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CONDITIONAL FUNCTION SPACE INTEGRALS WITH APPLICATIONS

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ABSTRACT. In the theory of the conditional Wiener integral, the integrand is a functional of the standard Wiener process. In this paper we first consider a conditional function space integral for functionals of more general stochastic process and obtain an evaluation formula of the conditional function space integral. We then use this formula to derive the generalized Kac-Feynman integral equation and also to obtain a Cameron-Martin type translation theorem for our conditional function space integrals. These results subsume similar known results obtained by Chung and Kang, Park and Skoug, and Yeh for the standard Wiener process.

1. Introduction. Let $(C_0[0,T], \mathcal{B}(C_0[0,T]), m_w)$ denote Wiener space where $C_0[0,T]$ is the space of all continuous functions x on $[0,T]$ with $x(0)=0$. Many physical problems can be formulated in terms of the conditional Wiener integral $E[F|X]$ of the functionals defined on $C_0[0,T]$ of the form

$$F(x) = \exp\bigg\{ -\int_0^t \theta(s, x(s))\, ds\bigg\} \tag{1.1}$$

where $X(x) = x(t)$ and $\theta(\cdot,\cdot)$ is a sufficiently smooth function on $[0,T]\times \mathbf{R}$. It is indeed known from a theorem of Kac [**8**] that the function $U(\cdot,\cdot)$ defined on $[0,T]\times\mathbf{R}$ by

$$U(t,\xi) = \frac{1}{\sqrt{2\pi t}} \exp\bigg\{ -\frac{(\xi-\xi_0)^2}{2t}\bigg\} E[F(x(t)+\xi_0)|x(t) = \xi-\xi_0] \tag{1.2}$$

is the solution of the partial differential equation

$$\frac{\partial U}{\partial t} = \frac{1}{2}\frac{\partial^2 U}{\partial \xi^2} - \theta U \tag{1.3}$$

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satisfying the condition $U(\xi,0)=\delta(\xi-\xi_0)$. In [**6**], Donsker and Lions showed that the function

$$U(t,\xi)=E[\delta_{t,\xi-\xi_0}(x)F(x)] \tag{1.4}$$

is the solution of the partial differential equation (1.3) where $\delta_{t,\xi}$ $(t>0,\xi\in\mathbf{R})$ is the Donsker's delta function formally defined by

$$\delta_{t,\xi}(x)=\frac{1}{2\pi}\int_{\mathbf{R}}\mathrm{e}^{iu(x(t)-\xi)}\,du,\qquad x\in C_0[0,T].$$

In [**15**], in order to provide a rigorous treatment of the function (1.4) involving the Donsker's delta function, Yeh introduced the concept of the conditional Wiener integral and derived a Fourier inversion formula for conditional Wiener integrals :

$$\begin{aligned} &E[F|x(t)=\xi]\,\frac{1}{\sqrt{2\pi t}}\exp\bigg\{-\frac{\xi^2}{2t}\bigg\} \\ &\qquad=\frac{1}{2\pi}\int_{\mathbf{R}}\mathrm{e}^{-iu\xi}E[\mathrm{e}^{iux(t)}F]\,du,\qquad \xi\in\mathbf{R} \end{aligned} \tag{1.5}$$

which gives a formula to obtain the explicit evaluation of the solution of the partial differential equation (1.3).

Using the inversion formula (1.5), Yeh [**15**] derived the Kac-Feynman integral equation for a time independent continuous potential function $\theta(\xi)$. In [**4**], Chung and Kang, using the Yeh's inversion formula, obtained similar results for a time dependent bounded potential function $\theta(s,\xi)$. In [**11**], Park and Skoug obtained a simple formula for expressing conditional Wiener integrals with a vector-valued conditioning function in terms of ordinary Wiener integral, and then used the formula to derive the Kac-Feynman integral equation for time independent potential function $\theta(\xi)$.

In this paper we extend the ideas of [**4, 10, 11**] from the Wiener processes to more general stochastic processes. We note that the Wiener process is free of drift and is stationary in time. However, the stochastic process considered in this paper is a process subject to drift and is nonstationary in time.

In Section 2, we consider the function space induced by a generalized Brownian motion process and define the conditional function space integral of Y given X as the conditional expectation $E[Y|X]$ given as the function on the value space of X. In Section 3, we obtain an evaluation formula of conditional function space integrals. In Section 4, we use the evaluation formula of conditional function space integrals to derive the generalized Kac-Feynman integral equation. In Section 5, we also use the evaluation formula of conditional function space integrals to derive a Cameron-Martin type translation theorem for our conditional function space integrals.

2. Preliminaries. Let $D = [0, T]$ and let $(\Omega, \mathcal{B}, P)$ be a probability measure space. A real valued stochastic process X on $(\Omega, \mathcal{B}, P)$ and D is called *a generalized Brownian motion process* if $X(0, \omega)=0$ almost everywhere and for $0 \le t_0 < t_1 < \cdots < t_n \le T$, the n-dimensional random vector $(X(t_1, \omega), \ldots, X(t_n, \omega))$ is normally distributed with the density function

$$(2.1) \quad K(t, \vec{\eta}) = \left((2\pi)^n \prod_{j=1}^{n} (b(t_j) - b(t_{j-1})) \right)^{-1/2} \cdot \exp\left\{ -\frac{1}{2} \sum_{j=1}^{n} \frac{((\eta_j - a(t_j)) - (\eta_{j-1} - a(t_{j-1})))^2}{b(t_j) - b(t_{j-1})} \right\}$$

where $\vec{\eta} = (\eta_1, \ldots, \eta_n)$, $\eta_0 = 0$ and $a(t)$ is a real valued function with $a(0) = 0$ and $b(t)$ is a strictly increasing real valued function with $b(0) = 0$.

We emphasize that no continuity or smoothness condition on $a(t)$ and $b(t)$ are assumed unless otherwise stated.

As explained in [**13**, pp. 18–20], X induces a probability measure μ on the measurable space $(\mathbf{R}^D, \mathcal{B}^D)$ where $\mathbf{R}^D$ is the space of all real valued functions $x(t)$, $t \in D$, and $\mathcal{B}^D$ is the smallest σ-algebra of subsets of $\mathbf{R}^D$ with respect to which all the coordinate evaluation maps $e_t(x) = x(t)$ defined on $\mathbf{R}^D$ are measurable. The triple $(\mathbf{R}^D, \mathcal{B}^D, \mu)$ is a probability measure space. This measure space is called *the function space induced by the generalized Brownian motion process X determined by $a(\cdot)$ and $b(\cdot)$*.

Let X be an $\mathbf{R}^n$-valued measurable function and Y a complex-valued μ-integrable function on $(\mathbf{R}^D, \mathcal{B}^D, \mu)$. Let $\mathcal{F}(X)$ denote the σ-algebra of subsets of $\mathbf{R}^D$ generated by X. Then by the definition of conditional expectation, the conditional expectation of Y given $\mathcal{F}(X)$, written $E[Y|X]$, is any $\mathbf{R}^n$-valued $\mathcal{F}(X)$-measurable function on $\mathbf{R}^D$ such that

$$\int_E Y\,d\mu = \int_E E[Y|X]\,d\mu \quad \text{for} \quad E \in \mathcal{F}(X).$$

It is well known that there exists a Borel measurable and μ_X-integrable function on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), \mu_X)$ such that $E[Y|X] = \psi \circ X$, where μ_X is the probability measure defined by $\mu_X(B) = \mu(X^{-1}(B))$ for $B \in \mathcal{B}(\mathbf{R}^n)$. The function $\psi(\vec{\xi})$, $\vec{\xi} \in \mathbf{R}^n$ is unique up to Borel null sets in $\mathbf{R}^n$. Following Yeh [**15**] the function $\psi(\vec{\xi})$, written $E[Y|X = \vec{\xi}]$, is called the *conditional function space integral* of Y given X.

The following proposition will be used in the sequel.

Proposition 2.1 [**15**]. *Let X be an $\mathbf{R}^n$-valued measurable function and Y a complex valued μ-integrable functional on $\mathbf{R}^D$. Let f be a complex valued measurable function on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$. Then*

$$E[(f \circ X)Y] = \int_{\mathbf{R}^n} f(\vec{\eta})E[Y|X = \vec{\eta}]\,d\mu_X(\vec{\eta}) \tag{2.2}$$

in the sense that the existence of one side implies that of the other as well as their equality of two.

3. Formula for Conditional Function Space Integrals. Let $(\mathbf{R}^D, \mathcal{B}^D, \mu)$ be the function space induced by the generalized Brownian motion process defined in Section 2. In this section we will obtain an evaluation formula of conditional function space integrals over $(\mathbf{R}^D, \mathcal{B}^D, \mu)$. Let W be a stochastic process on $(\mathbf{R}^D, \mathcal{B}^D, \mu)$ and D defined by $W(t, x) = x(t)$, $t \in D$, $x \in \mathbf{R}^D$. Then W is a generalized Brownian motion process whose sample space is $\mathbf{R}^D$. For a partition $\tau = \{t_1, \dots, t_n\}$ of $[0, T]$ with $0 = t_0 < t_1 < \cdots < t_n = T$, define a function $X_\tau : \mathbf{R}^D \longrightarrow \mathbf{R}^n$ by $X_\tau(x) = (x(t_1), \dots, x(t_n))$. For $x \in \mathbf{R}^D$, define the function $[X_\tau(x)] \equiv [x] : [0, T] \longrightarrow \mathbf{R}$ by

$$[x](t) = x(t_{j-1}) + \frac{b(t) - b(t_{j-1})}{b(t_j) - b(t_{j-1})}(x(t_j) - x(t_{j-1})) \tag{3.1}$$

for each $t \in [t_{j-1}, t_j]$, $j = 1, \dots, n$. Similarly, for $\vec{\xi} = (\xi_1, \dots, \xi_n) \in \mathbf{R}^n$, define the function $[\vec{\xi}] : [0,T] \longrightarrow \mathbf{R}$ by

$$[\vec{\xi}](t) = \xi_{j-1} + \frac{b(t) - b(t_{j-1})}{b(t_j) - b(t_{j-1})}(\xi_j - \xi_{j-1}) \tag{3.2}$$

for each $t \in [t_{j-1}, t_j]$, $j = 1, \dots, n$, and $\xi_0 = 0$.

Lemma 3.1. *Let $\{x(t), t \in [0,T]\}$ be a generalized Brownian motion process. Then the processes $\{x(t) - [x](t), t \in [0,T]\}$ and $\{[x](t), t \in [0,T]\}$ are independent.*

Proof. Since the processes are Gaussian, it suffices to show that for every $t, s \in [0,T]$,

$$E[e_t(x - [x]) \cdot e_s([x])] = E[e_t(x - [x])]E[e_s([x])]$$

where e_t is the coordinate evaluation map. To show this, we may assume that $t \le s$, $t_{i-1} \le t \le t_i$ and $t_{j-1} \le s \le t_j$, $i \le j$. So we have

$$\begin{aligned} &E[x(t_{j-1})(x(t) - x(t_{i-1}))] \\ &\quad = E\big[\big((x(t_{j-1}) - x(t)) + (x(t) - x(t_{i-1})) \\ &\qquad + x(t_{i-1})\big)\big(x(t) - x(t_{i-1})\big)\big] \\ &\quad = \big(a(t) - a(t_{i-1})\big)a(t_{j-1}) + b(t) - b(t_{i-1}) \end{aligned} \tag{3.3}$$

and similarly, we have

$$\begin{aligned} &E[x(t_{j-1})(x(t_i) - x(t_{i-1}))] \\ &\quad = \big(a(t_i) - a(t_{i-1})\big)a(t_{j-1}) + b(t_i) - b(t_{i-1}). \end{aligned} \tag{3.4}$$

Thus by simple calculations with (3.3) and (3.4), we have

$$\begin{aligned} E\big[e_t(x - [x]) \cdot e_s([x])\big] &= E\big[\big((x - [x])(t)\big) \cdot ([x](s))\big] \\ &= E\Big[\Big(x(t) - x(t_{i-1}) - \frac{b(t) - b(t_{i-1})}{b(t_i) - b(t_{i-1})} \\ &\qquad \cdot (x(t_i) - x(t_{i-1}))\Big)\Big(x(t_{j-1}) \\ &\qquad + \frac{b(s) - b(t_{j-1})}{b(t_j) - b(t_{j-1})}(x(t_j) - x(t_{j-1}))\Big)\Big] \end{aligned}$$

$$
\begin{aligned}
&= \big(a(t)-a(t_{i-1})\big)a(t_{j-1}) + \frac{b(s)-b(t_{j-1})}{b(t_j)-b(t_{j-1})} \\
&\quad \cdot \big(a(t)-a(t_{i-1})\big)\big(a(t_j)-a(t_{j-1})\big) \\
&\quad - \frac{b(t)-b(t_{i-1})}{b(t_i)-b(t_{i-1})}\big(a(t_i)-a(t_{i-1})\big)a(t_{j-1}) \\
&\quad - \frac{(b(t)-b(t_{i-1}))(b(s)-b(t_{j-1}))}{(b(t_i)-b(t_{i-1}))(b(t_j)-b(t_{j-1}))} \\
&\quad \cdot \big(a(t_j)-a(t_{j-1})\big)\big(a(t_i)-a(t_{i-1})\big) \\
&= E[x(t)-[x](t)]E[[x](s)].
\end{aligned}
$$

Theorem 3.2. *Let $F \in L^1(\mathbf{R}^D, \mathcal{B}^D, \mu)$. Then*

$$
\int_{\mathbf{R}^D} F(x)d\mu(x) = \int_{\mathbf{R}^n} E[F(x-[x]+[\vec{\xi}])]\, d(\mu \circ X_\tau^{-1})(\vec{\xi}). \tag{3.5}
$$

Proof. Let Y and Z be $\mathcal{B}^D$-measurable functions from $\mathbf{R}^D$ into $\mathbf{R}^D$ defined by $Y(x) = x - [x]$ and $Z(x) = [x]$. Let $C_1 = Y(\mathbf{R}^D)$ and $C_2 = Z(\mathbf{R}^D)$. As was shown in Lemma 3.1, since Y and Z are independent, we have

(3.6)
$$
\begin{aligned}
\int_{\mathbf{R}^D} F(x)\, d\mu(x) &= \int_{\mathbf{R}^D} F\big(x-[x]+[x]\big)\, d\mu(x) \\
&= \int_{C_1\times C_2} F(y+z)\, d(\mu\circ Y^{-1} \times \mu \circ Z^{-1})(y,z) \\
&= \int_{C_2}\int_{C_1} F(y+z)\, d(\mu\circ Y^{-1})(y)\, d(\mu\circ Z^{-1})(z).
\end{aligned}
$$

Since $Z(x) = [X_\tau(x)]$, by the change of variables theorem and (3.1), the last equality of (3.6) equals

$$\begin{aligned}(3.7)\quad \int_{C_2}\int_{C_1} F(y+z)\,d(\mu\circ Y^{-1})(y)\,d(\mu\circ[X_\tau]^{-1})(z) &= \int_{\mathbf{R}^n}\int_{C_1} F(y+[\vec{\xi}])\,d(\mu\circ Y^{-1})(y)\,d(\mu\circ X_\tau^{-1})(\vec{\xi})\\ &= \int_{\mathbf{R}^n}\int_{\mathbf{R}^D} F(x-[x]+[\vec{\xi}])\,d\mu(x)\,d(\mu\circ X_\tau^{-1})(\vec{\xi})\\ &= \int_{\mathbf{R}^n} E[F(x-[x]+[\vec{\xi}])]\,d(\mu\circ X_\tau^{-1})(\vec{\xi}).\end{aligned}$$

This, together with (3.6) and (3.7), completes the proof. □

Theorem 3.3. *Let* $F\in L^1(\mathbf{R}^D,\mathcal{B}^D,\mu)$. *Then*

$$(3.8)\qquad \int_{X_\tau^{-1}(B)} F(x)d\mu(x) = \int_B E\big[F(x-[x]+[\vec{\xi}])\big]\,d(\mu\circ X_\tau{}^{-1})(\vec{\xi})$$

for every $B\in\mathcal{B}(\mathbf{R}^n)$.

Proof. Using Theorem 3.2, for every $B\in\mathcal{B}(\mathbf{R}^n)$, we have

$$\begin{aligned}\int_{X_\tau^{-1}(B)} F(x)\,d\mu(x) &= \int_{\mathbf{R}^D} I_{X_\tau^{-1}(B)}(x)\cdot F(x)\,d\mu(x)\\ &= \int_{\mathbf{R}^D} \big((I_B\circ X_\tau)\cdot F\big)(x)\,d\mu(x)\\ &= \int_{\mathbf{R}^n} E\big[\big((I_B\circ X_\tau)\cdot F\big)\big(x-[x]+[\vec{\xi}]\big)\big]\,d(\mu\circ X_\tau{}^{-1})(\vec{\xi})\\ &= \int_{\mathbf{R}^n} E\big[I_B(\vec{\xi})\cdot F(x-[x]+[\vec{\xi}])\big]\,d(\mu\circ X_\tau{}^{-1})(\vec{\xi})\\ &= \int_{\mathbf{R}^n} I_B(\vec{\xi})\cdot E\big[F(x-[x]+[\vec{\xi}])\big]\,d(\mu\circ X_\tau{}^{-1})(\vec{\xi})\\ &= \int_{\mathbf{B}} E\big[F(x-[x]+[\vec{\xi}])\big]\,d(\mu\circ X_\tau{}^{-1})(\vec{\xi})\end{aligned}$$

where I_A is the indicator function of A, $A\in\mathcal{B}(\mathbf{R}^n)$. Hence (3.8) holds.

Theorem 3.4. *Let $F \in L^1(\mathbf{R}^D, \mathcal{B}^D, \mu)$. Then*

$$E[F(x)|X_\tau(x)=\vec{\xi}\,] = E[F(x-[x]+[\vec{\xi}\,])] \quad \textit{for} \quad \vec{\xi} \in \mathbf{R}^n. \tag{3.9}$$

Proof. By Proposition 2.1, we have for any $B \in \mathcal{B}(\mathbf{R}^n)$,

$$\begin{aligned}\int_{X_\tau^{-1}(B)} F(x)\,d\mu(x) &= \int_{\mathbf{R}^D} \big((I_B \circ X_\tau)F\big)(x)\,d\mu(x)\\ &= \int_{\mathbf{R}^n} I_B(\vec{\xi}\,)\cdot E[F(x)|X_\tau(x)=\vec{\xi}\,]\,d(\mu\circ X_\tau{}^{-1})(\vec{\xi}\,)\\ &= \int_B E[F(x)|X_\tau(x)=\vec{\xi}\,]\,d(\mu\circ X_\tau{}^{-1})(\vec{\xi}\,).\end{aligned}$$

Hence, using Theorem 3.3 and the definition of conditional expectation, we obtain equation (3.9) as desired. □

The following examples illustrate the usefulness of Theorem 3.4 in evaluating conditional function space integrals.

Example 1. For any $x \in \mathbf{R}^D$, let $F(x) = \int_0^T x(t)\,dt$. Then by using Theorem 3.4 and the Fubini theorem, we have

$$\begin{aligned}E[F(x)|X_\tau(x)=\vec{\xi}\,] = \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \Big[&a(t)-a(t_{j-1})\\ &-\frac{b(t)-b(t_{j-1})}{b(t_j)-b(t_{j-1})}(a(t_j)-a(t_{j-1}))\Big]\,dt\\ +\sum_{j=1}^n \int_{t_{j-1}}^{t_j} \Big[&\xi_{j-1}+\frac{b(t)-b(t_{j-1})}{b(t_j)-b(t_{j-1})}(\xi_j-\xi_{j-1})\Big]\,dt.\end{aligned}$$

In particular, if $\{x(t), t \in [0,T]\}$ is the standard Wiener process, then $a(t) \equiv 0$ and $b(t) = t$ and hence we have

$$E[F(x)|X_\tau(x)=\vec{\xi}\,] = \frac{1}{2}\sum_{j=1}^n (t_j - t_{j-1})(\xi_j + \xi_{j-1})$$

which is a result in [**11**].

Example 2. For any $x \in \mathbf{R}^D$, let $F(x) = \int_0^T x^2(t)\,dt$. Then by using Theorem 3.4 and the Fubini theorem, we have

$$
\begin{aligned}
(3.10)\quad E[F(x)|X_\tau(x) = \vec{\xi}\,] = \int_0^T & E\big[(x(t) - [x](t))^2 \\
& + 2(x(t) - [x](t))[\vec{\xi}\,](t) + ([\vec{\xi}\,](t))^2\big]\,dt.
\end{aligned}
$$

Proceeding as in Example 1, we obtain

$$
\begin{aligned}
E\bigg[\int_0^T & (x(t))^2\,dt|X_\tau(x) = \vec{\xi}\,\bigg] \\
= \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \bigg[& (a(t) - a(t_{j-1}))^2 + b(t) - b(t_{j-1}) \\
& - 2\frac{b(t) - b(t_{j-1})}{b(t_j) - b(t_{j-1})}\big[\big(a(t) - a(t_{j-1})\big) \\
& \cdot \big(a(t_j) - a(t_{j-1})\big) + b(t) - b(t_{j-1})\big] \\
& + \left(\frac{b(t) - b(t_{j-1})}{b(t_j) - b(t_{j-1})}\right)^2 \\
& \cdot \big[(a(t_j) - a(t_{j-1}))^2 + b(t_j) - b(t_{j-1})\big] \\
& + 2[\vec{\xi}\,](t)\bigg[a(t) - a(t_{j-1}) \\
& - \frac{b(t) - b(t_{j-1})}{b(t_j) - b(t_{j-1})}(a(t_j) - a(t_{j-1}))\bigg] \\
& + ([\vec{\xi}\,](t))^2\bigg]\,dt.
\end{aligned}
$$

In particular, if $\{x(t), t \in [0,T]\}$ is the standard Wiener process, then $a(t) \equiv 0$ and $b(t) = t$ and hence we have

$$
\begin{aligned}
E\bigg[\int_0^T (x(t))^2\,dt|X_\tau(x) = \vec{\xi}\,\bigg] = {} & \frac{1}{2}T^2 - \frac{1}{3}\sum_{j=1}^n (t_j - t_{j-1})(t_j + 2t_{j-1}) \\
& + \frac{1}{3}\sum_{j=1}^n (t_j - t_{j-1})(\xi_j^2 + \xi_{j-1}\xi_j + \xi_{j-1}^2)
\end{aligned}
$$

which is a result in [**11**].

Example 3. For any $x \in \mathbf{R}^D$ let $F(x) = \exp\{\int_0^T x(t)\,dt\}$. Then by using Theorem 3.4 and the Fubini theorem, we have

$$E[F(x)|X_\tau(x) = \vec{\xi}] = \exp\left\{\int_0^T [\vec{\xi}](t)\,dt\right\} \cdot \prod_{j=1}^n E\Big[\exp\ \Big\{\int_{t_{j-1}}^{t_j} (x(t) - x(t_{j-1})) - \frac{b(t)-b(t_{j-1})}{b(t_j)-b(t_{j-1})}\big(x(t_j) - x(t_{j-1}))\big)\,dt\Big\}\Big].$$

In particular, if $\{x(t), t \in [0,T]\}$ is the standard Wiener process, then $a(t) \equiv 0$ and $b(t) = t$ and hence we have

$$E\left[\exp\left\{\int_0^T x(t)\,dt\right\} \,\Big|\, X_\tau(x) = \vec{\xi}\right] = \prod_{j=1}^n \Big[\exp\Big\{\frac{(t_j - t_{j-1})^3}{24} + \frac{(\xi_j - \xi_{j-1})(t_j - t_{j-1})}{2} + \xi_{j-1}(t_j - t_{j-1})\Big\}\Big].$$

4. The generalized Kac-Feynman integral equation. For each $t \in [0,T]$ and $\xi \in \mathbf{R}$, let Y_t and X_t be $\mathcal{B}^D$-measurable functions on $\mathbf{R}^D$ defined by

$$(4.1) \qquad Y_t(x) = \exp\left\{\int_0^t \theta(s, x(s)+\xi)\,ds\right\} \quad \text{and} \quad X_t(x) = x(t) + \xi$$

where $\theta(\cdot,\cdot)$ is a complex valued Borel measurable function on $[0,T]\times\mathbf{R}$ for which Y_t is μ-integrable for each $(t,\xi) \in [0,T]\times\mathbf{R}$. Let us define a function U on $[0,T]\times\mathbf{R}\times\mathbf{R}$ by

$$(4.2) \quad U(t;\xi,\eta) = E[Y_t|X_t = \eta](2\pi b(t))^{-1/2} \exp\left\{-\frac{(\eta - a(t) - \xi)^2}{2b(t)}\right\}.$$

In this section, we will show that by using the evaluation formula developed in Section 3, the function U given by (4.2) satisfies the generalized Kac-Feynman integral equation

$$\begin{aligned} U(t;\xi,\eta) &= (2\pi b(t))^{-1/2}\exp\Big\{-\frac{(\eta-a(t)-\xi)^2}{2b(t)}\Big\} \\ &\quad + \int_0^t\int_{\mathbf{R}}\theta(s,\zeta)U(s;\xi,\zeta)(2\pi(b(t)-b(s)))^{-1/2} \\ &\quad \cdot\exp\Big\{-\frac{((\zeta-a(s))-(\eta-a(t)))^2}{2(b(t)-b(s))}\Big\}\,d\zeta\,ds. \end{aligned} \tag{4.3}$$

Lemma 4.1. *Let $\{x(t), t\in[0,T]\}$ be a generalized Brownian motion process. Then for $0<s<t$, the random variable $x(s)-(b(s)/b(t))x(t)$ is normally distributed with mean $a(s)-(b(s)/b(t))a(t)$ and variance $b(s)-b^2(s)/b(t)$. Moreover, for $0<u<s<t<T$, the two Gaussian random variables $x(s)-(b(s)/b(t))x(t)$ and $x(u)-(b(u)/b(s))x(s)$ are independent.*

Proof. For $s,t\in[0,T]$ with $s<t$, we have the facts that

$$E[x^2(s)] = b(s)+a^2(s) \quad\text{and}\quad E[x(s)x(t)] = b(s)+a(s)a(t).$$

Thus by using these, we see that the variance of $x(t)-(b(s)/b(t))x(t)$ is $b(s)-b^2(s)/b(t)$. Hence we have

$$x(s)-\frac{b(s)}{b(t)}x(t)\sim N\bigg(a(s)-\frac{b(s)}{b(t)}a(t)\,,\,b(s)-\frac{b^2(s)}{b(t)}\bigg).$$

Moreover, for $0<u<s<t$, we can show that

$$\begin{aligned} &E\bigg[\bigg(x(s)-\frac{b(s)}{b(t)}x(t)\bigg)\bigg(x(u)-\frac{b(u)}{b(s)}x(s)\bigg)\bigg] \\ &\qquad = E\bigg[x(s)-\frac{b(s)}{b(t)}x(t)\bigg]E\bigg[x(u)-\frac{b(u)}{b(s)}x(s)\bigg], \end{aligned}$$

which completes the proof of the lemma. □

Theorem 4.2. *For $t \in [0,T]$, let X_t and Y_t be as in* (4.1)*. Then the function $U(t;\xi,\eta)$ given by* (4.2) *satisfies the generalized Kac-Feynman integral equation* (4.3)*.*

Proof. For $(t,\eta) \in [0,T] \times \mathbf{R}$, let

$$I(t;\xi,\eta) = E\bigg[\exp\bigg\{\int_0^t \theta(s,x(s)+\xi)\,ds\bigg\}\,\big|\,x(t)+\xi=\eta\bigg]. \tag{4.4}$$

By differentiating the function $\exp\{\int_0^s \theta(u,x(u)+\xi)\,du\}$ with respect to s and then integrating the derivative on $[0,t]$, we obtain

$$\begin{aligned}\exp\bigg\{\int_0^t \theta(s,x(s)+\xi)\,ds\bigg\} \\ = 1+\int_0^t \exp\bigg\{\int_0^s \theta(u,x(u)+\xi)\,du\bigg\}\theta(s,x(s)+\xi)\,ds.\end{aligned} \tag{4.5}$$

Since the left hand side of (4.5) is μ-integrable, it follows that the second term of the right hand side of (4.5) is μ-integrable. Hence by taking conditional expectations, and by using Theorem 3.4 and the Fubini theorem, we obtain

(4.6)

$$\begin{aligned}I(t;\xi,\eta) &= E\bigg[1+\int_0^t \theta(s,x(s)+\xi) \\ &\qquad \cdot\exp\bigg\{\int_0^s \theta(u,x(u)+\xi)\,du\bigg\}\,ds\,\big|\,x(t)+\xi=\eta\bigg] \\ &= 1+E\bigg[\int_0^t \theta(s,x(s)+\xi) \\ &\qquad \cdot\exp\bigg\{\int_0^s \theta(u,x(u)+\xi)\,du\bigg\}\,ds\,\big|\,x(t)+\xi=\eta\bigg] \\ &= 1+\int_0^t E\big[\theta\big(s,x(s)-[x](s)+[\eta-\xi](s)+\xi\big) \\ &\qquad \cdot\exp\bigg\{\int_0^s \theta\big(u,x(u)-[x](u)+[\eta-\xi](u)+\xi\big)\,du\big\}\big]\,ds \\ &= 1+\int_0^t E\bigg[\theta\bigg(s,x(s)-\frac{b(s)}{b(t)}x(t)+\frac{b(s)}{b(t)}(\eta-\xi)+\xi\bigg)\end{aligned}$$

$$\cdot \exp\bigg\{\int_0^s \theta\bigg(u, x(u) - \frac{b(u)}{b(s)}x(s) + \frac{b(u)}{b(s)}\bigg[x(s) - \frac{b(s)}{b(t)}x(t) + \frac{b(s)}{b(t)}(\eta-\xi)\bigg] + \xi\bigg)\, du\bigg\}\bigg]\, ds.$$

Since the variance of $x(s) - (b(s)/b(t))x(t) + (b(s)/b(t))(\eta-\xi) + \xi$ is $\sigma^2 \equiv b(s) - b^2(s)/b(t)$, by applying Lemma 4.1 to the last equation in (4.6), we have

(4.7)

$$\begin{aligned}
I(t;\xi,\eta) &= 1 + \int_0^t \int_{\mathbf{R}} E\bigg[\exp\bigg\{\int_0^s \theta\bigg(u, x(u) - \frac{b(u)}{b(s)}x(s) + \frac{b(u)}{b(s)}(\zeta-\xi) + \xi\bigg)\, du\bigg\}\bigg]\theta(s,\zeta)(2\pi\sigma^2)^{-1/2} \\
&\quad \cdot \exp\bigg\{-\frac{\big(\zeta-(a(s)-(b(s)/b(t))a(t)+(b(s)/b(t))(\eta-\xi)+\xi)\big)^2}{2\sigma^2}\bigg\}\, d\zeta\, ds \\
&= 1 + \int_0^t \int_{\mathbf{R}} \theta(s,\zeta)(2\pi\sigma^2)^{-1/2} E\bigg[\exp\bigg\{\int_0^s \theta(u, x(u))\, du\bigg\} \,\Big|\, x(s)+\xi=\zeta\bigg] \\
&\quad \cdot \exp\bigg\{-\frac{\big(\zeta-(a(s)-(b(s)/b(t))a(t)+(b(s)/b(t))(\eta-\xi)+\xi)\big)^2}{2\sigma^2}\bigg\}\, d\zeta\, ds.
\end{aligned}$$

But we have that

(4.8)

$$\begin{aligned}
&(2\pi b(t))^{-1/2} \exp\bigg\{-\frac{(\eta-a(t)-\xi)^2}{2b(t)}\bigg\}(2\pi b(s))^{1/2} \exp\bigg\{\frac{(\zeta-a(s)-\xi)^2}{2b(s)}\bigg\} \\
&\cdot (2\pi\sigma^2)^{-1/2} \exp\bigg\{-\frac{\big(\zeta-(a(s)-(b(s)/b(t))a(t)+(b(s)/b(t))(\eta-\xi)+\xi)\big)^2}{2\sigma^2}\bigg\} \\
&\qquad = (2\pi(b(t)-b(s)))^{-1/2} \exp\bigg\{-\frac{((\zeta-a(s))-(\eta-a(t)))^2}{2(b(t)-b(s))}\bigg\}.
\end{aligned}$$

Hence by (4.7) and (4.8) we obtain

$$\begin{aligned}
U(t;\xi,\eta) &= (2\pi b(t))^{-1/2} \exp\bigg\{-\frac{(\eta-a(t)-\xi)^2}{2b(t)}\bigg\} \cdot I(t;\xi,\eta) \\
&= (2\pi b(t))^{-1/2} \exp\bigg\{-\frac{(\eta-a(t)-\xi)^2}{2b(t)}\bigg\}
\end{aligned}$$

$$+\int_0^t\int_{\mathbf{R}}\theta(s,\zeta)(2\pi b(t))^{-1/2}\exp\bigg\{-\frac{(\eta-a(t)-\xi)^2}{2b(t)}\bigg\}(2\pi\sigma^2)^{-1/2}$$
$$\cdot\exp\bigg\{-\frac{\big(\zeta-(a(s)-(b(s)/b(t))a(t)+(b(s)/b(t))(\eta-\xi)+\xi)\big)^2}{2\sigma^2}\bigg\}$$
$$\cdot E\bigg[\exp\bigg\{\int_0^s\theta(u,x(u))\,du\bigg\}\,\Big|\,x(s)+\xi=\zeta\bigg]\,d\zeta\,ds$$
$$=(2\pi b(t))^{-1/2}\exp\bigg\{-\frac{(\eta-a(t)-\xi)^2}{2b(t)}\bigg\}$$
$$+\int_0^t\int_{\mathbf{R}}\theta(s,\zeta)U(s;\xi,\zeta)(2\pi(b(t)-b(s)))^{-1/2}$$
$$\cdot\exp\bigg\{-\frac{((\zeta-a(s))-(\eta-a(t)))^2}{2(b(t)-b(s))}\bigg\}\,d\zeta\,ds,$$

which completes the proof of the theorem. □

Proposition 4.3. *For each* $(t,\xi)\in[0,T]\times\mathbf{R}$, *let*

$$F_{t,\xi}(x)=\exp\bigg\{\int_0^t\theta(t-s,x(s)+\xi)\,ds\bigg\},\qquad x\in\mathbf{R}^D$$

where $\theta(\cdot,\cdot)$ *is a complex valued measurable function for which* $\sup_{\xi\in\mathbf{R}}E[|F_{t,\xi}|^2]$ *is finite for each* $t\in[0,T]$. *For* $\psi\in L^2(\mathbf{R})$, *let*

$$G(t,\xi)=\int_{\mathbf{R}^D}F_{t,\xi}(x)\psi(x(t)+\xi)\,d\mu(x),\qquad(t,\xi)\in[0,T]\times\mathbf{R}.$$

Then $G(t,\xi)$ *exists and is finite for any* $(t,\xi)\in[0,T]\times\mathbf{R}$, *and* $G(t,\cdot)\in L^2(\mathbf{R})$ *for each* $t\in[0,T]$.

Proof. By Hölder's inequality and the change of variables theorem,

$$|G(t,\xi)|^2\le E[|F_{t,\xi}(x)|^2]\cdot\int_{\mathbf{R}^D}|\psi(x(t)+\xi)|^2\,d\mu(x)$$
$$=E[|F_{t,\xi}|^2]\cdot\int_{\mathbf{R}}\frac{1}{\sqrt{2\pi b(t)}}|\psi(\zeta)|^2\exp\bigg\{-\frac{(\zeta-a(t)-\xi)^2}{2b(t)}\bigg\}\,d\zeta$$
$$\le\frac{1}{\sqrt{2\pi b(t)}}E[|F_{t,\xi}|^2]\cdot\int_{\mathbf{R}}|\psi(\zeta)|^2\,d\zeta<\infty$$

and

$$\begin{aligned}\int_{\mathbf{R}} |G(t,\xi)|^2\,d\xi &\le \int_{\mathbf{R}} \left(E[|F_{t,\xi}|^2] \cdot \int_{\mathbf{R}^D} |\psi(x(t)+\xi)|^2\,d\mu(x) \right) d\xi \\ &\le \sup_{\xi\in\mathbf{R}} E[|F_{t,\xi}|^2] \cdot \int_{\mathbf{R}} \int_{\mathbf{R}} \frac{1}{\sqrt{2\pi b(t)}} |\psi(\zeta)|^2 \\ &\qquad\qquad \cdot \exp\left\{ -\frac{(\zeta - a(t) - \xi)^2}{2b(t)} \right\} d\zeta\,d\xi \\ &= \sup_{\xi\in\mathbf{R}} E[|F_{t,\xi}|^2] \cdot \int_{\mathbf{R}} |\psi(\zeta)|^2\,d\zeta < \infty.\end{aligned}$$

Remark. (1) Proposition 4.3 shows that the correspondence $\psi(\cdot) \mapsto G(t,\cdot)$ defines a bounded linear operator from $L^2(\mathbf{R})$ into $L^2(\mathbf{R})$.

(2) If $\theta(\cdot,\cdot)$ is bounded, or if $-\theta(\cdot,\cdot)$ is real and nonpositive, then the condition on $F_{t,\xi}$ in Proposition 4.3 is satisfied (see [**4, 11, 15**]).

Proposition 4.4. *Let $F_{t,\xi}$ and ψ be as in Proposition* 4.3. *Let*

$$K(t,\eta,\xi) = E[F_{t,\xi}|X_t = \eta](2\pi b(t))^{-1/2} \exp\left\{ -\frac{(\eta - a(t) - \xi)^2}{2b(t)} \right\}.$$

Then

$$\int_{\mathbf{R}^D} F_{t,\xi}(x)\psi(x(t)+\xi)\,d\mu(x) = \int_{\mathbf{R}} \psi(\eta) K(t;\eta,\xi)\,d\eta$$

and the integrals on both sides exist.

Proof. By Proposition 2.1, we have

$$\begin{aligned}&\int_{\mathbf{R}^D} F_{t,\xi}(x)\psi(x(t)+\xi)\,d\mu(x) \\ &\quad= \int_{\mathbf{R}^D} \exp\left\{ \int_0^t \theta(t-s, x(s)+\xi)\,ds \right\} \psi(x(t)+\xi)\,d\mu(x) \\ &\quad= E[(\psi\circ X_t)F_{t,\xi}] \\ &\quad= \int_{\mathbf{R}} \psi(\eta) E[F_{t,\xi} \mid X_t = \eta](2\pi b(t))^{-1/2} \cdot \exp\left\{ -\frac{(\eta - a(t) - \xi)^2}{2b(t)} \right\} d\eta\end{aligned}$$

$$= \int_{\mathbf{R}} \psi(\eta) K(t;\eta,\xi)\, d\eta$$

and by Proposition 4.3, the integrals on both sides exist. □

Theorem 4.5. *Let* $K(t;\eta,\xi)$ *be as in Proposition* 4.4, *and let* $\psi \in L^2(\mathbf{R})$. *Then the function*

$$\Phi(t,\xi) = \int_{\mathbf{R}} K(t;\eta,\xi)\psi(\eta)\, d\eta$$

satisfies the integral equation

$$\begin{aligned}\Phi(t,\xi) = \int_{\mathbf{R}} &(2\pi b(t))^{-1/2} \exp\bigg\{ - \frac{(\eta - a(t) - \xi)^2}{2b(t)}\bigg\}\psi(\eta)\, d\eta \\ &+ \int_0^t \int_{\mathbf{R}} (2\pi(b(t)-b(s)))^{-1/2}\theta(s,\zeta)\Phi(s,\zeta) \\ &\qquad \cdot \exp\bigg\{ - \frac{((\zeta - a(s)) - (\xi - a(t)))^2}{2(b(t)-b(s))}\bigg\}\, d\zeta\, ds.\end{aligned}$$

Proof. By Proposition 4.4, $\Phi(t,\xi)$ exists and is finite for all $(t,\xi) \in [0,T] \times \mathbf{R}$. By applying Theorem 4.2 to $K(t,\eta,\xi)$, we have

$$\begin{aligned}\int_{\mathbf{R}} K(t;\eta,\xi)\psi(\eta)\, d\eta &= \int_{\mathbf{R}} (2\pi b(t))^{-1/2} \cdot \exp\bigg\{ - \frac{(\eta - a(t) - \xi)^2}{2b(t)}\bigg\}\psi(\eta)\, d\eta \\ &\quad + \int_{\mathbf{R}}\int_0^t \int_{\mathbf{R}} \theta(s,\zeta)K(s;\eta,\zeta)((2\pi(b(t)-b(s)))^{-1/2} \\ &\quad \cdot \exp\bigg\{ - \frac{((\zeta - a(s)) - (\eta - a(t)))^2}{2(b(t)-b(s))}\bigg\}\psi(\eta)\, d\zeta\, ds\, d\eta \\ &= \int_{\mathbf{R}} (2\pi b(t))^{-1/2} \exp\bigg\{ - \frac{(\eta - a(t) - \xi)^2}{2b(t)}\bigg\}\psi(\eta)\, d\eta \\ &\quad + \int_0^t \int_{\mathbf{R}} ((2\pi(b(t)-b(s)))^{-1/2}\theta(s,\zeta) \\ &\quad \cdot \exp\bigg\{ - \frac{((\zeta - a(s)) - (\xi - a(t)))^2}{2(b(t)-b(s))}\bigg\}\end{aligned}$$

$$
\begin{aligned}
&\cdot\left\{\int_{\mathbf{R}} K(s;\eta,\zeta)\psi(\eta)d\eta\right\} d\zeta\, ds \\
&= \int_{\mathbf{R}} (2\pi b(t))^{-1/2} \exp\left\{-\frac{(\eta - a(t) - \xi)^2}{2b(t)}\right\}\psi(\eta)\, d\eta \\
&\quad + \int_0^t \int_{\mathbf{R}} ((2\pi(b(t) - b(s)))^{-1/2}\theta(s,\zeta)\Phi(s,\zeta) \\
&\quad \cdot \exp\left\{-\frac{((\zeta - a(s)) - (\xi - a(t)))^2}{2(b(t) - b(s))}\right\} d\zeta\, ds,
\end{aligned}
$$

which completes the proof of the theorem. □

Remark. Under the appropriate regularity conditions on $\theta(\cdot,\cdot)$, $a(\cdot)$ and $b(\cdot)$, we will show in a subsequent paper that the generalized Kac-Feynman integral equation is equivalent to a partial differential equation which is the generalized form of the equation (1.3).

5. Translation of conditional function space integral. In this section we will prove a translation theorem of conditional function space integral (see, Theorem 5.5) and then use it to evaluate a conditional function space integral.

The following theorem is due to Varberg [**12**, p. 805].

Theorem 5.1. *Let X be a separable Gaussian process on $(\Omega, \mathcal{B}, P)$ and $[0,T]$ with continuous covariance function $r(t,s)$ and mean function $a \in L^2[0,T]$. Let $x_0(t) = \int_0^T r(t,s)\, dp(s)$ where p is of bounded variation on $[0,T]$. Then for all μ-integrable function F,*

$$E[F(x)] = E[F(x + x_0)J(x_0, x)]$$

where $J(x_0, x) = \exp\{-(1/2)\int_0^T [2x(t) - 2a(t) + x_0(t)]\, dp(t)\}$.

Throughout this section we require that $a(\cdot)$ and $b(\cdot)$ be continuously differentiable on $[0,T]$ and that $b'(t) > 0$ on $(0,T)$ such that the function f defined by $f(t) = (b'(t))^{-1}$, $f(0) = f(T) = 0$ is of bounded variation on $[0,T]$.

We note that the generalized Brownian motion process X determined by $a(\cdot)$ and $b(\cdot)$ is a Gaussian process with mean function $a(t)$ and covariance function $r(s,t) = \min\{b(s), b(t)\}$. By Theorem 14.2 [**13**, p. 187], the probability measure μ induced by X, taking a separable version, is supported by $C_0[0,T]$ (which is equivalent to the Banach space of continuous functions x on $[0,T]$ with $x(0)=0$ under the sup norm). Hence $(C_0[0,T], \mathcal{B}(C_0[0,T]), \mu)$ is the function space induced by X where $\mathcal{B}(C_0[0,T])$ is the Borel σ-algebra of $C_0[0,T]$.

Lemma 5.2. *Let $x_0(t) = \int_0^t h(s)ds$ for every $t \in [0,T]$, where h is of bounded variation on $[0,T]$. If F is a μ-integrable function on $(C_0[0,T], \mathcal{B}(C_0[0,T]), \mu)$, then*

$$E[F(x)] = E[F(x_0 + x)J(x_0, x)]$$

where

$$J(x_0,x) = \exp\left\{ -\frac{1}{2}\int_0^T \frac{h^2(t)}{b'(t)}\,dt\right\} \cdot \exp\left\{ -\int_0^T \frac{h(t)}{b'(t)}\,dx(t) + \int_0^T \frac{h(t)}{b'(t)}\,da(t)\right\}.$$

Proof. Let $p : [0,T] \to \mathbf{R}$ be the function defined by

$$p(t) = \begin{cases} -\dfrac{h(t)}{b'(t)}, & \text{if } 0 < t < T \\ 0, & \text{otherwise.} \end{cases}$$

Then $p(\cdot)$ is a function of bounded variation on $[0,T]$ such that

$$x_0(t) = \int_0^T r(t,s)\,dp(s)$$

where $r(t,s) = \min\{b(t), b(s)\}$. Using Theorem 5.1, we have

$$E[F(x)] = E[F(x + x_0)J(x_0, x)]$$

where

$$J(x_0,x) = \exp\left\{ -\frac{1}{2}\int_0^T [2x(t) - 2a(t) + x_0(t)]\,dp(t)\right\}. \tag{5.1}$$

Since $2x(t) - 2a(t) + x_0(t) \in C_0[0,T]$, the integral in (5.1) is equal to

$$\int_0^T \frac{h(t)}{b'(t)} d[2x(t) - 2a(t) + x_0(t)] = 2\int_0^T \frac{h(t)}{b'(t)}\,dx(t) - 2\int_0^T \frac{h(t)}{b'(t)}\,da(t) + \int_0^T \frac{h^2(t)}{b'(t)}\,dt.$$

Using this, we have

$$J(x_0, x) = \exp\left\{-\frac{1}{2}\int_0^T \frac{h^2(t)}{b'(t)}\,dt\right\} \cdot \exp\left\{-\int_0^T \frac{h(t)}{b'(t)}\,dx(t) + \int_0^T \frac{h(t)}{b'(t)}\,da(t)\right\},$$

which completes the proof of the lemma. □

Lemma 5.3. *Let $h_n(\cdot)$ be real valued and of bounded variation on $[0,T]$, $n = 1, 2, \ldots,$ and let $h_n(\cdot)$ converge in the space $L^2[0,T]$ as $n \to \infty$. Then for any real number λ, $\exp\{\lambda \int_0^T h_n(t)\,dx(t)\}$ converges in the space $L^2(C_0[0,T])$ as $n \to \infty$.*

Proof. Let $h(\cdot)$ be of bounded variation on $[0,T]$ and $x \in C_0[0,T]$. Then it is easily shown that $\int_0^T h(t)\,dx(t)$ is a Gaussian random variable with mean and variance given by, respectively,

$$m(h) = \int_0^T h(t)\,da(t) \quad \text{and} \quad v(h) = \int_0^T h^2(t)\,db(t).$$

Thus by the change of variables theorem, we have

$$\int_{C_0[0,T]} \exp\left\{\lambda \int_0^T h(t)\,dx(t)\right\} d\mu(x) = \exp\left\{\lambda m(h) + \frac{\lambda^2}{2} v(h)\right\}.$$

Using this, we have

$$\int_{C_0[0,T]} \left[\exp\left\{\lambda \int_0^T h_i(t)\,dx(t)\right\} - \exp\left\{\lambda \int_0^T h_j(t)\,dx(t)\right\}\right]^2 d\mu(x)$$

$$
\begin{aligned}
&= \int_{C_0[0,T]} \exp\left\{2\lambda \int_0^T h_i(t)\,dx(t)\right\} d\mu(x) \\
&\quad - 2\int_{C_0[0,T]} \exp\left\{\lambda \int_0^T [h_i(t)+h_j(t)]\,dx(t)\right\} d\mu(x) \\
&\quad + \int_{C_0[0,T]} \exp\left\{2\lambda \int_0^T h_j(t)\,dx(t)\right\} d\mu(x) \\
&= \exp\left\{2\lambda m(h_i) + \frac{1}{2}(2\lambda)^2 v(h_i)\right\} \\
&\quad - 2\exp\left\{\lambda m(h_i+h_j) + \frac{\lambda^2}{2} v(h_i+h_j)\right\} \\
&\quad + \exp\left\{2\lambda m(h_j) + \frac{1}{2}(2\lambda)^2 v(h_j)\right\}.
\end{aligned}
$$

Since each exponent in the last equation approaches the same limit as i and j approach ∞, the proof of the lemma is complete. □

We now prove a translation theorem for our function space integrals.

Theorem 5.4. *Let $x_0(t) = \int_0^t h(s)\,ds$ for every $t \in [0,T]$ with $h \in L^2[0,T]$. Then for any μ-integrable function F, we have*

$$E[F(x)] = E[F(x+x_0)J(x_0,x)] \tag{5.2}$$

where

$$J(x_0,x) = \exp\left\{-\frac{1}{2}\int_0^T \frac{h^2(t)}{b'(t)}\,dt\right\} \cdot \exp\left\{-\int_0^T \frac{h(t)}{b'(t)}\,dx(t) + \int_0^T \frac{h(t)}{b'(t)}\,da(t)\right\}. \tag{5.3}$$

Proof. To prove the theorem, it suffices to show the equality (5.2) for a bounded and continuous function F in the uniform topology. Let $x_n(t) = \int_0^t h_n(s)\,ds$ where $\{h_n\}$ is a sequence of functions of bounded variation on $[0,T]$ such that h_n converges to h in the space $L^2[0,T]$ as $n \to \infty$. Then by Lemma 5.2, we have

$$E[F(x)] = E[F(x+x_n)J(x_n,x)] \tag{5.4}$$

where

$$J(x_n, x) = \exp\left\{ -\frac{1}{2}\int_0^T \frac{h_n^2(t)}{b'(t)}\,dt \right\} \cdot \exp\left\{ -\int_0^T \frac{h_n(t)}{b'(t)}\,dx(t) + \int_0^T \frac{h_n(t)}{b'(t)}\,da(t) \right\}. \tag{5.5}$$

Since x_n converges x_0 uniformly and F is continuous in the uniform topology, $F(x+x_n)$ converges to $F(x+x_0)$. Since F is bounded, by using the dominated convergence theorem and Lemma 5.3, we can show that the right member of (5.4) converges to the right member of (5.2) as $n \to \infty$, which completes the proof of the theorem.

The following result generalizes a theorem given in [**11**, p. 391].

Theorem 5.5. *Let $x_0(t) = \int_0^t h(s)\,ds$ for every $t \in [0,T]$ with $h \in L^2[0,T]$. Then for any μ-integrable function F,*

$$\begin{aligned} &E[F(x)|X_\tau(x) = \vec{\xi}] \\ &= E[F(x_0+x)J(x_0,x)|x(t_j) = \xi_j - x_0(t_j), \quad j = 1,2,\dots,n] \\ &\quad \cdot \prod_{j=1}^{n} \exp\left\{ -\frac{(x_0(t_j)-x_0(t_{j-1}))^2}{2(b(t_j)-b(t_{j-1}))} \right. \\ &\qquad \left. + \frac{(x_0(t_j)-x_0(t_{j-1}))((\xi_j-\xi_{j-1})-(a(t_j)-a(t_{j-1})))}{b(t_j)-b(t_{j-1})} \right\} \end{aligned}$$

where $J(x_0,x)$ is as in (5.3).

Proof. By using Theorem 3.4 and Theorem 5.4, we see that

$$E[F(x)|X_\tau(x) = \vec{\xi}] = E[F(x - [x] + [\vec{\xi}])] \tag{5.6}$$

and that

$$E[F(x - [x] + [\vec{\xi}])] = E[F(x + x_0 - [x] - [x_0] + [\vec{\xi}])J(x_0,x)]. \tag{5.7}$$

Next we rewrite $J(x_0, x)$ in the form

$$
\begin{aligned}
J(x_0,x) &= \exp\bigg\{\int_0^T \frac{h(t)}{b'(t)}\,da(t) - \frac{1}{2}\int_0^T \frac{h^2(t)}{b'(t)}\,dt\bigg\} \\
&\quad \cdot \exp\bigg\{-\int_0^T \frac{h(t)}{b'(t)}\,d(x(t)-[x](t)+[\vec{\xi}](t)-[x_0](t))\bigg\} \\
&\quad \cdot \exp\bigg\{-\int_0^T \frac{h(t)}{b'(t)}\,d([x](t))\bigg\} \\
&\quad \cdot \exp\bigg\{\int_0^T \frac{h(t)}{b'(t)}\,d([\vec{\xi}](t)-[x_0](t))\bigg\}.
\end{aligned}
\tag{5.8}
$$

But by a simple calculation, we have

$$
\int_0^T \frac{h(t)}{b'(t)}\,d([x](t)) = \sum_{j=1}^n \frac{x_0(t_j)-x_0(t_{j-1})}{b(t_j)-b(t_{j-1})}(x(t_j)-x(t_{j-1}))
\tag{5.9}
$$

and similarly, we have

$$
\begin{aligned}
&\int_0^T \frac{h(t)}{b'(t)}\,d([\vec{\xi}](t)-[x_0](t)) \\
&= \sum_{j=1}^n \frac{x_0(t_j)-x_0(t_{j-1})}{b(t_j)-b(t_{j-1})}\big((\xi_j-\xi_{j-1})-(x_0(t_j)-x_0(t_{j-1}))\big).
\end{aligned}
\tag{5.10}
$$

By using the fact that $\{x(t_j)-x(t_{j-1}), j=1,\dots,n\}$ is independent and then applying the change of variable theorem, we have

(5.11)

$$
\begin{aligned}
&E\bigg[\exp\bigg\{-\int_0^T \frac{h(t)}{b'(t)}\,d([x](t))\bigg\}\bigg] \\
&= \prod_{j=1}^n E\bigg[\exp\bigg\{-\frac{x_0(t_j)-x_0(t_{j-1})}{b(t_j)-b(t_{j-1})}(x(t_j)-x(t_{j-1}))\bigg\}\bigg] \\
&= \prod_{j=1}^n \exp\bigg\{\frac{(x_0(t_j)-x_0(t_{j-1}))^2-2(a(t_j)-a(t_{j-1}))(x_0(t_j)-x_0(t_{j-1}))}{2(b(t_j)-b(t_{j-1}))}\bigg\}.
\end{aligned}
$$

Since $\{x(t)-[x](t), t\in[0,T]\}$ and $\{[x](t), t\in[0,T]\}$ are independent processes by Lemma 3.1, we see that

$$
F(x+x_0-[x]-[x_0]+[\vec{\xi}])\exp\bigg\{-\int_0^T \frac{h(t)}{b'(t)}d(x(t)-[x](t)+[\vec{\xi}](t)-[x_0](t))\bigg\}
$$

and $\exp\{-\int_0^T (h(t)/b'(t))d([x](t))\}$ are independent. Therefore, using (5.8), (5.10) and (5.11), we have

$$
\begin{aligned}
&E[F(x+x_0-[x]-[x_0]+[\vec{\xi}])J(x_0,x)]\\
&= E\bigg[F(x+x_0-[x]-[x_0]+[\vec{\xi}])\\
&\qquad \cdot \exp\bigg\{-\int_0^T \frac{h(t)}{b'(t)}\,d(x(t)-[x](t)+[\vec{\xi}](t)-[x_0](t))\bigg\}\bigg]\\
&\quad \cdot \exp\bigg\{-\frac{1}{2}\int_0^T \frac{h^2(t)}{b'(t)}\,dt+\int_0^T \frac{h(t)}{b'(t)}\,da(t)\bigg\}\\
&\quad \cdot \prod_{j=1}^n \exp\bigg\{\frac{(x_0(t_j)-x_0(t_{j-1}))^2-2(a(t_j)-a(t_{j-1}))(x_0(t_j)-x_0(t_{j-1}))}{2(b(t_j)-b(t_{j-1}))}\bigg\}\\
&\quad \cdot \prod_{j=1}^n \exp\bigg\{\frac{x_0(t_j)-x_0(t_{j-1})}{b(t_j)-b(t_{j-1})}\big((\xi_j)-\xi_{j-1})-(x_0(t_j)-x_0(t_{j-1}))\big)\bigg\}.
\end{aligned}
$$

Therefore, using (5.6) and (5.7), we obtain

$$
\begin{aligned}
&E[F(x)|X_\tau(x)=\vec{\xi}]\\
&\qquad = E\bigg[F(x_0+x)\exp\bigg\{-\int_0^T \frac{h(t)}{b'(t)}\,dx(t)\bigg\}\bigg|\\
&\qquad\qquad x(t_j)=\xi_j-x_0(t_j),\quad j=1,2,\dots,n\bigg]\\
&\qquad\quad \cdot \exp\bigg\{-\frac{1}{2}\int_0^T \frac{h^2(t)}{b'(t)}\,dt+\int_0^T \frac{h(t)}{b'(t)}\,da(t)\bigg\}\\
&\qquad\quad \cdot \prod_{j=1}^n \exp\bigg\{-\frac{(x_0(t_j)-x_0(t_{j-1}))^2}{2(b(t_j)-b(t_{j-1}))}\\
&\qquad\qquad\quad +\frac{(x_0(t_j)-x_0(t_{j-1}))((\xi_j-\xi_{j-1})-(a(t_j)-a(t_{j-1})))}{b(t_j)-b(t_{j-1})}\bigg\},
\end{aligned}
$$

which completes the proof of the theorem. □

Corollary 5.6 [**11, 16**]. *In Theorem* 5.5, *if* $a(t)\equiv 0$ *and* $b(t)=t$, *then we have*

$$\begin{aligned}&E[F(x)|X_\tau(x)=\vec{\xi}]\\&=E[F(x_0+x)J(x_0,x)|x(t_j)=\xi_j-x_0(t_j),\quad j=1,2,\dots,n]\\&\cdot\prod_{j=1}^{n}\exp\bigg\{-\frac{(x_0(t_j)-x_0(t_{j-1}))^2}{2(t_j-t_{j-1})}+\frac{(x_0(t_j)-x_0(t_{j-1}))(\xi_j-\xi_{j-1})}{t_j-t_{j-1}}\bigg\}\end{aligned}$$

where

$$J(x_0,x)=\exp\bigg\{-\frac{1}{2}\int_0^T h^2(t)\,dt\bigg\}\exp\bigg\{-\int_0^T h(t)\,dx(t)\bigg\}.$$

We next use Theorem 5.5 to evaluate a conditional function space integral.

Corollary 5.7. *Let* h *and* x_0 *be as in Theorem* 5.5. *In Theorem* 5.5, *if* $a(t)\equiv 0$, *then for any* $\alpha\in\mathbf{R}$ *we have*

$$\begin{aligned}E\bigg[\exp\bigg\{\alpha\int_0^T\frac{h(t)}{b'(t)}\,dt\bigg\}\,\Big|\,x(t_j)=\xi_j,\quad j=1,2,\dots,n\bigg]&\\=\exp\bigg\{\frac{\alpha^2}{2}\int_0^T\frac{h^2(t)}{b'(t)}\,dt\bigg\}&\\\cdot\prod_{j=1}^{n}\exp\bigg\{-\frac{\alpha^2(x_0(t_j)-x_0(t_{j-1}))^2}{2(b(t_j)-b(t_{j-1}))}&\\+\frac{\alpha(x_0(t_j)-x_0(t_{j-1}))(\xi_j-\xi_{j-1})}{(b(t_j)-b(t_{j-1}))}\bigg\}.&\end{aligned}$$

Proof. Using Theorem 5.5 with $F\equiv 1$ on $C_0[0,T]$, we have

$$\begin{aligned}1&=E[J(x_0,x)|x(t_j)=\xi_j-x_0(t_j),\quad n=1,2,\dots,n]\\&\cdot\prod_{j=1}^{n}\exp\bigg\{-\frac{(x_0(t_j)-x_0(t_{j-1}))^2}{2(b(t_j)-b(t_{j-1}))}+\frac{(x_0(t_j)-x_0(t_{j-1}))(\xi_j-\xi_{j-1})}{(b(t_j)-b(t_{j-1}))}\bigg\}.\end{aligned}$$

Replacing ξ_j by $\xi_j + x_0(t_j)$ for $j = 1, 2, \dots, n$, we obtain

$$E\left[\exp\left\{-\int_0^T \frac{h(t)}{b'(t)}dx(t)\right\} \mid x(t_j) = \xi_j, \quad j = 1, 2, \dots, n\right]$$
$$= \exp\left\{\frac{1}{2}\int_0^T \frac{h^2(t)}{b'(t)}dt\right\}$$
$$\cdot \prod_{j=1}^{n} \exp\left\{-\frac{(x_0(t_j) - x_0(t_{j-1}))^2}{2(b(t_j) - b(t_{j-1}))} - \frac{(x_0(t_j) - x_0(t_{j-1}))(\xi_j - \xi_{j-1})}{(b(t_j) - b(t_{j-1}))}\right\}.$$

Thus the desired result follows by replacing $h(s)$ by $-\alpha h(s)$.

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