=hm p(x'7y)=lim u(xy'n)=p(xy").

noo n-,

The linearity and idempotency (x"=x") of the mapping x*x F (defined on L2(A)lA) are clear. Since LE(A) fl A is dense in L2(A), by (20) it is uniquely extended on the whole space L2(A). Further, tsince xE =x for every x E L2(A) n A, x-* xF satisfies the condition (2') in Corollary 1.2. Therefore, we obtain

COROLLARY 5.1 The mapping x--x (x E L2(A) (1 A) is uniquely extended
to the conditional expectation x-+xe relative to A

From Theorem 5 and this Corollary it follows that for every x E A, Px n +converges weakly to the xe, P being the maximal central projection in the W* subalgebra A of A.

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