

## Conditional Integral Transforms on a Function Space

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ABSTRACT. Let  $C^r[0, t]$  be the function space of the vector-valued continuous paths  $x : [0, t] \rightarrow \mathbb{R}^r$  and define  $X_t : C^r[0, t] \rightarrow \mathbb{R}^{(n+1)r}$  and  $Y_t : C^r[0, t] \rightarrow \mathbb{R}^{nr}$  by  $X_t(x) = (x(t_0), x(t_1), \dots, x(t_{n-1}), x(t_n))$  and  $Y_t(x) = (x(t_0), x(t_1), \dots, x(t_{n-1}))$ , respectively, where  $0 = t_0 < t_1 < \dots < t_n = t$ . In the present paper, using two simple formulas for the conditional expectations over  $C^r[0, t]$  with the conditioning functions  $X_t$  and  $Y_t$ , we establish evaluation formulas for the analogue of the conditional analytic Fourier-Feynman transform for the function of the form

$$\exp\left\{\int_0^t \theta(s, x(s)) d\eta(s)\right\} \psi(x(t)), \quad x \in C^r[0, t]$$

where  $\eta$  is a complex Borel measure on  $[0, t]$  and both  $\theta(s, \cdot)$  and  $\psi$  are the Fourier-Stieltjes transforms of the complex Borel measures on  $\mathbb{R}^r$ .

### 1. Introduction and an analogue of the r-dimensional Wiener space

Let  $C_0[0, t]$  be the space of real-valued continuous functions  $x$  on  $[0, t]$  with  $x(0) = 0$ . It is well known that the space  $C_0[0, t]$  is equipped with the Wiener measure which is a probability measure. On the space, Yeh [10, 11, 12] introduced an inversion formula that a conditional expectation can be found by a Fourier-transform. In [2], the author and his co-authors introduced a simple formula for the conditional Wiener integrals over  $C_0(\mathbb{B})$ , the space of the abstract Wiener space  $\mathbb{B}$ -valued continuous functions which vanish at 0. Using the formula, they established various evaluation formulas for the conditional analytic Wiener and Feynman integrals of the functionals on  $C_0(\mathbb{B})$  in a certain Banach algebra which corresponds to the Cameron and Storvick's Banach algebra  $S''$  [1]. In [3], the author evaluated conditional analytic Fourier-Feynman transform for functional of the form

$$\exp\left\{\int_0^t \theta(s, x(s)) d\zeta(s)\right\} \phi(x(t)), \quad x \in C_0(\mathbb{B})$$

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where  $\zeta$  is a complex Borel measure on  $[0, t]$  and both  $\theta(s, \cdot)$  and  $\phi$  are the Fourier-Stieltjes transforms of the complex Borel measures on the real separable Hilbert space embedded in  $\mathbb{B}$ . Note that  $\theta(s, \cdot)$  and  $\phi$  are defined on  $\mathbb{B}$ , the abstract Wiener space [8].

On the other hand, let  $C[0, t]$  denote the space of real-valued continuous functions on the interval  $[0, t]$ . Ryu and Im [6, 9] introduced a probability measure  $w_\varphi$  on  $(C[0, t], \mathcal{B}(C[0, t]))$ , where  $\mathcal{B}(C[0, t])$  denotes the Borel  $\sigma$ -algebra on  $C[0, t]$  and  $\varphi$  is a probability distribution on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ . This measure space is a generalization of the Wiener space  $C_0[0, t]$ . In the Wiener space, every path  $x$  starts at the origin, that is,  $x(0) = 0$ . If the paths  $x$  start at any points, that is, if  $x \in C[0, t]$ , certain properties on  $C_0[0, t]$  can not hold or some of them should be modified. Fortunately, in [4, 5], the author could derive two simple formulas for the conditional  $w_\varphi$ -integrals of the functions on  $C[0, t]$  with the vector-valued conditioning functions  $X : C[0, t] \rightarrow \mathbb{R}^{n+1}$  and  $Y : C[0, t] \rightarrow \mathbb{R}^n$  given by  $X(x) = (x(t_0), x(t_1), \dots, x(t_n))$  and  $Y(x) = (x(t_0), x(t_1), \dots, x(t_{n-1}))$ , where  $0 = t_0 < t_1 < \dots < t_{n-1} < t_n = t$  is a partition of  $[0, t]$ . These formulas express the conditional  $w_\varphi$ -integrals directly in terms of the non-conditional  $w_\varphi$ -integrals.

Let  $C^r[0, t]$  be the product space of  $C[0, t]$  and define  $X_t : C^r[0, t] \rightarrow \mathbb{R}^{(n+1)r}$  and  $Y_t : C^r[0, t] \rightarrow \mathbb{R}^{nr}$  by  $X_t(x) = (x(t_0), x(t_1), \dots, x(t_{n-1}), x(t_n))$  and  $Y_t(x) = (x(t_0), x(t_1), \dots, x(t_{n-1}))$ . In the present paper, with the conditioning functions  $X_t$  and  $Y_t$ , we introduce two simple formulas for the conditional expectations over  $C^r[0, t]$ , an  $r$ -dimensional analogue of Wiener space. We then establish evaluation formulas for an analogue of the conditional analytic Fourier-Feynman transform for the function of the form

$$\exp\left\{\int_0^t \theta(s, x(s)) d\eta(s)\right\} \psi(x(t)), \quad x \in C^r[0, t]$$

where  $\eta$  is a complex Borel measure on  $[0, t]$  and both  $\theta(s, \cdot)$  and  $\psi$  are the Fourier-Stieltjes transforms of the complex Borel measures on  $\mathbb{R}^r$ .

Throughout this paper, let  $\mathbb{C}$  and  $\mathbb{C}_+$  denote the sets of complex numbers and complex numbers with positive real parts, respectively.

Now, we introduce the probability measure  $w_\varphi$  on  $(C[0, t], \mathcal{B}(C[0, t]))$ .

For a positive real  $t$ , let  $C = C[0, t]$  be the space of all real-valued continuous functions on the closed interval  $[0, t]$  with the supremum norm. For  $\vec{t} = (t_0, t_1, \dots, t_n)$  with  $0 = t_0 < t_1 < \dots < t_n \leq t$ , let  $J_{\vec{t}} : C[0, t] \rightarrow \mathbb{R}^{n+1}$  be the function given by

$$J_{\vec{t}}(x) = (x(t_0), x(t_1), \dots, x(t_n)).$$

For  $B_j$  ( $j = 0, 1, \dots, n$ ) in  $\mathcal{B}(\mathbb{R})$ , the subset  $J_{\vec{t}}^{-1}(\prod_{j=0}^n B_j)$  of  $C[0, t]$  is called an interval and let  $\mathcal{J}$  be the set of all such intervals. For a probability measure  $\varphi$  on

$(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ , let

$$m_\varphi \left( J_{\bar{t}}^{-1} \left( \prod_{j=0}^n B_j \right) \right) = \left[ \prod_{j=1}^n \frac{1}{2\pi(t_j - t_{j-1})} \right]^{\frac{1}{2}} \int_{B_0} \int_{\prod_{j=1}^n B_j} \exp \left\{ -\frac{1}{2} \sum_{j=1}^n \frac{(u_j - u_{j-1})^2}{t_j - t_{j-1}} \right\} d(u_1, \dots, u_n) d\varphi(u_0).$$

Then  $\mathcal{B}(C[0, t])$  coincides with the smallest  $\sigma$ -algebra generated by  $\mathcal{J}$  and there exists a unique probability measure  $w_\varphi$  on  $(C[0, t], \mathcal{B}(C[0, t]))$  such that  $w_\varphi(I) = m_\varphi(I)$  for all  $I$  in  $\mathcal{J}$ . This measure  $w_\varphi$  is called an analogue of the Wiener measure associated with the probability measure  $\varphi$  [6, 9]. Let  $r$  be a positive integer and  $C^r = C^r[0, t]$  be the product space of  $C[0, t]$  with the product measure  $w_\varphi^r$ . Since  $C[0, t]$  is a separable Banach space, we have  $\mathcal{B}(C^r[0, t]) = \prod_{j=1}^r \mathcal{B}(C[0, t])$ . This probability measure space  $(C^r[0, t], \mathcal{B}(C^r[0, t]), w_\varphi^r)$  is called an analogue of the  $r$ -dimensional Wiener space.

**Lemma 1.1** ([6, Lemma 2.1]). *If  $f : \mathbb{R}^{n+1} \rightarrow \mathbb{C}$  is a Borel measurable function, then we have*

$$\begin{aligned} & \int_C f(x(t_0), x(t_1), \dots, x(t_n)) dw_\varphi(x) \\ & \stackrel{*}{=} \left[ \prod_{j=1}^n \frac{1}{2\pi(t_j - t_{j-1})} \right]^{\frac{1}{2}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} f(u_0, u_1, \dots, u_n) \exp \left\{ -\frac{1}{2} \sum_{j=1}^n \frac{(u_j - u_{j-1})^2}{t_j - t_{j-1}} \right\} \\ & \quad d(u_1, \dots, u_n) d\varphi(u_0) \end{aligned}$$

where  $\stackrel{*}{=}$  means that if either side exists, then both sides exist and they are equal.

**Definition 1.2.** Let  $F : C^r[0, t] \rightarrow \mathbb{C}$  be  $w_\varphi^r$ -integrable and let  $Z$  be a random vector on  $C^r[0, t]$  assuming that the value space of  $Z$  is a normed space with the Borel  $\sigma$ -algebra. Then we have the conditional expectation  $E[F|Z]$  of  $F$  given  $Z$  from a well known probability theory. Further, there exists a  $P_Z$ -integrable complex-valued function  $\Xi$  on the value space of  $Z$  such that  $E[F|Z](x) = (\Xi \circ Z)(x)$  for  $w_\varphi^r$ -a.e.  $x \in C^r[0, t]$ , where  $P_Z$  is the probability distribution of  $Z$ . The function  $\Xi$  is called the conditional  $w_\varphi^r$ -integral of  $F$  given  $Z$  and it is also denoted by  $E[F|Z]$ .

**2. The simple formulas for conditional  $w_\varphi^r$ -integrals**

Let  $0 = t_0 < t_1 < \dots < t_{n-1} < t_n = t$  be a partition of  $[0, t]$ . For any  $x$  in  $C^r[0, t]$ , define the polygonal function  $[x]$  of  $x$  on  $[0, t]$  by

$$(2.1) \quad [x](s) = \sum_{j=1}^n \chi_{(t_{j-1}, t_j]}(s) \left( \frac{t_j - s}{t_j - t_{j-1}} x(t_{j-1}) + \frac{s - t_{j-1}}{t_j - t_{j-1}} x(t_j) \right) + \chi_{\{0\}}(s) x(0)$$

for  $s \in [0, t]$ , where  $\chi_{(t_{j-1}, t_j]}$  and  $\chi_{\{0\}}$  denote the indicator functions of  $(t_{j-1}, t_j]$  and  $\{0\}$ , respectively. For  $\vec{\xi}_n = (\xi_0, \xi_1, \dots, \xi_{n-1}, \xi_n) \in \mathbb{R}^{(n+1)r}$ , we define the polygonal function  $[\vec{\xi}_n]$  of  $\vec{\xi}_n$  as (2.1) replacing  $x(t_j)$  by  $\xi_j$  for  $j = 0, 1, \dots, n$ .

Now, we introduce a lemma which is useful to prove several theorems. The proof follows immediately from Corollary 2.5 in [4].

**Lemma 2.1.** *The processes  $\{x(s) - [x](s) : t_{j-1} \leq s \leq t_j\}$  on  $C^r[0, t]$ , for  $j = 1, \dots, n$ , are stochastically independent.*

In the following two theorems, we introduce two simple formulas for the conditional  $w_\varphi^r$ -integrals on  $C^r[0, t]$ . Their proofs follow immediately from Theorem 2.9 in [4] and Theorem 2.5 in [5].

**Theorem 2.2.** *Let  $F : C^r[0, t] \rightarrow \mathbb{C}$  be  $w_\varphi^r$ -integrable and  $X_t : C^r[0, t] \rightarrow \mathbb{R}^{(n+1)r}$  be given by*

$$(2.2) \quad X_t(x) = (x(t_0), x(t_1), \dots, x(t_{n-1}), x(t_n))$$

for  $x \in C^r[0, t]$ . Then we have for  $P_{X_t}$ -a.e.  $\vec{\xi}_n \in \mathbb{R}^{(n+1)r}$

$$(2.3) \quad E[F|X_t](\vec{\xi}_n) = E[F(x - [x] + [\vec{\xi}_n])],$$

where  $P_{X_t}$  is the probability distribution of  $X_t$  on  $(\mathbb{R}^{(n+1)r}, \mathcal{B}(\mathbb{R}^{(n+1)r}))$ .

**Theorem 2.3.** *Let  $F : C^r[0, t] \rightarrow \mathbb{C}$  be  $w_\varphi^r$ -integrable and  $Y_t : C^r[0, t] \rightarrow \mathbb{R}^{nr}$  be given by*

$$(2.4) \quad Y_t(x) = (x(t_0), x(t_1), \dots, x(t_{n-1}))$$

for  $x \in C^r[0, t]$ . Then we have for  $P_{Y_t}$ -a.e.  $\vec{\xi}_{n-1} = (\xi_0, \xi_1, \dots, \xi_{n-1}) \in \mathbb{R}^{nr}$

$$(2.5) \quad E[F|Y_t](\vec{\xi}_{n-1}) = \left[ \frac{1}{2\pi(t - t_{n-1})} \right]^{\frac{r}{2}} \int_{\mathbb{R}^r} E[F(x - [x] + [\vec{\xi}_n])] \times \exp\left\{ -\frac{\|\xi_n - \xi_{n-1}\|_{\mathbb{R}^r}^2}{2(t - t_{n-1})} \right\} d\xi_n$$

where  $P_{Y_t}$  is the probability distribution of  $Y_t$  on  $(\mathbb{R}^{nr}, \mathcal{B}(\mathbb{R}^{nr}))$  and  $[\vec{\xi}_n]$  denotes the polygonal function of  $(\xi_0, \xi_1, \dots, \xi_{n-1}, \xi_n) \in \mathbb{R}^{(n+1)r}$ .

For a function  $F : C^r[0, t] \rightarrow \mathbb{C}$  and  $\lambda > 0$ , let  $F^\lambda(x) = F(\lambda^{-\frac{1}{2}}x)$ ,  $X_t^\lambda(x) = X_t(\lambda^{-\frac{1}{2}}x)$  and  $Y_t^\lambda(x) = Y_t(\lambda^{-\frac{1}{2}}x)$ , where  $X_t$  and  $Y_t$  are given by (2.2) and (2.4), respectively. Suppose that  $E[F^\lambda]$  exists for each  $\lambda > 0$ . By the definition of the conditional  $w_\varphi^r$ -integral and (2.3), we have

$$E[F^\lambda|X_t^\lambda](\vec{\xi}_n) = E[F(\lambda^{-\frac{1}{2}}(x - [x]) + [\vec{\xi}_n])]$$

for  $P_{X_t^\lambda}$ -a.e.  $\vec{\xi}_n \in \mathbb{R}^{(n+1)r}$ , where  $P_{X_t^\lambda}$  is the probability distribution of  $X_t^\lambda$  on the Borel class of  $\mathbb{R}^{(n+1)r}$ . Throughout this paper, for  $y \in C^r[0, t]$  let

$$I_F^\lambda(y, \vec{\xi}_n) = E[F(y + \lambda^{-\frac{1}{2}}(x - [x]) + [\vec{\xi}_n])]$$

unless otherwise specified, where the expectation is taken over the variable  $x$ . Moreover, under the notations used in Theorem 2.3, we have by (2.5)

$$(2.6) \quad E[F^\lambda | Y_t^\lambda](\vec{\xi}_{n-1}) = \left[ \frac{\lambda}{2\pi(t - t_{n-1})} \right]^{\frac{r}{2}} \int_{\mathbb{R}^r} I_F^\lambda(0, \vec{\xi}_n) \exp\left\{ -\frac{\lambda}{2} \frac{\|\xi_n - \xi_{n-1}\|_{\mathbb{R}^r}^2}{t - t_{n-1}} \right\} d\xi_n$$

for  $P_{Y_t^\lambda}$ -a.e.  $\vec{\xi}_{n-1} \in \mathbb{R}^{nr}$ , where  $P_{Y_t^\lambda}$  is the probability distribution of  $Y_t^\lambda$  on the Borel class of  $\mathbb{R}^{nr}$ . From now on, for  $y \in C^r[0, t]$  let  $K_F^\lambda(y, \vec{\xi}_{n-1})$  be given by (2.6) replacing 0 by  $y$ .

If  $I_F^\lambda(0, \vec{\xi}_n)$  has the analytic extension  $J_\lambda^*(F)(\vec{\xi}_n)$  on  $\mathbb{C}_+$  as a function of  $\lambda$ , then it is called the conditional analytic Wiener  $w_\varphi^r$ -integral of  $F$  given  $X_t$  with the parameter  $\lambda$  and denoted by

$$E^{anw_\lambda}[F|X_t](\vec{\xi}_n) = J_\lambda^*(F)(\vec{\xi}_n)$$

for  $\vec{\xi}_n \in \mathbb{R}^{(n+1)r}$ . Moreover, if for a nonzero real  $q$ ,  $E^{anw_\lambda}[F|X_t](\vec{\xi}_n)$  has the limit as  $\lambda$  approaches to  $-iq$  through  $\mathbb{C}_+$ , then it is called the conditional analytic Feynman  $w_\varphi^r$ -integral of  $F$  given  $X_t$  with the parameter  $q$  and denoted by

$$E^{anf_q}[F|X_t](\vec{\xi}_n) = \lim_{\lambda \rightarrow -iq} E^{anw_\lambda}[F|X_t](\vec{\xi}_n).$$

Similarly, the definitions of  $E^{anw_\lambda}[F|Y_t](\vec{\xi}_{n-1})$  and  $E^{anf_q}[F|Y_t](\vec{\xi}_{n-1})$  are understood with  $K_F^\lambda(0, \vec{\xi}_{n-1})$ .

### 3. Time-dependent conditional Fourier-Feynman transform

For a given extended real number  $p$  with  $1 < p \leq \infty$ , suppose that  $p$  and  $p'$  are related by  $\frac{1}{p} + \frac{1}{p'} = 1$  (possibly  $p' = 1$  if  $p = \infty$ ). Let  $F_n$  and  $F$  be measurable functions such that for  $\rho > 0$

$$\lim_{n \rightarrow \infty} \int_{C^r} |F_n(\rho y) - F(\rho y)|^{p'} dw_\varphi^r(y) = 0.$$

Then we write

$$\text{l.i.m.}_{n \rightarrow \infty} (w^{p'}) (F_n) \approx F$$

and call  $F$  the limit in the mean of order  $p'$ . A similar definition is understood when  $n$  is replaced by a continuously varying parameter.

Now, we define an analytic conditional Fourier-Feynman transform of the functions on  $C^r[0, t]$ .

**Definition 3.1.** Let  $F$  be defined on  $C^r[0, t]$  and let  $X_t$  be given by (2.2). For  $\lambda \in \mathbb{C}_+$  and for  $w_\varphi^r$ -a.e.  $y \in C^r[0, t]$ , let

$$T_\lambda[F|X_t](y, \vec{\xi}_n) = E^{anw_\lambda}[F(y + \cdot)|X_t](\vec{\xi}_n)$$

for  $P_{X_t}$ -a.e.  $\vec{\xi}_n \in \mathbb{R}^{n+1}$  if it exists. For a non-zero real  $q$  and for  $w_\varphi^r$ -a.e.  $y \in C^r[0, t]$ , we define the  $L_1$  analytic conditional Fourier-Feynman transform  $T_q^{(1)}[F|X_t]$  of  $F$  by the formula

$$T_q^{(1)}[F|X_t](y, \vec{\xi}_n) = E^{anf_q}[F(y + \cdot)|X_t](\vec{\xi}_n)$$

for  $P_{X_t}$ -a.e.  $\vec{\xi}_n \in \mathbb{R}^{n+1}$  if it exists. For  $1 < p \leq \infty$  we define the  $L_p$  analytic conditional Fourier-Feynman transform  $T_q^{(p)}[F|X_t]$  of  $F$  by the formula

$$T_q^{(p)}[F|X_t](\cdot, \vec{\xi}_n) \approx \text{l.i.m.}_{\lambda \rightarrow -iq} (w^{p'}) (T_\lambda[F|X_t](\cdot, \vec{\xi}_n))$$

for  $P_{X_t}$ -a.e.  $\vec{\xi}_n \in \mathbb{R}^{n+1}$ , where  $\lambda$  approaches to  $-iq$  through  $\mathbb{C}_+$ .

Similar definitions are understood with  $K_F^\lambda(y, \vec{\xi}_{n-1})$  if we replace  $X_t$  by  $Y_t$  which is given by (2.4).

Let  $\eta$  be a complex valued Borel measure on  $[0, t]$ . Then  $\eta = \mu + \nu$  can be decomposed uniquely into the sum of a continuous measure  $\mu$  and a discrete measure  $\nu$ . Further, let  $\delta_{p_{l,j}}$  denote the Dirac measure with total mass 1 concentrated at  $p_{l,j}$ .

Let  $\mathcal{M}(\mathbb{R}^r)$  be the class of all complex Borel measures on  $\mathbb{R}^r$  and  $\mathcal{G}^*$  be the set of all  $\mathbb{C}$ -valued functions  $\theta$  on  $[0, \infty) \times \mathbb{R}^r$  which have the form

$$(3.1) \quad \theta(s, \vec{u}) = \int_{\mathbb{R}^r} \exp\{i\langle \vec{u}, \vec{v} \rangle\} d\sigma_s(\vec{v})$$

where  $\langle \cdot, \cdot \rangle$  denotes the dot product on  $\mathbb{R}^r$  and  $\{\sigma_s : s \in [0, \infty)\}$  is the family from  $\mathcal{M}(\mathbb{R}^r)$  satisfying the following conditions;

- (i) for each Borel subset  $E$  of  $\mathbb{R}^r$ ,  $\sigma_s(E)$  is a Borel measurable function of  $s$  on  $[0, t]$ ,
- (ii)  $\|\sigma_s\| \in L_1([0, t], \mathcal{B}([0, t]), |\eta|)$ .

Now we have the following theorem.

**Theorem 3.2.** Let  $X_t$  be given by (2.2) and let  $\eta = \mu + \sum_{l=1}^n \sum_{j=1}^{r_l} w_{l,j} \delta_{p_{l,j}}$ , where  $w_{l,j} \in \mathbb{C}$  for all  $(l, j)$  and  $0 = t_0 < p_{1,1} < p_{1,2} < \dots < p_{1,r_1} < t_1 < p_{2,1} < \dots <$

$p_{2,r_2} < t_2 < \dots < t_{n-1} < p_{n,1} < \dots < p_{n,r_n} < t_n = t$ . Further, let  $k$  be a positive integer and let

$$(3.2) \quad F_k(x) = \left[ \int_0^t \theta(s, x(s)) d\eta(s) \right]^k \text{ for } x \in C^r[0, t],$$

where  $\theta \in \mathcal{G}^*$  is given by (3.1). Then for  $\lambda \in \mathbb{C}_+$ ,  $y \in C^r[0, t]$  and  $\vec{\xi}_n = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{(n+1)r}$ ,  $T_\lambda[F_k|X_t](y, \vec{\xi}_n)$  exists and it is given by

$$T_\lambda[F_k|X_t](y, \vec{\xi}_n) = k! \sum_{q_1 + \dots + q_n = k} \prod_{l=1}^n A(l, \lambda, y, \xi_{l-1}, \xi_l, q_l)$$

where

$$\begin{aligned} & A(l, \lambda, y, \xi_{l-1}, \xi_l, q_l) \\ &= \sum_{m_{l,0} + m_{l,1} + \dots + m_{l,r_l} = q_l} \left( \prod_{j=1}^{r_l} \frac{w_{l,j}^{m_{l,j}}}{m_{l,j}!} \right) \sum_{j_0 + j_1 + \dots + j_{r_l} = m_{l,0}} \int_{\Delta_{m_{l,0}; j_0, \dots, j_{r_l}}} \int_{\mathbb{R}^{q_l r}} D(l, y, \\ & \xi_{l-1}, \xi_l, \vec{v}_l, \vec{h}_l, \vec{s}_l) \exp \left\{ -\frac{1}{2\lambda} \sum_{u=0}^{r_l} \sum_{v=1}^{j_u+1} (s_{l,u,v} - s_{l,u,v-1}) \right\} \left\| \sum_{\beta=u+1}^{r_l} \sum_{\gamma=1}^{j_\beta+1} \frac{t_l - s_{l,\beta,\gamma}}{t_l - t_{l-1}} \right. \\ & \times \vec{v}_{l,\beta,\gamma} + \sum_{\gamma=v}^{j_u+1} \frac{t_l - s_{l,u,\gamma}}{t_l - t_{l-1}} \vec{v}_{l,u,\gamma} + \sum_{\gamma=1}^{v-1} \frac{t_{l-1} - s_{l,u,\gamma}}{t_l - t_{l-1}} \vec{v}_{l,u,\gamma} + \sum_{\beta=0}^{u-1} \sum_{\gamma=1}^{j_\beta+1} \frac{t_{l-1} - s_{l,\beta,\gamma}}{t_l - t_{l-1}} \\ & \left. \times \vec{v}_{l,\beta,\gamma} \right\|_{\mathbb{R}^r}^2 \Bigg\} d \left( \prod_{u=0}^{r_l} \prod_{v=1}^{j_u} \sigma_{s_{l,u,v}} \times \prod_{u=1}^{r_l} \sigma_{p_{l,u}}^{m_{l,u}} \right) (\vec{v}_l, \vec{h}_l) d\mu^{m_{l,0}}(\vec{s}_l) \end{aligned}$$

with the conventions those  $s_{l,0,0} = t_{l-1}$ ,  $s_{l,u,0} = p_{l,u} = s_{l,u-1, j_{u-1}+1}$  for  $u = 1, \dots, r_l$ ,  $s_{l,r_l, j_{r_l}+1} = t_l$ ,  $\vec{s}_l = (s_{l,0,1}, \dots, s_{l,0, j_0}, \dots, s_{l,r_l,1}, \dots, s_{l,r_l, j_{r_l}})$ ,  $\Delta_{m_{l,0}; j_0, \dots, j_{r_l}} = \{ \vec{s}_l : t_{l-1} < s_{l,0,1} < \dots < s_{l,0, j_0} < p_{l,1} < s_{l,1,1} < \dots < s_{l,1, j_1} < p_{l,2} < \dots < p_{l,r_l} < s_{l,r_l,1} < \dots < s_{l,r_l, j_{r_l}} < t_l \}$ ,  $\vec{v}_l = (\vec{v}_{l,0,1}, \dots, \vec{v}_{l,0, j_0}, \vec{v}_{l,1,1}, \dots, \vec{v}_{l,1, j_1}, \dots, \vec{v}_{l,r_l,1}, \dots, \vec{v}_{l,r_l, j_{r_l}})$ ,  $\vec{v}_{l,r_l, j_{r_l}+1} = \vec{0} \in \mathbb{R}^r$ ,  $\vec{h}_l = (\vec{h}_{l,1,1}, \dots, \vec{h}_{l,1, m_{l,1}}, \vec{h}_{l,2,1}, \dots, \vec{h}_{l,2, m_{l,2}}, \dots, \vec{h}_{l,r_l,1}, \dots, \vec{h}_{l,r_l, m_{l,r_l}})$ ;  $\vec{v}_{l,u-1, j_{u-1}+1} = \sum_{v=1}^{m_{l,u}} \vec{h}_{l,u,v}$  for  $u = 1, \dots, r_l$ ,

$$(3.3) \quad \begin{aligned} & D(l, y, \xi_{l-1}, \xi_l, \vec{v}_l, \vec{h}_l, \vec{s}_l) \\ &= \exp \left\{ i \sum_{u=0}^{r_l} \sum_{v=1}^{j_u+1} \left\langle y(s_{l,u,v}) + \frac{t_l - s_{l,u,v}}{t_l - t_{l-1}} \xi_{l-1} + \frac{s_{l,u,v} - t_{l-1}}{t_l - t_{l-1}} \xi_l, \vec{v}_{l,u,v} \right\rangle \right\}, \end{aligned}$$

$\sum_{\beta=u+1}^{r_l} \sum_{\gamma=1}^{j_\beta+1} \frac{t_l - s_{l,\beta,\gamma}}{t_l - t_{l-1}} \vec{v}_{l,\beta,\gamma} = \vec{0}$  if  $u = r_l$ ,  $\sum_{\gamma=1}^{v-1} \frac{t_{l-1} - s_{l,u,\gamma}}{t_l - t_{l-1}} \vec{v}_{l,u,\gamma} = \vec{0}$  if  $v = 1$  and  $\sum_{\beta=0}^{u-1} \sum_{\gamma=1}^{j_\beta+1} \frac{t_{l-1} - s_{l,\beta,\gamma}}{t_l - t_{l-1}} \vec{v}_{l,\beta,\gamma} = \vec{0}$  if  $u = 0$ .

*Proof.* For  $\lambda > 0$ ,  $y \in C^r[0, t]$  and  $\vec{\xi}_n = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{(n+1)r}$ , we have by Lemma 2.1 and the binomial expansion

$$\begin{aligned} & I_{F_k}^\lambda(y, \vec{\xi}_n) \\ &= \sum_{q_1 + \dots + q_n = k} \frac{k!}{q_1! \dots q_n!} \prod_{l=1}^n \int_{C^r} \left[ \int_{t_{l-1}}^{t_l} \theta(s, y(s) + \lambda^{-\frac{1}{2}}(x(s) - [x](s)) + [\vec{\xi}_n](s)) \right. \\ & \quad \left. d\mu(s) + \sum_{j=1}^{r_l} w_{l,j} \theta(p_{l,j}, y(p_{l,j}) + \lambda^{-\frac{1}{2}}(x(p_{l,j}) - [x](p_{l,j})) + [\vec{\xi}_n](p_{l,j})) \right]^{q_l} dw_\varphi^r(x). \end{aligned}$$

Using the simplex method [7] and the Fubini's theorem we have by (3.1)

$$\begin{aligned} & I_{F_k}^\lambda(y, \vec{\xi}_n) \\ &= k! \sum_{q_1 + \dots + q_n = k} \prod_{l=1}^n \sum_{m_{l,0} + m_{l,1} + \dots + m_{l,r_l} = q_l} \left( \prod_{j=1}^{r_l} \frac{w_{l,j}^{m_{l,j}}}{m_{l,j}!} \right) \sum_{j_0 + j_1 + \dots + j_{r_l} = m_{l,0}} \\ & \quad \int_{\Delta_{m_{l,0}; j_0, \dots, j_{r_l}}} \int_{C^r} \left[ \int_{\mathbb{R}^{m_{l,0}r}} \exp \left\{ i \sum_{u=0}^{r_l} \sum_{v=1}^{j_u} \langle y(s_{l,u,v}) + \lambda^{-\frac{1}{2}}(x(s_{l,u,v}) - [x](s_{l,u,v})) + [\vec{\xi}_n](s_{l,u,v}), \vec{v}_{l,u,v} \rangle \right\} d \left( \prod_{u=0}^{r_l} \prod_{v=1}^{j_u} \sigma_{s_{l,u,v}} \right) (\vec{v}_l) \right] \left[ \int_{\mathbb{R}^{(m_{l,1} + \dots + m_{l,r_l})r}} \right. \\ & \quad \left. \exp \left\{ i \sum_{u=1}^{r_l} \sum_{v=1}^{m_{l,u}} \langle y(s_{l,u,0}) + \lambda^{-\frac{1}{2}}(x(s_{l,u,0}) - [x](s_{l,u,0})) + [\vec{\xi}_n](s_{l,u,0}), \vec{h}_{l,u,v} \rangle \right\} \right. \\ & \quad \left. d \left( \prod_{u=1}^{r_l} \sigma_{p_{l,u}}^{m_{l,u}} \right) (\vec{h}_l) \right] dw_\varphi^r(x) d\mu^{m_{l,0}}(\vec{s}_l), \end{aligned}$$

where  $\vec{s}_l$ ,  $\vec{v}_l$ ,  $\vec{h}_l$  and  $\Delta_{m_{l,0}; j_0, \dots, j_{r_l}}$  are given by the assumptions, and  $s_{l,u,0} = p_{l,u}$  for  $u = 1, \dots, r_l$ . For  $u = 1, \dots, r_l$ , let  $s_{l,u-1, j_{u-1}+1} = s_{l,u,0}$ ,  $\vec{v}_{l,u-1, j_{u-1}+1} = \sum_{v=1}^{m_{l,u}} \vec{h}_{l,u,v}$ ,  $s_{l,r_l, j_{r_l}+1} = t_l$ ,  $s_{l,0,0} = t_{l-1}$  and  $\vec{v}_{l,r_l, j_{r_l}+1} = \vec{0} \in \mathbb{R}^r$ . Then we have

$$\begin{aligned} & I_{F_k}^\lambda(y, \vec{\xi}_n) \\ &= k! \sum_{q_1 + \dots + q_n = k} \prod_{l=1}^n \sum_{m_{l,0} + m_{l,1} + \dots + m_{l,r_l} = q_l} \left( \prod_{j=1}^{r_l} \frac{w_{l,j}^{m_{l,j}}}{m_{l,j}!} \right) \sum_{j_0 + j_1 + \dots + j_{r_l} = m_{l,0}} \\ & \quad \int_{\Delta_{m_{l,0}; j_0, \dots, j_{r_l}}} \int_{\mathbb{R}^{q_l r}} D(l, y, \xi_{l-1}, \xi_l, \vec{v}_l, \vec{h}_l, \vec{s}_l) \int_{C^r} \exp \left\{ i \lambda^{-\frac{1}{2}} \sum_{u=0}^{r_l} \sum_{v=1}^{j_u+1} \right. \\ & \quad \left. \left\langle \frac{t_l - s_{l,u,v}}{t_l - t_{l-1}} (x(s_{l,u,v}) - x(t_{l-1})) + \frac{t_{l-1} - s_{l,u,v}}{t_l - t_{l-1}} (x(t_l) - x(s_{l,u,v})), \right. \right. \\ & \quad \left. \left. \vec{v}_{l,u,v} \right\rangle \right\} dw_\varphi^r(x) d \left( \prod_{u=0}^{r_l} \prod_{v=1}^{j_u} \sigma_{s_{l,u,v}} \times \prod_{u=1}^{r_l} \sigma_{p_{l,u}}^{m_{l,u}} \right) (\vec{v}_l, \vec{h}_l) d\mu^{m_{l,0}}(\vec{s}_l) \end{aligned}$$



where  $D(l, y, \xi_{l-1}, \xi_l, \vec{v}_l, \vec{h}_l, \vec{s}_l)$  is given by (3.3). Let

$$S(1) = \left[ \prod_{u=0}^{r_1} \prod_{v=1}^{j_u+1} \frac{1}{2\pi(s_{1,u,v} - s_{1,u,v-1})} \right]^{\frac{r}{2}}$$

and for  $l = 2, \dots, n$  let

$$S(l) = \left[ \frac{1}{2\pi t_{l-1}} \prod_{u=0}^{r_l} \prod_{v=1}^{j_u+1} \frac{1}{2\pi(s_{l,u,v} - s_{l,u,v-1})} \right]^{\frac{r}{2}}.$$

Let  $m(1) = m_{1,0} + r_1 + 1$  and  $m(l) = m_{l,0} + r_l + 2$  if  $l = 2, \dots, n$ . Then we have by an application of Lemma 1.1

$$\begin{aligned} & I_{F_k}^\lambda(y, \vec{\xi}_n) \\ = & k! \sum_{q_1+\dots+q_n=k} \prod_{l=1}^n \sum_{m_{l,0}+m_{l,1}+\dots+m_{l,r_l}=q_l} \left( \prod_{j=1}^{r_l} \frac{w_{l,j}^{m_{l,j}}}{m_{l,j}!} \right) \sum_{j_0+j_1+\dots+j_{r_l}=m_{l,0}} \\ & \int_{\Delta_{m_{l,0};j_0,\dots,j_{r_l}}} \int_{\mathbb{R}^{q_l r}} D(l, y, \xi_{l-1}, \xi_l, \vec{v}_l, \vec{h}_l, \vec{s}_l) S(l) \int_{\mathbb{R}^r} \int_{\mathbb{R}^{m(l)r}} \exp \left\{ i\lambda^{-\frac{1}{2}} \sum_{u=0}^{r_l} \sum_{v=1}^{j_u+1} \right. \\ & \left. \left\langle \frac{t_l - s_{l,u,v}}{t_l - t_{l-1}} (\vec{\zeta}_{l,u,v} - \vec{\zeta}_{l,0,0}) + \frac{t_{l-1} - s_{l,u,v}}{t_l - t_{l-1}} (\vec{\zeta}_{l,r_l,j_{r_l}+1} - \vec{\zeta}_{l,u,v}), \vec{v}_{l,u,v} \right\rangle - \frac{1}{2} \sum_{u=0}^{r_l} \right. \\ & \left. \sum_{v=1}^{j_u+1} \frac{\|\vec{\zeta}_{l,u,v} - \vec{\zeta}_{l,u,v-1}\|_{\mathbb{R}^r}^2}{s_{l,u,v} - s_{l,u,v-1}} - \frac{\|\vec{\zeta}_{l,0,0} - \vec{\zeta}_{1,0,0}\|_{\mathbb{R}^r}^2}{2t_{l-1} + \delta_{l1}} \right\} d\vec{\zeta}_l d\varphi^r(\vec{\zeta}_{1,0,0}) d \left( \prod_{u=0}^{r_l} \prod_{v=1}^{j_u} \sigma_{s_{l,u,v}} \right. \\ & \left. \times \prod_{u=1}^{r_l} \sigma_{p_{l,u}}^{m_{l,u}} \right) (\vec{v}_l, \vec{h}_l) d\mu^{m_{l,0}}(\vec{s}_l) \end{aligned}$$

where  $\vec{\zeta}_{l,u-1,j_{u-1}+1} = \vec{\zeta}_{l,u,0}$  for  $u = 1, \dots, r_l$  and  $\vec{\zeta}_1 = (\vec{\zeta}_{1,0,1}, \dots, \vec{\zeta}_{1,0,j_0+1}, \vec{\zeta}_{1,1,1}, \dots, \vec{\zeta}_{1,1,j_1+1}, \dots, \vec{\zeta}_{1,r_1,1}, \dots, \vec{\zeta}_{1,r_1,j_{r_1}+1})$ ,  $\vec{\zeta}_l = (\vec{\zeta}_{l,0,0}, \vec{\zeta}_{l,0,1}, \dots, \vec{\zeta}_{l,0,j_0+1}, \vec{\zeta}_{l,1,1}, \dots, \vec{\zeta}_{l,1,j_1+1}, \dots, \vec{\zeta}_{l,r_l,1}, \dots, \vec{\zeta}_{l,r_l,j_{r_l}+1})$  if  $l = 2, \dots, n$ . Let  $\vec{\eta}_{l,u,v} = \vec{\zeta}_{l,u,v} - \vec{\zeta}_{l,u,v-1}$  for  $u = 0, 1, \dots, r_l; v = 1, \dots, j_u + 1$  and  $\vec{\eta}_{1,0,0} = \vec{0}$ ,  $\vec{\eta}_{l,0,0} = \vec{\zeta}_{l,0,0} - \vec{\zeta}_{1,0,0}$  if  $l = 2, \dots, n$ . Then we have by the change of variable theorem

$$\begin{aligned} & I_{F_k}^\lambda(y, \vec{\xi}_n) \\ = & k! \sum_{q_1+\dots+q_n=k} \prod_{l=1}^n \sum_{m_{l,0}+m_{l,1}+\dots+m_{l,r_l}=q_l} \left( \prod_{j=1}^{r_l} \frac{w_{l,j}^{m_{l,j}}}{m_{l,j}!} \right) \sum_{j_0+j_1+\dots+j_{r_l}=m_{l,0}} \\ & \int_{\Delta_{m_{l,0};j_0,\dots,j_{r_l}}} \int_{\mathbb{R}^{q_l r}} D(l, y, \xi_{l-1}, \xi_l, \vec{v}_l, \vec{h}_l, \vec{s}_l) S(l) \int_{\mathbb{R}^{m(l)r}} \exp \left\{ i\lambda^{-\frac{1}{2}} \sum_{u=0}^{r_l} \sum_{v=1}^{j_u+1} \right. \end{aligned}$$

$$\begin{aligned} & \left\langle \frac{t_l - s_{l,u,v}}{t_l - t_{l-1}} \left( \sum_{\beta=0}^{u-1} \sum_{\gamma=1}^{j_{\beta}+1} \bar{\eta}_{l,\beta,\gamma} + \sum_{\gamma=1}^v \bar{\eta}_{l,u,\gamma} \right) + \frac{t_{l-1} - s_{l,u,v}}{t_l - t_{l-1}} \left( \sum_{\gamma=v+1}^{j_u+1} \bar{\eta}_{l,u,\gamma} + \right. \right. \\ & \left. \left. \sum_{\beta=u+1}^{r_l} \sum_{\gamma=1}^{j_{\beta}+1} \bar{\eta}_{l,\beta,\gamma} \right), \vec{v}_{l,u,v} \right\rangle - \frac{1}{2} \sum_{u=0}^{r_l} \sum_{v=1}^{j_u+1} \frac{\|\bar{\eta}_{l,u,v}\|_{\mathbb{R}^r}^2}{s_{l,u,v} - s_{l,u,v-1}} - \frac{\|\bar{\eta}_{l,0,0}\|_{\mathbb{R}^r}^2}{2t_{l-1} + \delta_{l1}} \Big\} \\ & d\bar{\eta}_l d \left( \prod_{u=0}^{r_l} \prod_{v=1}^{j_u} \sigma_{s_{l,u,v}} \times \prod_{u=1}^{r_l} \sigma_{p_{l,u}}^{m_{l,u}} \right) (\vec{v}_l, \vec{h}_l) d\mu^{m_{l,0}}(\vec{s}_l) \end{aligned}$$

where  $\bar{\eta}_l = (\bar{\eta}_{l,0,1}, \dots, \bar{\eta}_{l,0,j_0+1}, \bar{\eta}_{l,1,1}, \dots, \bar{\eta}_{l,1,j_1+1}, \dots, \bar{\eta}_{l,r_1,1}, \dots, \bar{\eta}_{l,r_1,j_{r_1}+1})$ ,  $\bar{\eta}_l = (\bar{\eta}_{l,0,0}, \bar{\eta}_{l,0,1}, \dots, \bar{\eta}_{l,0,j_0+1}, \bar{\eta}_{l,1,1}, \dots, \bar{\eta}_{l,1,j_1+1}, \dots, \bar{\eta}_{l,r_l,1}, \dots, \bar{\eta}_{l,r_l,j_{r_l}+1})$  if  $l \in \{2, \dots, n\}$ ,  $\sum_{\beta=0}^{u-1} \sum_{\gamma=1}^{j_{\beta}+1} \bar{\eta}_{l,\beta,\gamma} = \vec{0}$  if  $u = 0$ ,  $\sum_{\gamma=v+1}^{j_u+1} \bar{\eta}_{l,u,\gamma} = \vec{0}$  if  $v = j_u + 1$  and  $\sum_{\beta=u+1}^{r_l} \sum_{\gamma=1}^{j_{\beta}+1} \bar{\eta}_{l,\beta,\gamma} = \vec{0}$  if  $u = r_l$ . Now we have

$$\begin{aligned} & I_{F_k}^\lambda(y, \vec{\xi}_n) \\ &= k! \sum_{q_1+\dots+q_n=k} \prod_{l=1}^n \sum_{m_{l,0}+m_{l,1}+\dots+m_{l,r_l}=q_l} \left( \prod_{j=1}^{r_l} \frac{w_{l,j}^{m_{l,j}}}{m_{l,j}!} \right) \sum_{j_0+j_1+\dots+j_{r_l}=m_{l,0}} \\ & \int_{\Delta_{m_{l,0};j_0,\dots,j_{r_l}}} \int_{\mathbb{R}^{q_l r}} D(l, y, \xi_{l-1}, \xi_l, \vec{v}_l, \vec{h}_l, \vec{s}_l) S(l) \int_{\mathbb{R}^{m(l)r}} \exp \left\{ i\lambda^{-\frac{1}{2}} \sum_{u=0}^{r_l} \sum_{v=1}^{j_u+1} \left\langle \right. \right. \\ & \bar{\eta}_{l,u,v}, \sum_{\beta=u+1}^{r_l} \sum_{\gamma=1}^{j_{\beta}+1} \frac{t_l - s_{l,\beta,\gamma}}{t_l - t_{l-1}} \vec{v}_{l,\beta,\gamma} + \sum_{\gamma=v}^{j_u+1} \frac{t_l - s_{l,u,\gamma}}{t_l - t_{l-1}} \vec{v}_{l,u,\gamma} + \sum_{\gamma=1}^{v-1} \frac{t_{l-1} - s_{l,u,\gamma}}{t_l - t_{l-1}} \\ & \left. \left. \vec{v}_{l,u,\gamma} + \sum_{\beta=0}^{u-1} \sum_{\gamma=1}^{j_{\beta}+1} \frac{t_{l-1} - s_{l,\beta,\gamma}}{t_l - t_{l-1}} \vec{v}_{l,\beta,\gamma} \right\rangle - \frac{1}{2} \sum_{u=0}^{r_l} \sum_{v=1}^{j_u+1} \frac{\|\bar{\eta}_{l,u,v}\|_{\mathbb{R}^r}^2}{s_{l,u,v} - s_{l,u,v-1}} - \right. \\ & \left. \frac{\|\bar{\eta}_{l,0,0}\|_{\mathbb{R}^r}^2}{2t_{l-1} + \delta_{l1}} \right\} d\bar{\eta}_l d \left( \prod_{u=0}^{r_l} \prod_{v=1}^{j_u} \sigma_{s_{l,u,v}} \times \prod_{u=1}^{r_l} \sigma_{p_{l,u}}^{m_{l,u}} \right) (\vec{v}_l, \vec{h}_l) d\mu^{m_{l,0}}(\vec{s}_l) \\ &= k! \sum_{q_1+\dots+q_n=k} \prod_{l=1}^n A(l, \lambda, y, \xi_{l-1}, \xi_l, q_l) \end{aligned}$$

where the last equality follows from the well known integration formula

$$(3.4) \quad \int_{\mathbb{R}} \exp\{-au^2 + ibu\} du = \left(\frac{\pi}{a}\right)^{\frac{1}{2}} \exp\left\{-\frac{b^2}{4a}\right\}$$

for  $a \in \mathbb{C}_+$  and any real  $b$ . By the Morera's theorem, the theorem follows. □

**Corollary 3.3.** *Under the assumptions given as in Theorem 3.2, with one exception  $\eta = \mu$ , that is, assuming that  $\eta$  has no discrete part, we have for  $\lambda \in \mathbb{C}_+$ ,  $y \in C^r[0, t]$*

and  $\vec{\xi}_n \in \mathbb{R}^{(n+1)r}$

$$T_\lambda[F_k|X_t](y, \vec{\xi}_n) = k! \sum_{q_1+\dots+q_n=k} \prod_{l=1}^n A_\mu(l, \lambda, y, \vec{\xi}_n, q_l)$$

where

$$\begin{aligned} & A_\mu(l, \lambda, y, \vec{\xi}_n, q_l) \\ &= \int_{\Delta_{q_l}} \int_{\mathbb{R}^{q_l r}} \exp \left\{ i \sum_{u=1}^{q_l} \langle y(s_{l,u}) + [\vec{\xi}_n](s_{l,u}), \vec{v}_{l,u} \rangle - \frac{1}{2\lambda} \sum_{u=1}^{q_l+1} (s_{l,u} - s_{l,u-1}) \right. \\ & \quad \left. \times \left\| \sum_{\beta=u}^{q_l} \frac{t_l - s_{l,\beta}}{t_l - t_{l-1}} \vec{v}_{l,\beta} + \sum_{\beta=1}^{u-1} \frac{t_{l-1} - s_{l,\beta}}{t_l - t_{l-1}} \vec{v}_{l,\beta} \right\|_{\mathbb{R}^r}^2 \right\} d \left( \prod_{u=1}^{q_l} \sigma_{s_{l,u}} \right) (\vec{v}_l) d\mu^{q_l}(\vec{s}_l) \end{aligned}$$

with the conventions those  $s_{l,0} = t_{l-1}$ ,  $s_{l,q_l+1} = t_l$ ,  $\vec{s}_l = (s_{l,1}, s_{l,2}, \dots, s_{l,q_l})$ ,  $\Delta_{q_l} = \{\vec{s}_l : t_{l-1} < s_{l,1} < s_{l,2} < \dots < s_{l,q_l} < t_l\}$ ,  $\vec{v}_l = (\vec{v}_{l,1}, \vec{v}_{l,2}, \dots, \vec{v}_{l,q_l})$ ,  $\sum_{\beta=u}^{q_l} \frac{t_l - s_{l,\beta}}{t_l - t_{l-1}} \vec{v}_{l,\beta} = \vec{0}$  if  $u = q_l + 1$  and  $\sum_{\beta=1}^{u-1} \frac{t_{l-1} - s_{l,\beta}}{t_l - t_{l-1}} \vec{v}_{l,\beta} = \vec{0}$  if  $u = 1$ .

**Corollary 3.4.** Under the assumptions given as in Theorem 3.2 with one exception  $\eta = \sum_{l=1}^n \sum_{j=1}^{r_l} w_{l,j} \delta_{p_{l,j}}$ , that is, assuming that  $\eta$  has no continuous part, we have for  $\lambda \in \mathbb{C}_+$ ,  $y \in C^r[0, t]$  and  $\vec{\xi}_n \in \mathbb{R}^{(n+1)r}$

$$T_\lambda[F_k|X_t](y, \vec{\xi}_n) = k! \sum_{q_1+\dots+q_n=k} \prod_{l=1}^n A_\nu(l, \lambda, y, \vec{\xi}_n, q_l)$$

where

$$\begin{aligned} & A_\nu(l, \lambda, y, \vec{\xi}_n, q_l) \\ &= \sum_{m_{l,1}+\dots+m_{l,r_l}=q_l} \left( \prod_{j=1}^{r_l} \frac{w_{l,j}^{m_{l,j}}}{m_{l,j}!} \right) \int_{\mathbb{R}^{q_l r}} \exp \left\{ i \sum_{u=1}^{r_l} \sum_{v=1}^{m_{l,u}} \langle y(p_{l,u}) + [\vec{\xi}_n](p_{l,u}), \vec{h}_{l,u,v} \rangle \right. \\ & \quad \left. - \frac{1}{2\lambda} \sum_{u=1}^{r_l+1} (p_{l,u} - p_{l,u-1}) \left\| \sum_{\beta=u}^{r_l} \sum_{\gamma=1}^{m_{l,\beta}} \frac{t_l - p_{l,\beta}}{t_l - t_{l-1}} \vec{h}_{l,\beta,\gamma} + \sum_{\beta=1}^{u-1} \sum_{\gamma=1}^{m_{l,\beta}} \frac{t_{l-1} - p_{l,\beta}}{t_l - t_{l-1}} \vec{h}_{l,\beta,\gamma} \right\|_{\mathbb{R}^r}^2 \right. \\ & \quad \left. \right\} d \left( \prod_{u=1}^{r_l} \sigma_{p_{l,u}} \right) (\vec{h}_l) \end{aligned}$$

with the conventions those  $p_{l,0} = t_{l-1}$ ,  $p_{l,r_l+1} = t_l$ ,  $\vec{h}_l = (\vec{h}_{l,1,1}, \dots, \vec{h}_{l,1,m_{l,1}}, \vec{h}_{l,2,1}, \dots, \vec{h}_{l,2,m_{l,2}}, \dots, \vec{h}_{l,r_l,1}, \dots, \vec{h}_{l,r_l,m_{l,r_l}})$ ,  $\sum_{\beta=u}^{r_l} \sum_{\gamma=1}^{m_{l,\beta}} \frac{t_l - p_{l,\beta}}{t_l - t_{l-1}} \vec{h}_{l,\beta,\gamma} = \vec{0}$  if  $u = r_l + 1$  and  $\sum_{\beta=1}^{u-1} \sum_{\gamma=1}^{m_{l,\beta}} \frac{t_{l-1} - p_{l,\beta}}{t_l - t_{l-1}} \vec{h}_{l,\beta,\gamma} = \vec{0}$  if  $u = 1$ .

**Theorem 3.5.** Let  $1 \leq p \leq \infty$  and  $q$  be a nonzero real number. Then, under the

assumptions given as in Theorem 3.2,  $T_q^{(p)}[F_k|X_t](y, \vec{\xi}_n)$  exists for  $y \in C^r[0, t]$  and  $\vec{\xi}_n = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{(n+1)r}$ , and it is given by

$$(3.5) \quad T_q^{(p)}[F_k|X_t](y, \vec{\xi}_n) = k! \sum_{q_1 + \dots + q_n = k} \prod_{l=1}^n A(l, -iq, y, \xi_{l-1}, \xi_l, q_l)$$

where  $A$  is given as in Theorem 3.2.

*Proof.* For  $1 \leq p \leq \infty$  let  $T_q^{(p)}[F_k|X_t](y, \vec{\xi}_n)$  be given by the right hand side of (3.5). If  $\lambda \in \mathbb{C}_+$  or  $\lambda = -iq$ , we have by the simplex method and the binomial expansion

$$\begin{aligned} & \left| k! \sum_{q_1 + \dots + q_n = k} \prod_{l=1}^n A(l, \lambda, y, \xi_{l-1}, \xi_l, q_l) \right| \\ & \leq k! \sum_{q_1 + \dots + q_n = k} \prod_{l=1}^n \left[ \sum_{m_{l,0} + m_{l,1} + \dots + m_{l,r_l} = q_l} \frac{1}{m_{l,0}!} \left[ \int_{t_{l-1}}^{t_l} \|\sigma_{s_l}\| |d|\mu|(s_l) \right]^{m_{l,0}} \right. \\ & \quad \left. \times \prod_{j=1}^{r_l} \frac{(|w_{l,j}| \|\sigma_{p_{l,j}}\|)^{m_{l,j}}}{m_{l,j}!} \right] \\ & = k! \sum_{q_1 + \dots + q_n = k} \prod_{l=1}^n \frac{1}{q_l!} \left[ \int_{t_{l-1}}^{t_l} \|\sigma_{s_l}\| |d|\eta|(s_l) \right]^{q_l} = \left[ \int_0^t \|\sigma_s\| |d|\eta|(s) \right]^k \end{aligned}$$

so that for  $\lambda \in \mathbb{C}_+$  we have

$$|T_\lambda[F_k|X_t](y, \vec{\xi}_n) - T_q^{(p)}[F_k|X_t](y, \vec{\xi}_n)| \leq 2 \left[ \int_0^t \|\sigma_s\| |d|\eta|(s) \right]^k.$$

Hence when  $1 < p \leq \infty$ , for  $\frac{1}{p} + \frac{1}{p'} = 1$  and  $\rho > 0$ , we have

$$\begin{aligned} & \int_{C^r} |T_\lambda[F_k|X_t](\rho y, \vec{\xi}_n) - T_q^{(p)}[F_k|X_t](\rho y, \vec{\xi}_n)|^{p'} dw_\varphi^r(y) \\ & \leq \int_{C^r} \left[ 2 \left[ \int_0^t \|\sigma_s\| |d|\eta|(s) \right]^k \right]^{p'} dw_\varphi^r(y) = 2^{p'} \left[ \int_0^t \|\sigma_s\| |d|\eta|(s) \right]^{kp'} < \infty. \end{aligned}$$

Letting  $\lambda \rightarrow -iq$  through  $\mathbb{C}_+$ , we have the result by the dominated convergence theorem.  $\square$

**Theorem 3.6.** For  $x \in C^r[0, t]$  let

$$(3.6) \quad F(x) = \exp \left\{ \int_0^t \theta(s, x(s)) d\eta(s) \right\}.$$

Then under the assumptions given as in Theorem 3.5,  $T_q^{(p)}[F|X_t](y, \vec{\xi}_n)$  exists for  $y \in C^r[0, t]$  and  $\vec{\xi}_n \in \mathbb{R}^{(n+1)r}$ , and it is given by

$$(3.7) \quad T_q^{(p)}[F|X_t](y, \vec{\xi}_n) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} T_q^{(p)}[F_k|X_t](y, \vec{\xi}_n)$$

where  $T_q^{(p)}[F_k|X_t](y, \vec{\xi}_n)$  is given as in Theorem 3.5.

*Proof.* By the Maclaurin series of the exponential function, we have

$$F(x) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} F_k(x)$$

and for  $\lambda \in \mathbb{C}_+$  or  $\lambda = -iq$ ,  $y \in C^r[0, t]$  and  $\vec{\xi}_n = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{(n+1)r}$  we have

$$\begin{aligned} & \left| 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \left| k! \sum_{q_1+\dots+q_n=k} \prod_{l=1}^n A(l, \lambda, y, \xi_{l-1}, \xi_l, q_l) \right| \right| \\ & \leq 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \left[ \int_0^t \|\sigma_s\| |d\eta|(s) \right]^k = \exp \left\{ \int_0^t \|\sigma_s\| |d\eta|(s) \right\} \end{aligned}$$

where  $A$  is given as in Theorems 3.2 and 3.5. Hence we have for  $\lambda \in \mathbb{C}_+$

$$(3.8) \quad T_\lambda[F|X_t](y, \vec{\xi}_n) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} T_\lambda[F_k|X_t](y, \vec{\xi}_n)$$

since the convergence of (3.8) is uniform with respect to both  $\lambda$  and  $y$ . For  $1 \leq p \leq \infty$  let  $T_q^{(p)}[F|X_t](y, \vec{\xi}_n)$  be given by the right hand side of (3.7). For  $\lambda \in \mathbb{C}_+$  we have

$$|T_\lambda[F|X_t](y, \vec{\xi}_n) - T_q^{(p)}[F|X_t](y, \vec{\xi}_n)| \leq 2 \exp \left\{ \int_0^t \|\sigma_s\| |d\eta|(s) \right\}$$

and hence when  $1 < p \leq \infty$ , for  $\frac{1}{p} + \frac{1}{p'} = 1$  and  $\rho > 0$ , we have

$$\begin{aligned} & \int_{C^r} |T_\lambda[F|X_t](\rho y, \vec{\xi}_n) - T_q^{(p)}[F|X_t](\rho y, \vec{\xi}_n)|^{p'} dw_\varphi^r(y) \\ & \leq \int_{C^r} \left[ 2 \exp \left\{ \int_0^t \|\sigma_s\| |d\eta|(s) \right\} \right]^{p'} dw_\varphi^r(y) = 2^{p'} \exp \left\{ p' \int_0^t \|\sigma_s\| |d\eta|(s) \right\} < \infty. \end{aligned}$$

Letting  $\lambda \rightarrow -iq$  through  $\mathbb{C}_+$ , we have the result by the dominated convergence theorem.  $\square$

For  $\nu \in \mathcal{M}(\mathbb{R}^r)$  define  $\psi$  on  $\mathbb{R}^r$  by

$$(3.9) \quad \psi(\vec{u}) = \int_{\mathbb{R}^r} \exp\{i\langle \vec{u}, \vec{v} \rangle\} d\nu(\vec{v}).$$

Then we have for  $\vec{\xi}_n = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{(n+1)r}$  and  $\lambda > 0$

$$(3.10) \quad \psi(y(t) + \lambda^{-\frac{1}{2}}(x(t) - [x](t)) + [\vec{\xi}_n](t)) = \psi(y(t) + \xi_n)$$

and

$$(3.11) \quad |\psi(y(t) + \xi_n)| \leq \|\nu\|.$$

By (3.10), (3.11), Theorems 3.2, 3.5 and 3.6, we have the following theorem.

**Theorem 3.7.** *Let  $G_k$  and  $G$  be given by*

$$G_k(x) = F_k(x)\psi(x(t))$$

and

$$(3.12) \quad G(x) = F(x)\psi(x(t))$$

for  $x \in C^r[0, t]$ , where  $F_k$ ,  $F$  and  $\psi$  are given by (3.2), (3.6) and (3.9), respectively. Then, under the assumptions and notations given as in Theorems 3.2, 3.5 and 3.6, we have for  $y \in C^r[0, t]$ , nonzero real  $q$  and  $\vec{\xi}_n = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{(n+1)r}$

$$T_q^{(p)}[G_k|X_t](y, \vec{\xi}_n) = \psi(y(t) + \xi_n)T_q^{(p)}[F_k|X_t](y, \vec{\xi}_n)$$

and

$$\begin{aligned} T_q^{(p)}[G|X_t](y, \vec{\xi}_n) &= \psi(y(t) + \xi_n)T_q^{(p)}[F|X_t](y, \vec{\xi}_n) \\ &= \psi(y(t) + \xi_n) + \sum_{k=1}^{\infty} \frac{1}{k!} T_q^{(p)}[G_k|X_t](y, \vec{\xi}_n). \end{aligned}$$

**Remark 3.8.**

- If  $F_k$ ,  $G_k$ ,  $F$  and  $G$  are defined on  $r$ -dimensional Wiener space, then we can obtain the same results in Theorems 3.2, 3.5, 3.6 and 3.7 with  $\xi_0 = \vec{0} \in \mathbb{R}^r$  in the expression of  $\vec{\xi}_n$ .
- If some of the  $p_{l,j}$ s are in the set  $\{t_0, t_1, \dots, t_n\}$ , we can obtain all the results in the present section with minor modifications.
- If  $\eta = \mu + \sum_{l=1}^n \sum_{j=1}^{r_l} w_{l,j} \delta_{p_{l,j}}$  and some of the  $r_l$ s are  $\infty$ , then, using the following version of the  $\aleph_0$ -nomial formula [7, p.41]

$$(3.13) \quad \left( \sum_{p=0}^{\infty} b_p \right)^n = \sum_{h=0}^{\infty} \sum_{q_0+q_1+\dots+q_h=n, q_h \neq 0} \frac{n!}{q_0!q_1! \dots q_h!} b_0^{q_0} b_1^{q_1} \dots b_h^{q_h},$$

we can show that  $T_q^{(p)}[G|X_t](y, \vec{\xi}_n)$  exists in Theorem 3.7.

#### 4. Time-independent conditional Fourier-Feynman transform

In the present section we evaluate the conditional analytic Fourier-Feynman

transform of  $G$  given  $Y_t$ , where  $Y_t$  and  $G$  are given by (2.4) and (3.12), respectively. For the purpose we need the following lemma.

**Lemma 4.1.** For  $\lambda > 0$ ,  $\vec{v} \in \mathbb{R}^r$ ,  $y \in C^r[0, t]$  and  $\xi_{n-1} \in \mathbb{R}^r$  let

$$\begin{aligned} \Psi(n, \lambda, y, \xi_{n-1}, \vec{v}, \vec{v}_n, \vec{h}_n, \vec{s}_n) &= \left[ \frac{\lambda}{2\pi(t-t_{n-1})} \right]^{\frac{r}{2}} \int_{\mathbb{R}^r} D(n, y, \xi_{n-1}, \xi_n, \vec{v}_n, \vec{h}_n, \vec{s}_n) \\ &\quad \times \exp \left\{ i \langle \xi_n, \vec{v} \rangle - \frac{\lambda \|\xi_n - \xi_{n-1}\|_{\mathbb{R}^r}^2}{2(t-t_{n-1})} \right\} d\xi_n \end{aligned}$$

where  $\vec{v}_n, \vec{h}_n, \vec{s}_n$  and  $D(n, y, \xi_{n-1}, \xi_n, \vec{v}_n, \vec{h}_n, \vec{s}_n)$  are given as in Theorem 3.2 with  $l = n$ . Then we have

$$\begin{aligned} (4.1) \quad \Psi(n, \lambda, y, \xi_{n-1}, \vec{v}, \vec{v}_n, \vec{h}_n, \vec{s}_n) &= \exp \left\{ i \langle \xi_{n-1}, \vec{v} \rangle + i \sum_{u=0}^{r_n} \sum_{v=1}^{j_u+1} \langle y(s_{n,u,v}) + \xi_{n-1}, \vec{v}_{n,u,v} \rangle \right. \\ &\quad \left. - \frac{t-t_{n-1}}{2\lambda} \left\| \vec{v} + \sum_{u=0}^{r_n} \sum_{v=1}^{j_u+1} \frac{s_{n,u,v} - t_{n-1}}{t-t_{n-1}} \vec{v}_{n,u,v} \right\|_{\mathbb{R}^r}^2 \right\}. \end{aligned}$$

*Proof.* For  $\lambda > 0$ , we have by the change of variable theorem and (3.4)

$$\begin{aligned} &\Psi(n, \lambda, y, \xi_{n-1}, \vec{v}, \vec{v}_n, \vec{h}_n, \vec{s}_n) \\ &= \left[ \frac{\lambda}{2\pi(t-t_{n-1})} \right]^{\frac{r}{2}} \int_{\mathbb{R}^r} \exp \left\{ i \sum_{u=0}^{r_n} \sum_{v=1}^{j_u+1} \left\langle y(s_{n,u,v}) + \xi_{n-1} + \frac{s_{n,u,v} - t_{n-1}}{t-t_{n-1}} (\xi_n \right. \right. \\ &\quad \left. \left. - \xi_{n-1}), \vec{v}_{n,u,v} \right\rangle + i \langle \xi_{n-1}, \vec{v} \rangle + i \langle \xi_n - \xi_{n-1}, \vec{v} \rangle - \frac{\lambda \|\xi_n - \xi_{n-1}\|_{\mathbb{R}^r}^2}{2(t-t_{n-1})} \right\} d\xi_n \\ &= \exp \left\{ i \langle \xi_{n-1}, \vec{v} \rangle + i \sum_{u=0}^{r_n} \sum_{v=1}^{j_u+1} \langle y(s_{n,u,v}) + \xi_{n-1}, \vec{v}_{n,u,v} \rangle - \frac{t-t_{n-1}}{2\lambda} \left\| \vec{v} + \sum_{u=0}^{r_n} \right. \right. \\ &\quad \left. \left. \sum_{v=1}^{j_u+1} \frac{s_{n,u,v} - t_{n-1}}{t-t_{n-1}} \vec{v}_{n,u,v} \right\|_{\mathbb{R}^r}^2 \right\} \end{aligned}$$

which completes the proof. □

**Theorem 4.2.** Let the assumptions and notations be given as in Theorem 3.2. Suppose that  $Y_t$  is given by (2.4). Then for  $\lambda \in \mathbb{C}_+$ ,  $y \in C^r[0, t]$  and  $\vec{\xi}_{n-1} = (\xi_0, \xi_1, \dots, \xi_{n-1}) \in \mathbb{R}^{nr}$ ,  $T_\lambda[F_k|Y_t](y, \vec{\xi}_{n-1})$  exists and it is given by

$$T_\lambda[F_k|Y_t](y, \vec{\xi}_{n-1}) = k! \sum_{q_1 + \dots + q_n = k} \left[ \prod_{l=1}^{n-1} A(l, \lambda, y, \xi_{l-1}, \xi_l, q_l) \right] B(n, \lambda, y, \vec{0}, \xi_{n-1}, q_n)$$

where for  $\vec{v} \in \mathbb{R}^r$ ,  $B(n, \lambda, y, \vec{v}, \xi_{n-1}, q_n)$  is given by the expression of  $A(n, \lambda, y, \xi_{n-1}, \xi_n, q_n)$  replacing  $D(n, y, \xi_{n-1}, \xi_n, \vec{v}_n, \vec{h}_n, \vec{s}_n)$  by  $\Psi(n, \lambda, y, \xi_{n-1}, \vec{v}, \vec{v}_n, \vec{h}_n, \vec{s}_n)$  which is given by (4.1).

*Proof.* For  $\vec{\xi}_{n-1} = (\xi_0, \xi_1, \dots, \xi_{n-1}) \in \mathbb{R}^{nr}$ , let  $(\vec{\xi}_{n-1}, \xi_n) = (\xi_0, \xi_1, \dots, \xi_{n-1}, \xi_n) \in \mathbb{R}^{(n+1)r}$ , where  $\xi_n \in \mathbb{R}^r$ . Then for  $\lambda > 0$  and  $y \in C^r[0, t]$  we have by Theorem 3.2

$$\begin{aligned} K_{F_k}^\lambda(y, \vec{\xi}) &= \left[ \frac{\lambda}{2\pi(t - t_{n-1})} \right]^{\frac{r}{2}} \int_{\mathbb{R}^r} I_{F_k}^\lambda(y, (\vec{\xi}_{n-1}, \xi_n)) \exp\left\{ -\frac{\lambda \|\xi_n - \xi_{n-1}\|_{\mathbb{R}^r}^2}{2(t - t_{n-1})} \right\} d\xi_n \\ &= k! \left[ \frac{\lambda}{2\pi(t - t_{n-1})} \right]^{\frac{r}{2}} \int_{\mathbb{R}^r} \left[ \sum_{q_1 + \dots + q_n = k} \prod_{l=1}^n A(l, \lambda, y, \xi_{l-1}, \xi_l, q_l) \right] \\ &\quad \times \exp\left\{ -\frac{\lambda \|\xi_n - \xi_{n-1}\|_{\mathbb{R}^r}^2}{2(t - t_{n-1})} \right\} d\xi_n \\ &= k! \sum_{q_1 + \dots + q_n = k} \left[ \prod_{l=1}^{n-1} A(l, \lambda, y, \xi_{l-1}, \xi_l, q_l) \right] \left[ \frac{\lambda}{2\pi(t - t_{n-1})} \right]^{\frac{r}{2}} \\ &\quad \times \int_{\mathbb{R}^r} A(n, \lambda, y, \xi_{n-1}, \xi_n, q_n) \exp\left\{ -\frac{\lambda \|\xi_n - \xi_{n-1}\|_{\mathbb{R}^r}^2}{2(t - t_{n-1})} \right\} d\xi_n \\ &= k! \sum_{q_1 + \dots + q_n = k} \left[ \prod_{l=1}^{n-1} A(l, \lambda, y, \xi_{l-1}, \xi_l, q_l) \right] B(n, \lambda, y, \vec{0}, \xi_{n-1}, q_n) \end{aligned}$$

where the last equality follows from Lemma 4.1. Now, by the Morera's theorem, we have the theorem. □

For  $\lambda \in \mathbb{C}_+$  it is easy to prove

$$(4.2) \quad |\Psi(n, \lambda, y, \xi_{n-1}, \vec{v}, \vec{v}_n, \vec{h}_n, \vec{s}_n)| \leq 1$$

where  $\Psi$  is given by (4.1). Applying the same method used in the proof of Theorems 3.5, 3.6 with (4.2), we can prove the following theorem.

**Theorem 4.3.** *Let the assumptions and notations be given as in Theorems 3.2, 3.5, 3.6 and 4.2. Furthermore, let  $q$  be a nonzero real number and  $1 \leq p \leq \infty$ . Then for  $y \in C^r[0, t]$  and  $\vec{\xi}_{n-1} = (\xi_0, \xi_1, \dots, \xi_{n-1}) \in \mathbb{R}^{nr}$ ,  $T_q^{(p)}[F_k|Y_t](y, \vec{\xi}_{n-1})$  and  $T_q^{(p)}[F|Y_t](y, \vec{\xi}_{n-1})$  exist and they are given by*

$$\begin{aligned} T_q^{(p)}[F_k|Y_t](y, \vec{\xi}_{n-1}) &= k! \sum_{q_1 + \dots + q_n = k} \left[ \prod_{l=1}^{n-1} A(l, -iq, y, \xi_{l-1}, \xi_l, q_l) \right] \\ &\quad \times B(n, -iq, y, \vec{0}, \xi_{n-1}, q_n) \end{aligned}$$

and

$$T_q^{(p)}[F|Y_t](y, \vec{\xi}_{n-1}) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} T_q^{(p)}[F_k|Y_t](y, \vec{\xi}_{n-1}).$$



**Theorem 4.4.** *Under the assumptions and notations given as in Theorems 3.2, 3.7, 4.2 and 4.3, we have for  $y \in C^r[0, t]$  and  $\vec{\xi}_{n-1} = (\xi_0, \xi_1, \dots, \xi_{n-1}) \in \mathbb{R}^{nr}$*

$$T_q^{(p)}[G_k|Y_t](y, \vec{\xi}_{n-1}) = k! \sum_{q_1 + \dots + q_n = k} \left[ \prod_{l=1}^{n-1} A(l, -iq, y, \xi_{l-1}, \xi_l, q_l) \right] \\ \times \int_{\mathbb{R}^r} B(n, -iq, y, \vec{v}, \xi_{n-1}, q_n) \exp\{i\langle y(t), \vec{v} \rangle\} d\nu(\vec{v})$$

and

$$T_q^{(p)}[G|Y_t](y, \vec{\xi}_{n-1}) = \int_{\mathbb{R}^r} \exp\left\{i\langle y(t) + \xi_{n-1}, \vec{v} \rangle + \frac{t - t_{n-1}}{2qi} \|\vec{v}\|_{\mathbb{R}^r}^2\right\} d\nu(\vec{v}) \\ + \sum_{k=1}^{\infty} \frac{1}{k!} T_q^{(p)}[G_k|Y_t](y, \vec{\xi}_{n-1}).$$

*Proof.* For  $\vec{\xi}_{n-1} = (\xi_0, \xi_1, \dots, \xi_{n-1}) \in \mathbb{R}^{nr}$ , let  $(\vec{\xi}_{n-1}, \xi_n) = (\xi_0, \xi_1, \dots, \xi_{n-1}, \xi_n) \in \mathbb{R}^{(n+1)r}$ , where  $\xi_n \in \mathbb{R}^r$ . Then for  $\lambda > 0$  we have by Theorem 3.7

$$K_{G_k}^\lambda(y, \vec{\xi}_{n-1}) \\ = \left[ \frac{\lambda}{2\pi(t - t_{n-1})} \right]^{\frac{r}{2}} \int_{\mathbb{R}^r} I_{G_k}^\lambda(y, (\vec{\xi}_{n-1}, \xi_n)) \exp\left\{-\frac{\lambda \|\xi_n - \xi_{n-1}\|_{\mathbb{R}^r}^2}{2(t - t_{n-1})}\right\} d\xi_n \\ = k! \left[ \frac{\lambda}{2\pi(t - t_{n-1})} \right]^{\frac{r}{2}} \int_{\mathbb{R}^r} \psi(y(t) + \xi_n) \left[ \sum_{q_1 + \dots + q_n = k} \prod_{l=1}^n A(l, \lambda, y, \xi_{l-1}, \xi_l, q_l) \right] \\ \times \exp\left\{-\frac{\lambda \|\xi_n - \xi_{n-1}\|_{\mathbb{R}^r}^2}{2(t - t_{n-1})}\right\} d\xi_n \\ = k! \sum_{q_1 + \dots + q_n = k} \left[ \prod_{l=1}^{n-1} A(l, \lambda, y, \xi_{l-1}, \xi_l, q_l) \right] \left[ \frac{\lambda}{2\pi(t - t_{n-1})} \right]^{\frac{r}{2}} \int_{\mathbb{R}^r} \exp\{i\langle y(t), \vec{v} \rangle\} \\ \times \int_{\mathbb{R}^r} A(n, \lambda, y, \xi_{n-1}, \xi_n, q_n) \exp\left\{i\langle \xi_n, \vec{v} \rangle - \frac{\lambda \|\xi_n - \xi_{n-1}\|_{\mathbb{R}^r}^2}{2(t - t_{n-1})}\right\} d\xi_n d\nu(\vec{v}).$$

Now by Theorem 3.2 and Lemma 4.1 we have

$$\left[ \frac{\lambda}{2\pi(t - t_{n-1})} \right]^{\frac{r}{2}} \int_{\mathbb{R}^r} A(n, \lambda, y, \xi_{n-1}, \xi_n, q_n) \\ \times \exp\left\{i\langle \xi_n, \vec{v} \rangle - \frac{\lambda \|\xi_n - \xi_{n-1}\|_{\mathbb{R}^r}^2}{2(t - t_{n-1})}\right\} d\xi_n = B(n, \lambda, y, \vec{v}, \xi_{n-1}, q_n)$$

where  $B(n, \lambda, y, \vec{v}, \xi_{n-1}, q_n)$  is given as in Theorem 4.2. We also have by (3.4)

$$\begin{aligned} & \left[ \frac{\lambda}{2\pi(t-t_{n-1})} \right]^{\frac{r}{2}} \int_{\mathbb{R}^r} \psi(y(t) + \xi_n) \exp \left\{ -\frac{\lambda \|\xi_n - \xi_{n-1}\|_{\mathbb{R}^r}^2}{2(t-t_{n-1})} \right\} d\xi_n \\ &= \left[ \frac{\lambda}{2\pi(t-t_{n-1})} \right]^{\frac{r}{2}} \int_{\mathbb{R}^r} \exp \{ i \langle y(t) + \xi_{n-1}, \vec{v} \rangle \} \int_{\mathbb{R}^r} \exp \left\{ i \langle \xi_n - \xi_{n-1}, \vec{v} \rangle \right. \\ & \quad \left. - \frac{\lambda \|\xi_n - \xi_{n-1}\|_{\mathbb{R}^r}^2}{2(t-t_{n-1})} \right\} d\xi_n d\nu(\vec{v}) \\ &= \int_{\mathbb{R}^r} \exp \left\{ i \langle y(t) + \xi_{n-1}, \vec{v} \rangle - \frac{t-t_{n-1}}{2\lambda} \|\vec{v}\|_{\mathbb{R}^r}^2 \right\} d\nu(\vec{v}). \end{aligned}$$

By the Morera's theorem and the dominated convergence theorem, we have the theorem.  $\square$

**Remark 4.5.**

- If  $F_k, G_k, F$  and  $G$  are defined on  $r$ -dimensional Wiener space, then we can obtain the same results in Theorems 4.2, 4.3 and 4.4 with  $\xi_0 = \vec{0} \in \mathbb{R}^r$  in the expression of  $\vec{\xi}_{n-1}$ .
- If  $\eta = \mu$  or  $\eta = \sum_{l=1}^n \sum_{j=1}^{r_l} w_{l,j} \delta_{p_{l,j}}$ , we can obtain more simple expressions in Theorems 4.2, 4.3 and 4.4.
- If some of the  $p_{l,j}$ s are in the set  $\{t_0, t_1, \dots, t_n\}$ , we can obtain all the results in the present section with minor modifications.
- If  $\eta = \mu + \sum_{l=1}^n \sum_{j=1}^{r_l} w_{l,j} \delta_{p_{l,j}}$  and some of the  $r_l$ s are  $\infty$ , then, using (3.13), we can show that  $T_q^{(p)}[G|Y_t](y, \vec{\xi}_{n-1})$  exists in Theorem 4.4.

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